African Diaspora Journal of Mathematics
Special Volume in Honor of Prof. Gaston M. N'Guérékata ISSN 1539-854X
Volume 15, Number 2, pp. 76–86 (2013) www.math-res-pub.org/adjm

Weakly Almost Periodic Functions in Topological Vector Spaces

Saud M. Alsulami*
Department of Mathematics, King Abdulaziz University,
P. O. Box 80203, Jeddah-21589, Saudi Arabia

Liaqat Ali Khan†
Department of Mathematics, King Abdulaziz University,
P. O. Box 80203, Jeddah-21589, Saudi Arabia

Abstract

Harald Bohr was the founder of the theory of almost periodicity. The theory of almost periodic functions taking values in locally convex spaces was studied by G.M. N'Guérékata in his papers [9, 10]. Khan and Alsulami [12] studied the concept of almost periodicity in the general topological vector spaces. In this paper, we pursue their study further and extend the concept of weakly almost periodicity to topological vector spaces having non-trivial duals.

AMS Subject Classification: Primary 47H10; Secondary 54H25,54H10.

Keywords: weakly almost periodic functions, topological vector spaces.

1 Introduction

The topological vector spaces (TVSs) of continuous vector-valued functions have been the objects of intensive study for the last several decades in the realm of topological algebraic analysis and have played significant role in unification and classification of results in the broader areas of functional analysis. Harald Bohr was the founder of the theory of almost periodicity which had rapidly led to a strong development of harmonic analysis on groups and compact topological semigroups of linear operators. The theory has attracted many mathematicians for decades. The concept became one of the most attractive topics in the qualitative theory of differential equations because of their significance and applications in physics, mathematical biology, control theory, and other related fields. Some generalizations of the concept have been introduced successfully by V.V. Stepanov and A.S. Besicovitch. Almost periodic functions defined on the real line with values in a Banach space were studied by S. Bochner and developed by several mathematicians including C. Corduneanu,

*E-mail address: alsulami@kau.edu.sa
†E-mail address: lkhan@kau.edu.sa

S. Zaidman, J.A. Goldstein, L. Amerio, G. Prouse, K. Deleeuw, I. Glicksberg, A.M. Fink and others. Applications include ordinary, partial as well as abstract differential equations, topological and smooth dynamical systems, statistics, etc. The theory of almost periodic functions taking values in locally convex spaces (LCSs) was studied by G.M. N'Guérékata in his papers [9, 10].

Khan and Alsulami [12] had studied and generalized the concept of almost periodicity to general topological vector spaces, not necessarily locally convex. In this paper, we pursue their study further and extend the concept of weakly almost periodicity to general topological vector spaces, not necessarily locally convex. We hope that our results will open the door to many applications of differential and integro-differential equations in general topological vector spaces. In order to make this paper easy to be read, we will recall some results in section 2 and 3 from [12].

2 Preliminaries

In this section, we recall some results of [12]. Throughout this paper, E denotes a Hausdorff topological vector space (in short, a TVS) over the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) and having a base $\mathcal{W}$ of neighbourhoods of 0 consisting of balanced sets.

2.1 Differentiation in TVSs

Definition 2.1. Let E be a TVS. A function $f : (a,b) \to E$ is said to be *differentiable* at $t_0 \in (a,b)$ if there exists an element $z \in E$, denoted by $f^{'}(t_0)$, such that, given any balanced neighbourhood V of 0 in E, there exists a $\delta > 0$ satisfying,

$$\frac{f(t_0+h)-f(t_0)}{h} - f^{'}(t_0) \in V,$$

whenever $0 < |h| < \delta$; $f^{'}(t_0)$ is called the *Gateaux derivative* of f at t_0 and we briefly write as,

$$f^{'}(t_0) = \lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h}.$$

The following theorems extend results of [14].

