African Diaspora Journal of Mathematics
Special Volume in Honor of Prof. Gaston M. N'Guérékata
Volume 15, Number 2, pp. 26–34 (2013)

The Existence Results for Abstract Fractional Differential Equations with Nonlocal Conditions

Fang Li*
School of Mathematics, Yunnan Normal University,
Kunming 650092, P. R. China

Huiwen Wang†
School of Mathematics, Yunnan Normal University,
Kunming 650092, P. R. China

Abstract

This paper is concerned with the abstract fractional differential equations with nonlocal condition. By using the contraction mapping principle and the theory of the measures of noncompactness and the condensing maps, we obtain the existence results of mild solutions for the above equations.

AMS Subject Classification: 34K05; 47D06

Keywords: fractional differential equations; mild solution; existence; measures of noncompactness

1 Introduction

Fractional differential equations are increasingly used for many mathematical models in engineering, physics, economics, etc., so the theory of fractional differential equations has been extensively studied by several authors ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]).

On the other hand, Cauchy problems with nonlocal conditions are appropriate models for describing many natural phenomena, which cannot be described using classical Cauchy problems. This is why they have been studied extensively(cf., e.g., [5, 10, 11, 12, 13] and references therein).

Of concern is the following fractional differential equation on a separable Banach space X

$$\begin{aligned} &\frac{d^q}{dt^q}x(t) = Ax(t) + f(t, x(t)),\ t \in (0, T], \\ &x(0) = g(x), \end{aligned} \tag{1.1}$$

*E-mail address: fangli860@gmail.com
†E-mail address: hwwang114@gmail.com

where $T > 0,\ 0 < q < 1$. The fractional derivative is understood here in the Caputo sense. A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X, that is, there exists $M \geq 1$ such that $\|S(t)\| \leq M$.

In this paper, the mild solutions of (1.1) will be established under various conditions of the functions f, g. Firstly, we assume that f, g satisfy the Lipschitz conditions. Secondly, we establish the existence theorem based on a special measure of noncompactness without the assumptions that the nonlinearity f satisfies a Lipschitz type condition and the semigroup $\{S(t)\}_{t\geq 0}$ generated by A is compact.

2 Preliminaries

In this paper, we set $J := [0, T]$ and denote by X a separable Banach space with norm $\|\cdot\|$, by $L(X)$ the Banach space of all linear and bounded operators on X, and by $C([a, b], X)$ the space of all X-valued continuous functions on $[a, b]$ with the supremum norm as follows:

$$\|x\|_{[a,b]} = \|x\|_{C([a,b],X)} = \sup\{\|x(t)\| : t \in [a, b]\}, \text{ for any } x \in C([a, b], X).$$

Moreover, we abbreviate $\|\mu\|_{L^p([0,T],\mathbf{R}^+)}$ with $\|\mu\|_{L^p}$, for any $\mu \in L^p([0, T], \mathbf{R}^+)$.

Now we recall some basic concepts in the theory of measures of noncompactness and the condensing maps. (see, e.g., [14, 15]).

Definition 2.1. Let E be a Banach space, 2^E the family of all nonempty subsets of E, $(\mathcal{A}, \geq)$ a partially ordered set, $\beta : 2^E \to \mathcal{A}$. If for every $\Omega \in 2^E$:

$$\beta(\overline{co}(\Omega)) = \beta(\Omega) \quad \text{for every} \quad \Omega \in 2^E,$$

then β is called a measure of noncompactness (MNC) in E.

A MNC β is called:

(i) monotone, if $\Omega_0, \Omega_1 \in 2^E$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E$, $\Omega \in 2^E$;

(iii) invariant with respect to union with compact sets, if $\beta(\{D\} \cup \Omega) = \beta(\Omega)$ for every relatively compact set $D \subset E$, $\Omega \in 2^E$.

If $\mathcal{A}$ is a cone in a normed space, we say that the MNC β is

(iv) algebraically semiadditive, if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for each $\Omega_0, \Omega_1 \in 2^E$;

(v) regular, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω;

(vi) real, if $\mathcal{A}$ is $[0, +\infty)$ with the natural order.