Theorem 2.2. *Let E be a TVS. A function $f : (a,b) \to E$ can have at most one derivative at any point $t = t_0$.*

Proof. Suppose $f : [a,b] \to E$ has two derivatives $z_1, z_2 \in E$ at $t = t_0$, and let $W \in \mathcal{W}$. Choose balanced $V \in \mathcal{W}$ with $V + V \subseteq W$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$\frac{f(t_0+h)-f(t_0)}{h} - z_1 \in V \text{ for } 0 < |h| < \delta_1.$$

$$\frac{f(t_0+h)-f(t_0)}{h} - z_2 \in V \text{ for } 0 < |h| < \delta_2.$$

Then, for $0 < |h| < \min\{\delta_1, \delta_2\}$,

$$\begin{aligned} z_1 - z_2 &= [z_1 - \frac{f(t_0+h)-f(t_0)}{h}] + [\frac{f(t_0+h)-f(t_0)}{h} - z_2] \\ &\in -V + V \subseteq W. \end{aligned}$$

Since $W \in \mathcal{W}$ is arbitrary, $z_1 - z_2 \in \cap_{W\in\mathcal{W}} W = \{0\}$ as E is assumed to be Hausdorff. Thus $z_1 = z_2$.

Theorem 2.3. *Let E be a TVS. If a function $f : (a,b) \to E$ is differentiable at $t = t_0 \in (a,b)$, then f is continuous at t_0.*

Proof. Let $W \in \mathcal{W}$. Choose $V \in \mathcal{W}$ with $V + V \subseteq W$. Since f is differentiable at $t = t_0$, there exists a $\delta = \delta_V > 0$ satisfying

$$\frac{f(t_0+h) - f(t_0)}{h} - f^{'}(t_0) \in V \text{ for } 0 < |h| < \delta.$$

This implies that

$$f(t_0+h) - f(t_0) - hf^{'}(t_0) \in hV \text{ for } 0 < |h| < \delta,$$

and hence

$$\begin{aligned} f(t_0+h) - f(t_0) &= f(t_0+h) - f(t_0) - hf^{'}(t_0) + hf^{'}(t_0) \\ &\in hV + hf^{'}(t_0) \text{ for } 0 < |h| < \delta. \end{aligned}$$

Since V is absorbing, choose $0 < \delta^{'} < 1$ such that $hf^{'}(t_0) \in V$ for $|h| < \delta^{'}$ Hence, if $|h| < \min\{\delta, \delta^{'}\}$,

$$f(t_0+h) - f(t_0) \in V + V \subseteq W.$$

Thus f is continuous at t_0.

2.2 Almost Periodic Functions in TVSs

Definition 2.4. (i) A subset J of R is called *relatively dense* in R if there exists a number $\ell > 0$ such that every interval $[t, t+\ell]$ contains at least one point of J.

(ii) Let (E, τ) be TVS with a base $\mathcal{W}$ of balanced neighbourhoods of 0, and let $f : R \to E$ and $W \in \mathcal{W}$. Then a number $\tau = \tau_W$ is called a *W-almost period* of f if

$$f(t+\tau) - f(t) \in W \text{ for all } t \in R.$$

Let $P_{W,f}$ denote the set of all W-almost period of f.

Definition 2.5. $f : R \to E$ is called *almost periodic* if it is continuous and, for each $W \in \mathcal{W}$, there exists a number $\ell = \ell_W > 0$ such that each interval $[t, t+\ell_W]$ contains at least one point $\tau_W \in P_{W,f}$ such that

$$f(t+\tau_W) - f(t) \in W \text{ for all } t \in R.$$

Clearly, $P_{W,f}$ is relatively dense in R.

Equivalently:

Definition 2.6. A continuous function $f : R \to E$ is called *almost periodic* if, for each $W \in \mathcal{W}$, there exists a set $P_{W,f} \subseteq R$ such that

(i) $P_{W,f}$ is relatively dense in R; i.e. there exists a number $\ell = \ell_W > 0$ such that each interval $[t, t+\ell_W] \subseteq R$ contains at least one point $\tau_W \in P_{W,f}$,

(ii)

$$f(t+\tau_W) - f(t) \in W \text{ for all } \tau_W \in P_{W,f} \text{ and } t \in R.$$

τ_W is then called a W-translation number of the function f.