Now, let $G : [0, h] \to 2^E$ be a multifunction. It is called:

(i) integrable, if it admits a Bochner integrable selection $k : [0, h] \to E$, $k(t) \in G(t)$ for a.e. $t \in [0, h]$;

(ii) integrably bounded, if there exists a function $\vartheta \in L^1([0,h], E)$ such that

$$\|G(t)\| := \sup\{\|k\| : k \in G(t)\} \leq \vartheta(t) \ \text{ a.e. } \ t \in [0,h].$$

As an example of the MNC possessing all these properties, we consider the Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

We present the following assertion about χ-estimates for a multivalued integral (Theorem 4.2.3 of [15]).

Proposition 2.2. *For an integrable, integrably bounded multifunction $G : [0,h] \to 2^X$ where X is a separable Banach space, let*

$$\chi(G(t)) \leq m(t), \qquad \text{for a.e. } t \in [0,h],$$

where $m \in L^1_+([0,h])$. Then $\chi(\int_0^t G(s)ds) \leq \int_0^t m(s)ds$ for all $t \in [0,h]$.

Definition 2.3. A continuous map $\mathfrak{F} : Y \subseteq E \to E$ is called condensing with respect to a MNC β (or β-condensing) if for every bounded set $\Omega \subseteq Y$ which is not relatively compact, we have

$$\beta(\mathfrak{F}(\Omega)) \ngeq \beta(\Omega).$$

The following fixed point principle (see, e.g., [14, 15]) will be used later.

Theorem 2.4. *Let $\mathfrak{M}$ be a bounded convex closed subset of E and $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ a β-condensing map. Then $Fix\mathfrak{F} = \{x : x = \mathfrak{F}(x)\}$ is nonempty.*

Based on the works in [1, 2, 16], we set the following definition.

Definition 2.5. Let

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad \sigma \in (0,\infty)$$

be a one-side stable probability density, and

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0, \qquad \sigma \in (0,\infty).$$

For any $x \in X$, we define operators $\{Q(t)\}_{t\geq 0}$ and $\{R(t)\}_{t\geq 0}$ by

$$\begin{aligned} Q(t)x &= \int_0^\infty \xi_q(\sigma)S(t^q\sigma)x d\sigma, \\ R(t)x &= q\int_0^\infty \sigma t^{q-1}\xi_q(\sigma)S(t^q\sigma)x d\sigma. \end{aligned}$$

Remark 2.6. ([16]) It is not difficult to verify that for $v \in [0,1]$,

$$\int_0^\infty \sigma^v \xi_q(\sigma)d\sigma = \int_0^\infty \sigma^{-qv}\varpi_q(\sigma)d\sigma = \frac{\Gamma(1+v)}{\Gamma(1+qv)}.$$

Then, we can see

$$\|Q(t)\| \le M, \quad \|R(t)\| \le \frac{M}{\Gamma(q)} t^{q-1}, \quad t > 0.$$

We define the mild solution for problem (1.1) as follows.

Definition 2.7. A function $x \in C(J, X)$ satisfying the equation

$$x(t) = Q(t)g(x) + \int_0^t R(t-s) f(s, x(s)) ds, \quad t \in J, \tag{2.1}$$

is called a mild solution of problem (1.1).

3 Lipschitz conditions

Here, we will obtain mild solutions under the following assumptions.

(A1) $f : J \times X \to X$ is continuous. There exist constant $L, G > 0$ such that

$$\begin{aligned} \|f(t, x) - f(t, y)\| &\le L\|x - y\|, \ x, y \in X, \\ \|g(u) - g(v)\| &\le G\|u - v\|_J, \quad u, v \in C(J, X) \end{aligned}$$

for all $t \in J$.

(A2)

$$M\left(G + \frac{LT^q}{\Gamma(q+1)}\right) < 1.$$

Under these assumptions, we can prove the following result.