We recall the following properties of almost periodic functions.

Theorem 2.7. (i) *Any almost periodic function $f : R \to E$ has precompact range $f(R)$; hence f is bounded.*

(ii) *If $f : R \to E$ is almost periodic, then f is uniformly continuous on R.*

(iii) *If $\{f_n\}$ is a sequence of almost periodic functions which converges uniformly on R to a function f, then f is also almost periodic.*

(iv) *Let E be a complete TVS. If $f : R \to E$ is almostperiodic, then the functions $\lambda f (\lambda \in \mathbb{K})$ and $\overline{f}(t) \equiv f(-t)$ are also almost periodic.*

Moreover, the almost periodic functions satisfy the Bochner's criterion., see [12].

Theorem 2.8. *Let E be a quasi-complete metrizable TVS and $f : R \to E$. Then, f is almost periodic if and only if for every real sequence $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that $\{f(t+s_n)\}$ is uniformly convergent in $t \in R$.*

3 Weakly Almost Periodic Functions

Recall that, if X is a topological space and E a TVS with dual E^*, then a function $f : X \to E$ is said to be *weakly continuous* if, if for each $\varphi \in E^*$, the function $\varphi \circ f : X \to \mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) is continuous. Clearly, any continuous function (i.e. a strongly continuous function) $f : X \to E$ is weakly continuous; the converse holds if E is finite dimensional. We mention that the concept of weak continuity is well-defined in the non-locally convex setting, at least in the cases of $E = \ell_p$ and $E = H_p$, $0 < p < 1$. Indeed in such cases, the dual spaces $(\ell_p)^*$ and $(H_p)^*$ are non-trivial and in fact separate the points of ℓ_p and H_p, respectively. However, it is not well-defined in the case of $E = L_p$, $0 < p < 1$, since $(L_p)^*$ may be the trivial space $\{0\}$.

Let E be a TVS with non-trivial dual E^*.

Definition 3.1. A function $f : R \to E$ is called **weakly almost periodic** (we write **w.a.p.**) in E if, for every $x^* \in E^*$, the numerical function $x^* f : R \to \mathbb{K}$ is a.p.

Obviously,

(1) Every a.p. function $f : \mathbb{R} \to E$ is w.a.p.

(2) If $f : \mathbb{R} \to E$ is w.a.p., then it is weakly continuous and weakly bounded.

The following example [7] shows that the w.a.p. function may not be almost periodic even in the scalar case.

Example 3.2. Consider

$$f_0(t) = \begin{cases} e^{2i\pi\alpha n^2} & \text{if } 0 \le n < t < n+1; \\ 0 & \text{if } t < 0; \end{cases}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Let $c \in \mathbb{N}$, then the convolution product

$$f = \frac{\sin(4\pi\alpha ct)}{t} * f_0 \in WAP(\mathbb{R}, \mathbb{R}).$$

is weakly almost periodic and not almost periodic.

Definition 3.3. For any continuous bounded function $f : \mathbb{R} \to E$, its **primitive** (or **indefinite integral**) is defined as the function $F : \mathbb{R} \to E$ given by

$$F(t) = \int_0^t f(s)ds, \quad t \in \mathbb{R}.$$

Theorem 3.4. *Let E be a complete TVS. Let $f : \mathbb{R} \to E$ be a w.a.p. and continuous function. If F is the primitive of f with $F(\mathbb{R})$ weakly bounded, then F is w.a.p..*

Proof. We first note existence of the integral because of continuity of f over $\mathbb{R}$. Take any $x^* \in E^*$. Since f is w.a.p, $(x^* f)(t)$ is a.p. By continuity of x^*, we have

$$(x^* F)(t) = \int_0^t (x^* f)(\sigma)d\sigma.$$

Since, by our assumption, $F(\mathbb{R})$ is weakly bounded, it follows that $x^* F$ is bounded. Now $(x^* F)(t)$ is a.p.

Before proving the next result, we need to prove the following lemma.

Lemma 3.5. *Let E be a complete metrizable TVS and $\varphi : \mathbb{R} \to E$ be a.p.. Let $\{s_n\}$ be a real sequence such that $\lim_{n\to\infty} \varphi(s_n + a_k)$ exists for each $k = 1, 2, ...$, where $\{a_k\}$ is dense in $\mathbb{R}$. Then $\{\varphi(t + s_n)\}$ is uniformly convergent in $t \in \mathbb{R}$.*

Proof. Suppose by contradiction $\{\varphi(t + s_n)\}$ is not uniformly convergent in t; then there exists a $W \in \mathcal{W}$ such that for every $N = 1, 2, ...$, there exists n_N, $m_N \in \mathbb{N}$ and $t_N \in \mathbb{R}$ such that:

$$\varphi(t_N + s_{n_N}) - \varphi(t_N + s_{m_N}) \notin W. \tag{3.1}$$

By Bochner's criterion we can extract two subsequences $(s'_{n_N}) \subset (s_{n_N})$ and $(s'_{m_N}) \subset (s_{m_N})$ such that

$$\begin{aligned} \lim_{N\to\infty} \varphi(t + s'_{n_N}) &= g_1(t) \text{ uniformly in } t \in \mathbb{R}, \\ \lim_{N\to\infty} \varphi(t + s'_{m_N}) &= g_1(t) \text{ uniformly in } t \in \mathbb{R}. \end{aligned}$$

Let $V \in \mathcal{W}$ be balanced with $V + V + V \subset W$. Then there exists $N_0 = N_{0_V}$ such that if $N > N_0$,

$$\varphi(t_N + s'_{n_N}) - g_1(t_N) \in V,$$
$$\varphi(t_N + s'_{m_N}) - g_2(t_N) \in V.$$

We conclude $g_1(t_N) - g_2(t_N) \notin V$. For, if not, we should get

$$\begin{aligned} \varphi(t_N + s'_{n_N}) - \varphi(t_N + s'_{m_N}) &= \varphi(t_N + s'_{n_N}) - g_1(t_N) \\ &\quad + g_1(t_N) - g_2(t_N) \\ &\quad + g_2(t_N) - \varphi(t_N + s'_{m_N}) \\ &\in V + V + V \subseteq W, \end{aligned}$$

which contradicts (3.1).

We have found V with the property that if N is large enough, there exists $t_N \in \mathbb{R}$ such that

$$g_1(t_N) - g_2(t_N) \notin V.$$

But this is impossible. In fact, if we take a subsequence $\{b_k\}_{k=1}^{\infty} \subset \{a_k\}_{k=1}^{\infty}$ and $b_k \to t_N$, then we have

$$\lim_{N\to\infty} \varphi(b_k + s'_{n_N}) = \lim_{N\to\infty} \varphi(b_k + s'_{m_N})$$

for every k, and therefore $g_1(b_k) = g_2(b_k)$ for every k; by continuity of g_1 and g_2, $g_1(t_N) = g_2(t_N)$; thus

$$g_1(t_N) - g_2(t_N) = 0 \in U \text{ for every } U \in \mathcal{W}. \quad \square$$

Theorem 3.6. *Let E be a metrizable TVS with point separating dual E^* and $f : \mathbb{R} \to E$ a given function. Then the following are equivalent:*

(a) f is a.p.

(b) If f is w.a.p. and $f(\mathbb{R})$ is relatively compact in E.