Theorem 3.1. *Let (A1)-(A2) be satisfied. Then the problem (1.1) has a unique mild solution.*

Proof. Define an operator $\mathcal{H}$ on $C(J, X)$ by

$$(\mathcal{H}x)(t) = Q(t)g(x) + \int_0^t R(t-s) f(s, x(s)) ds.$$

Then it is clear that $\mathcal{H} : C(J, X) \to C(J, X)$. Moreover, we have from Assumption (A1),

$$\begin{aligned} \|(\mathcal{H}x)(t) - (\mathcal{H}y)(t)\| &\le \|Q(t)\| \cdot \|g(x) - g(y)\| + \int_0^t \|R(t-s)\| \|f(s, x(s)) - f(s, y(s))\| ds \\ &\le MG\|x - y\|_J + \frac{LM}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s) - y(s)\| ds \\ &\le M\left(G + \frac{LT^q}{\Gamma(q+1)}\right)\|x - y\|_J, \quad x, y \in C(J, X). \end{aligned}$$

From (A2), we find that $\mathcal{H}$ is a contraction operator on $C(J, X)$, thus $\mathcal{H}$ has a unique fixed point, which gives rise to a unique mild solution. This completes the proof. $\square$

4 g is compact

In this section, we will derive mild solutions under the following assumptions.

(H1) $f : J\times X \to X$ satisfies $f(\cdot, w) : J \to X$ is measurable for all $w \in X$ and $f(t,\cdot) : X \to X$ is continuous for a.e. $t \in J$, and there exists a functions $\mu(\cdot) \in L^p(J, \mathbf{R}^+)$ $(p > \frac{1}{q})$ such that

$$\|f(t, w)\| \le \mu(t)\|w\|$$

for almost all $t \in J$.

(H2) There exists a function $\eta \in L^p(J,\mathbf{R}^+)$ such that for any bounded set $D \subset X$,

$$\chi(f(t, D)) \le \eta(t)\chi(D), \qquad \text{a.e.} \quad t \in J.$$

(H3) (i)The function $g : C(J, X) \to X$ is continuous and compact.

(ii) There exists a constant $N > 0$ such that

$$\|g(x)\| \le N \quad \text{for all } x \in C(J, X).$$

Theorem 4.1. *Assume that (H1)-(H3) are satisfied. Then there exists at least a mild solution of problem (1.1) on $C(J, X)$, provided that*

$$\frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \max\{\|\mu\|_{L^p}, \|\eta\|_{L^p}\} < 1, \tag{4.1}$$

where $l_{p,q} = (\frac{p-1}{pq-1})^{1-\frac{1}{p}}$.

Proof. Define the operator $\mathcal{G} : C(J, X) \to C(J, X)$ in the following way:

$$(\mathcal{G}x)(t) = Q(t)g(x) + \int_0^t R(t-s)f(s, x(s))ds, \quad t \in J.$$

Clearly, the operator $\mathcal{G}$ is well defined, and the fixed point of $\mathcal{G}$ is the mild solution of problem (1.1). We will show $\mathcal{G}$ is β-condensing.

Obviously, (H3) and the Lebesgue dominated convergence theorem enable us to prove that $\mathcal{G}$ is continuous.

Consider the set

$$B_r = \{x \in C(J, X) : \|x\|_J \le r\},$$

where r is a constant chosen so that

$$r > \frac{MN}{1 - \frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \|\mu\|_{L^p}}.$$

By the Hölder inequality, we have

$$\int_0^t (t-s)^{q-1}\mu(s)ds \le t^{\frac{pq-1}{p}} l_{p,q}\|\mu\|_{L^p} \le T^{\frac{pq-1}{p}} l_{p,q}\|\mu\|_{L^p}. \tag{4.2}$$

For $t \in J$, $x \in B_r$, by (4.2) we get

$$\begin{aligned}
\|(\mathcal{G}x)(t)\| &\leq \|Q(t)g(x)\| + \int_0^t \|R(t-s)f(s, x(s))\| ds \\
&\leq MN + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mu(s) \|x(s)\| ds \\
&\leq MN + \frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \|\mu\|_{L^p} \cdot r < r.
\end{aligned}$$

Hence for some positive number r, $\mathcal{G}B_r \subset B_r$.