Proof. Clearly, by earlier remarks, $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$ (I) First we show that f is continuous. Suppose there exists t_0 such that f is discontinuous at t_0. Then we can find a $W \in \mathcal{W}$ and two sequences $\{s'_{n_1}\}$ and $\{s'_{n_2}\}$ such that $\lim_{n\to\infty} s'_{n_1} = 0 = \lim_{n\to\infty} s'_{n_2}$ and

$$f(t_0 + s'_{n_1}) - f(t_0 + s'_{n_2}) \notin W \tag{3.2}$$

for every $n \in \mathbb{N}$. By relative compactness of $f(\mathbb{R})$ we can extract $\{s'_{n_1}\}$ and $\{s'_{n_2}\}$ from the respective first two sequences such that $\lim_{n\to\infty} f(t_0 + s_{n_1}) = a_1 \in E$ and $\lim_{n\to\infty} f(t_0 + s_{n_2}) = a_2 \in E$. Consequently, using (3.2), we get $a_1 \neq a_2$ in E. Therefore, since E^* is point separating dual, there exists $x^* \in E^*$ such that

$$x^*(a_1) \neq x^*(a_2). \tag{3.3}$$

By continuity of x^*, we have:

$$\begin{aligned} x^*(a_1) &= x^*[\lim_{n\to\infty} f(t_0 + s_{n_1})] = \lim_{n\to\infty} x^* f(t_0 + s_{n_1}) \\ &= \lim_{n\to\infty} x^* f(t_0 + s_{n_2}) = x^*[\lim_{n\to\infty} x^* f(t_0 + s_{n_2})] = x^*(a_2), \end{aligned}$$

which contradicts (3.3). f is therefore continuous over $\mathbb{R}$.

(II) We are now going to show almost-periodicity of f:

Consider arbitrary real sequences $\{h_n\}$ and $\{\eta_r\}_{r=1}^{\infty}$ the rational numbers.

By relative compacity of $f(\mathbb{R})$, we can extract a subsequence $\{h_n\}$ (we do not change notation) such that for each r,

$$\lim_{n\to\infty} f(\eta_r + h_n) = x_r \text{ exists in } E \tag{3.4}$$

Now $f(\eta_r + h_n)\}$ is uniformly convergent in r. For, if it is not; then we find a $U \in \mathcal{W}$ and three subsequences $\{\xi_r\}_{r=1}^{\infty} \subset \{\eta_r\}_{r=1}^{\infty}$, $\{h'_r\}_{r=1}^{\infty} \subset \{h_r\}_{r=1}^{\infty}$, $\{h''_r\}_{r=1}^{\infty} \subset \{h_r\}_{r=1}^{\infty}$ and

$$f(\xi_r + h'_r) - f(\xi_r + h''_r) \notin U. \tag{3.5}$$

By relative compactness of $f(\mathbb{R})$ we may say

$$\lim_{r\to\infty} f(\xi_r + h'_r) = b' \in E, \quad \lim_{r\to\infty} f(\xi_r + h''_r) = b'' \in E, \tag{3.6}$$

and using (3.5), we get

$$b' - b'' \notin U.$$

Since the dual E^* is point separating, there exists $x^* \in E^*$ such that

$$x^*(b') \neq x^*(b''). \tag{3.7}$$

But $f(t)$ is w.a.p. hence $(x^* f)(t)$ is a.p. and consequently it is uniformly continuous over $\mathbb{R}$.

Consider the sequence of functions $(\varphi_n\}$ defined by:

$$\varphi_n(t) = (x^* f)(t + h_n),\ n = 1, 2, ...$$

The equality $\varphi_n(t+\tau) - \varphi_n(t) = x^* f(t+\tau+h_n) - x^* f(t+h_n)$ shows almost-periodicity of each φ_n. Also $\{\varphi_n\}_{n=1}^{\infty}$ is equi-uniformly continuous over $\mathbb{R}$ because $(x^* f)$ is uniformly continuous over $\mathbb{R}$, as it is easy to see. Using (3.4), we can say

$$\lim_{n\to\infty} x^* f(\eta_r + h_n) = x^*(x_r) \text{ for every } r.$$

Therefore, by Lemma 3.4, $\{x^* f(t + h_n)\}$ is uniformly convergent in t.

Consider now the sequences $\{\xi_r + h'_r\}$ and $\{\xi_r + h''_r\}$. By Bochner's criterion, we extract two subsequences (we use the same notations) such that $\{x^* f(t + \xi_r + h'_r))$ and $\{x^* f(t + \xi_r + h''_r)\}$ are uniformly convergent in $t \in \mathbb{R}$.