Let χ be a Hausdorff MNC in X. For every bounded subset $\Omega \subset C(J, X)$, we consider the measure of noncompactness β in the space $C(J, X)$ with values in the cone $\mathbf{R}_+^2$ of the following way:

$$\beta(\Omega) = (\Psi(\Omega), mod_c(\Omega)),$$

where

$$\Psi(\Omega) = \sup_{t \in J} \chi(\Omega(t)),$$

and $mod_c(\Omega)$ is the module of equicontinuity of Ω given by:

$$mod_c(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \leq \delta} \|x(t_1) - x(t_2)\|.$$

Next, we show that the operator $\mathcal{G}$ is β-condensing on every bounded subset of $C(J, X)$. Let $\Omega \subset C(J, X)$ be a nonempty, bounded set such that

$$\beta(\mathcal{G}(\Omega)) \geq \beta(\Omega). \tag{4.3}$$

Firstly, we estimate $\Psi(\Omega)$. For any $t \in J$, we set

$$\widetilde{F}(\Omega)(t) = \left\{ \int_0^t R(t-s) f(s, x(s)) ds : x \in \Omega \right\}.$$

We consider the multifunction $s \in [0, t] \multimap F(s)$,

$$F(s) = \{R(t-s) f(s, x(s)) : x \in \Omega\}.$$

Obviously, F is integrable, and from (H1) and (4.2) it follows that it is integrably bounded. Moreover, noting that (H2) we have the following estimate for a.e. $s \in [0, t]$:

$$\begin{aligned}
\chi(F(s)) &\leq \frac{M}{\Gamma(q)} (t-s)^{q-1} \chi(\{f(s, x(s)) : x \in \Omega\}) \\
&\leq \frac{M}{\Gamma(q)} (t-s)^{q-1} \eta(s) \chi(\Omega(s)).
\end{aligned}$$

Applying Proposition 2.2, we have

$$\begin{aligned}
\chi(\widetilde{F}(\Omega)(t)) &= \chi\left(\int_0^t F(s) ds \right) \\
&\leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \eta(s) \chi(\Omega(s)) ds \\
&\leq \frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \|\eta\|_{L^p} \Psi(\Omega).
\end{aligned}$$

This, together with (H3)(i), shows

$$\chi(\mathcal{G}(\Omega)(t)) \le \frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \|\eta\|_{L^p} \Psi(\Omega).$$

Furthermore

$$\Psi(\mathcal{G}\Omega) \le \frac{M}{\Gamma(q)} l_{p,q} T^{\frac{pq-1}{p}} \|\eta\|_{L^p} \Psi(\Omega),$$

which implies, by (4.1) and (4.3), $\Psi(\Omega) = 0$.

Secondly, we will prove $mod_c(\Omega) = 0$. For $0 < t_2 < t_1 < T$ and $x \in \Omega$, we have

$$\begin{aligned}
& \int_0^{t_2} \|[R(t_1 - s) - R(t_2 - s)] f(s, x(s))\| ds \\
\le\ & q \int_0^{t_2} \int_0^{\infty} \sigma \left\| [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] \xi_q(\sigma) S((t_1 - s)^q \sigma) f(s, x(s)) \right\| d\sigma ds \\
& + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \|S((t_1 - s)^q \sigma) - S((t_2 - s)^q \sigma)\| \cdot \|f(s, x(s))\| d\sigma ds \\
\le\ & \frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| \cdot \mu(s) \|x(s)\| ds \\
& + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \|S((t_1 - s)^q \sigma) - S((t_2 - s)^q \sigma)\| \cdot \mu(s) \|x(s)\| d\sigma ds \\
\le\ & \|x\|_J \cdot \Big[\frac{M}{\Gamma(q)} \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| \cdot \mu(s) ds \\
& + q \int_0^{t_2} \int_0^{\infty} \sigma (t_2 - s)^{q-1} \xi_q(\sigma) \|S((t_1 - s)^q \sigma) - S((t_2 - s)^q \sigma)\| \cdot \mu(s) d\sigma ds \Big].
\end{aligned}$$

Obviously, the first term on the right-hand side tends to 0 as $t_2 \to t_1$. The second term on the right-hand side tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$.

Moreover, by (4.2), we can see $(t - \cdot)^{q-1} \mu(\cdot) \in L^1$ and

$$\begin{aligned}
\int_{t_2}^{t_1} \|R(t_1 - s)\| \|f(s, x(s))\| ds &\le \frac{M}{\Gamma(q)} \cdot \|x\|_J \int_{t_2}^{t_1} (t_1 - s)^{q-1} \mu(s) ds \\
&\to 0, \quad \text{as} \quad t_2 \to t_1.
\end{aligned}$$