Let us now prove

$$\lim_{r\to\infty} x^* f(t + \xi_r + h'_r) = \lim_{r\to\infty} x^* f(t + \xi_r + h''_r). \tag{3.8}$$

Consider the inequality for each $r = 1, 2, ...$

$$\begin{aligned} \left|x^* f(t+\xi_r+h'_r) - x^* f(t+\xi_r+h''_r)\right| \quad \leq \quad & \left|x^* f(t+\xi_r+h'_r) - x^* f(t+\xi_r+h_r)\right| \\ & + \left|x^* f(t+\xi_r+h_r) - x^* f(t+\xi_r+h''_r)\right| \end{aligned} \tag{3.9}$$

Let $\varepsilon > 0$ be given; as $(x^* f(t+h_r))_{r=1}^{\infty}$ is uniformly convergent in t, we choose η_ε such that for $r, s > \eta_\varepsilon$, we have $|x^* f(t+h_s) - x^* f(t+h_r)| < \frac{\varepsilon}{2}$, for $t \in \mathbb{R}$; then for $r, s > \eta_\varepsilon$, we get

$$\left| x^* f(t+\xi_r + h_s) - x^* f(t+\xi_r + h_r) \right| < \frac{\varepsilon}{2}.$$

Consequently, for $r > \eta_\varepsilon$, we get:

$$\begin{aligned} \left| x^* f(t+\xi_r + h'_r) - x^* f(t+\xi_r + h_r) \right| &< \frac{\varepsilon}{2}, \\ \left| x^* f(t+\xi_r + h''_r) - x^* f(t+\xi_r + h_r) \right| &< \frac{\varepsilon}{2}, \end{aligned}$$

and the inequality (3.9) gives:

$$\left| x^* f(t+\xi_r + h'_r) - x^* f(t+\xi_r + h''_r) \right| < \varepsilon$$

for $t \in \mathbb{R}$. Then, (3.8) is proved.

Now take $t = 0$; then using (3.6) we get:

$$x^*(b') = \lim_{r\to\infty} x^* f(t+\xi_r + h'_r) = \lim_{r\to\infty} x^* f(\xi_r + h''_r) = x^*(b'')$$

which contradicts (3.7). This proves uniform convergence in r for $\{f(\eta_r + h_n)\}$. If $i, j > N$, we have

$$f(\eta_r + h_n) - f(\eta_r + h_u) \in U \text{ for every } r \tag{3.10}$$

Therefore if $t \in \mathbb{R}$, we take a subsequence of $\{\eta_r\}$ which converges to t. Then using continuity of f and the relation (3.10), we obtain, for $i.j > N$,

$$f(t+h_i) - f(t+h_j) \in U.$$

Thus, f is a.p.

Theorem 3.7. *Let E be a metrizable TVS with point separating dual E^* If $f : \mathbb{R} \to E$ is a.p. and the range $F(\mathbb{R})$ of its primitive is relatively compact in E, then F is a.p.*

Proof. Immediate from Theorems 3.3 and 3.5.

Theorem 3.8. *Let E be a complete LCS. If f is a.p. and its derivative f' uniformly continuous on $\mathbb{R}$, then f' is also a.p.*

Proof. Consider the functions

$$g_n(t) = \frac{f(t+\frac{1}{n}) - f(t)}{\frac{1}{n}} = n[f(t+\frac{1}{n}) - f(t)], \quad n = 1, 2, ...,$$

which are clearly periodic. Further, since the derivative f' exists,

$$f'(t) = \lim_{n\to\infty} g_n(t).$$

It suffices to prove that the sequence $\{g_n(t)\}$ converges uniformly over $\mathbb{R}$ to $f'(t)$.

Let $U \in \mathcal{W}$ be balanced. Then there exist $\varepsilon > 0$ and seminorms $p_i, 1 \leq i \leq n$, on E such that

$$U = \{x \in E : p_i(x) < \varepsilon \text{ for all } 1 \leq i \leq n\}.$$

By uniform continuity of f', we can choose $\delta = \delta_U > 0$ such that

$$f'(t_1) - f'(t_2) \in U \text{ for every } t_1, t_2 \text{ with } |t_1 - t_2| < \delta. \tag{3.11}$$

We write

$$\begin{aligned} n \int_0^{1/n} [f'(t+\sigma) - f'(t)]d\sigma &= n[f(t+\sigma) - f'(t)\sigma]_0^{1/n} \\ &= n\{[f(t+\frac{1}{n}) - f'(t)\frac{1}{n}] - [f(t) - 0]\}, \end{aligned}$$

so

$$g_n(t) - f'(t) = [f(t+\frac{1}{n}) - f'[t] - f'(t) = n \int_0^{1/n} [f'(t+\sigma) - f'(t)]d\sigma.$$

Therefore, if we take $N = N_U > \frac{1}{\delta}$, then for $n \geq N$, we have $|t + \frac{1}{n}) - t| = \frac{1}{n} < \frac{1}{N} < \delta$, hence, by (3.11),

$$p_i\,[g_n(t) - f'(t)] \leq n \int_0^{1/n} p_i[f'(t+\sigma) - f'(t)]d\sigma < n[\varepsilon\sigma]_0^{1/n} = \varepsilon$$

for every semi-norm $p_i, 1 \leq i \leq n$, and every $t \in \mathbb{R}$.

Finally, we prove the following generalization of ([9], Theorem 4) without metrizability and ([10], Theorem1) from LCSs to TVSs.

Theorem 3.9. *Let E be a complete TVS. If $f : \mathbb{R} \to E$ is a.p., then for every real sequence $\{s_n\}$, there exists a subsequence $\{s'_n\}$ such that for every $U \in W$,*

$$f(t + s'_n) - f(t + s'_m) \in U$$

for all $t \in \mathbb{R}$, m and n.

Proof. Let $U \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V + V \subseteq U$. By the definition of almost-periodicity, there exists $l = l(V)$ (therefore l depends on U) such that in every real interval of length l, there exists τ such that

$$f(t+\tau) - f(t) \in V$$

for every $t \in \mathbb{R}$.

Now for each s_n, we can find τ_n and σ_n such that $s_n = \tau_n + \sigma_n$ with τ_n a V-translation number of f and $\sigma_n \in [0, l]$ (it suffices to take $\tau_n \in [s_n - l, s_n]$ and then $\sigma_n = s_n - \tau_n$).

As f is uniformly continuous on $\mathbb{R}$, there exists $\delta = \delta(\varepsilon)$ such that

$$f(t') - f(t'') \in V \tag{3.12}$$

for all t', t'', $|t' - t''| < 2\delta$.

Also $0 \le \sigma_n \le l$ for every n; we can then subtract from $\{\sigma_n\}$, a convergent subsequence $\{\sigma_{n_k}\}$, by the Bolzano-Weierstrass theorem.

Let $\sigma = \lim\limits_{k\to\infty} \sigma_{n_k}$, with $0 \le \sigma_n \le l$. Now consider the subsequence $\{\sigma_{n_k}\}_{k=1}^{\infty}$ with

$$\sigma - \delta < \sigma_{n_k} < \sigma + \delta,\ k = 1, 2, ...$$

and let $(s_{n_k})_{k=1}^{\infty}$ be the corresponding subsequence where

$$s_{n_k} = \tau_{n_k} + \sigma_{n_k},\ k = 1, 2, ...$$

Let us prove the relation

$$f(t + s_{n_k}) - f(t + s_{n_j}) \in U \text{ for all } t \in \mathbb{R}. \tag{3.13}$$

For this, write

$$\begin{aligned} f(t + s_{n_k}) - f(t + s_{n_j}) &= f(t + \tau_{n_k} + \sigma_{n_k}) - f(t + \sigma_{n_k}) \\ &\quad + f(t + \sigma_{n_k}) - f(t + \sigma_{n_j}) \\ &\quad + f(t + \sigma_{n_j}) - f(t + \tau_{n_j} + \sigma_{n_j}). \end{aligned} \tag{3.14}$$

Because τ_{n_k} and τ_{n_j} are V-translation numbers of f, we shall get

$$\begin{aligned} &f(t + \tau_{n_k} + \sigma_{n_k}) - f(t + \sigma_{n_k}) \in V, \text{ for every } t \in \mathbb{R} \\ &f(t + \tau_{n_j} + \sigma_{n_j}) - f(t + \sigma_{n_j}) \in V, \text{ for every } t \in \mathbb{R}. \end{aligned} \tag{3.15}$$

On the other hand

$$\left|(t + \sigma_{n_k}) - (t + \sigma_{n_j})\right| = \left|\sigma_{n_k} - \sigma_{n_j}\right| < 2\delta;$$

therefore, by using relation (3.12), we get

$$f(t + \sigma_{n_k}) - f(t + \sigma_{n_k}) \in V, \text{ for every } t \in \mathbb{R}. \tag{3.16}$$

Finally we can deduce (3.13) by putting (3.15) and (3.16) in (3.14):

$$f(t + s_{n_k}) - f(t + s_{n_j}) \in V + V + V \subseteq U.$$

Then, by taking $s'_n = s_{n_k}$, $k = 1, 2, ...$

$$f(t + s'_n) - f(t + s'_m) = f(t + s_{n_k}) - f(t + s_{n_j}) \in U. \qquad \square$$

Acknowledgement

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under the Project No. (78/130/1432). The authors, therefore acknowledge with thanks DSR technical and financial support. The authors also wish to thank Professor G.M. N'Guérékata for providing us his reprints and other related material on almost periodic functions. Moreover, the authors would like to thank the editor for his useful comments.

References

[1] S. M. Alsulami, Abstract Cauchy Problems in TVS, J.King Abdulaziz Univ. (accepted)

[2] L. Amerio, G. Prouse, *Almost Periodic Functions and Functional Equations*, Van Nostrand-Reinhold, NewYork. 1971.

[3] D. Bugajewski, On topological properties of solution sets for differential equations in locally convex spaces, *Nonlinear Anal*, **47** (2001), 1211-1220.

[4] S. Bochner, Abstrakte fastperiodische Funktionen, *Acta Math.* (1933), 149 - 184.

[5] D. Bugajewski and G. M. N'Guérékata, Almost periodicity in Fréchét spaces, *J. Math. Anal. Appl.* **299** (2004), 534-549.

[6] C. Corduneanu, *Almost-Periodic Functions*, Interscience, 1968.

[7] F. Cherif, A Various Types of Almost Periodic Functions on Banach Spaces: Part II, *Int. Math. Forum*, Vol. **6**, no. 20 (2011), 953 - 985.

[8] A.M. Fink, *Almost Periodic Differential Equations. in: Lecture Notes in Math.*, vol.**37**, Springer-Verlag, Berlin, 1974.

[9] G.M. N'Guérékata, Almost periodicity in linear topological spaces and applications to abstract differential equations, *Internat. J. Math. Math. Sci.* **7** (1984), 529-540.

[10] G.M. N'Guérékata, Notes On Almost-Periodicity in Topological Vector Spaces, *Internat. J. Math. and Math. Sci.* Vol.**9** (1986), 201-204.

[11] G.M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic. Plenum, New York, 2001.

[12] L.A. Khan and S. M. Alsulami, Almost periodicity in linear topological spaces - Revisited, *Commun. Math. Anal.* **13**(2012), 54-63.

[13] A. I. Shtern, Almost Periodic functions and representations in locally convex spaces, *Russian Math. Surveys* Vol. **60:3** (2005), 489-557

[14] S. Zaidman, *Almost Periodic Functions in Abstract Spaces*, Pitman Advanced Publishing Program, Boston, 1985.