Combining with the continuity of $S(t)$ in the uniform operator topology for $t > 0$, we have

$$\begin{aligned}
& \|(\mathcal{G}x)(t_1) - (\mathcal{G}x)(t_2)\| \\
\le\ & \|Q(t_1) - Q(t_2)\| N + \int_0^{t_2} \|[R(t_1 - s) - R(t_2 - s)] f(s, x(s))\| ds \\
& + \int_{t_2}^{t_1} \|R(t_1 - s)\| \|f(s, x(s))\| ds \\
\to\ & 0, \quad \text{as} \quad t_2 \to t_1,
\end{aligned}$$

then $mod_c(\mathcal{G}\Omega) = 0$. By (4.3), we have $mod_c(\Omega) = 0$. Hence

$$\beta(\Omega) = (0, 0).$$

The regularity property of β implies the relative compactness of Ω.

Now, it follows from Definition 2.3 that $\mathcal{G}$ is β-condensing.

According to Theorem 2.4, problem (1.1) has at least one mild solution. $\square$

Finally, we give examples to illustrate our abstract results above. Let $k > 0$ be an integer, $0 < s_1 < s_2 < \cdots < s_m < T$, $c_j \in \mathbf{R}(j = 0, 1, 2, \cdots, m)$, $h(\cdot) \in L^1([0, T], \mathbf{R})$. Define

(1) $X = L^2([0, T])$, $Au = u''$ with $D(A) = H^2([0, \pi]) \cap H_0^1([0, \pi])$.

(2) $f(t, x(t))(\xi) = c_0 \sin x(t)(\xi)$.

(3) $f(t, x(t))(\xi) = \frac{1}{\sqrt[k]{t}} \sin x(t)(\xi)$.

(4) $g(\varphi(t, \xi)) = \sum\limits_{j=1}^{m} c_j \varphi(s_j, \xi), \varphi \in C([0, T], X)$.

(5) $g(\varphi(t, \xi)) = \int_0^T h(s) \sin(1 + \varphi(s, \xi)) ds, \varphi \in C([0, T], X)$.

Then, we obtain

- A generates an analytic and uniformly bounded semigroup $\{S(t)\}_{t \geq 0}$ on X with $\|S(t)\| \leq 1$.
- When c_j, $j = 0, 1, \cdots, m$ is small enough, "(1)+(2)+(4)" makes the assumptions in Theorem 3.1 satisfied. Therefore, the corresponding nonlinear nonlocal problem (1.1) has a unique mild solution.
- "(1)+(3)+(5)" makes the assumptions (H1)-(H3) in Theorem 4.1 satisfied. Therefore, the corresponding nonlinear nonlocal problem (1.1) has at least a mild solution if (4.1) holds.

Acknowledgements

This work was partly supported by the NSF of China (11201413), the NSF of Yunnan Province (2009ZC054M), the Educational Commission of Yunnan Province (2012Z010).

References

[1] M. M. El-Borai, Semigroups and some nonlinear fractional differential equations. *Appl. Math. Comput.* **149** (2004), 823-831.

[2] M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations. *Chaos Soliton. Fract.* **14** (2002), 433-440.

[3] D. Henry, Geometric Theory of Semilinear Parabolic Partial Differential Equations, *Springer, Berlin, Germany*, 1989.

[4] E. Hernández, D. O'Regan, and K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives. *Nonlinear Anal.* **73** (2010), 3462-3471.

[5] F. Li and G. M. N'Guérékata, An existence result for neutral delay integrodifferential equations with fractional order and nonlocal conditions. *Abstr. Appl. Anal.* **2011** (2011), Article ID 952782, 8 pages.

[6] G. M. Mophou and G. M. N'Guérékata, Existence of mild solution for some fractional differential equations with nonlocal conditions. *Semigroup Forum* **79** (2009), no.2, 315-322.

[7] G. M. Mophou and G. M. N'Guérékata, Existence of mild solutions for some semilinear neutral fractional functional evolution equations with infinite delay. *Appl. Math. Comput.* **216** (2010), 61-69.

[8] G. M. Mophou and G. M. N'Guérékata, A note on a semilinear fractional differential equation of neutral type with infinite delay. *Adv. Differ. Equ.* **2010** (2010), Article ID 674630, 8 pages.

[9] G. M. Mophou and G. M. N'Guérékata, Optimal control of a fractional diffusion equation with state constraints. *Comput. Math. Appl.* **62** (2011), 1413-1426.

[10] G. M. N'Guérékata, A Cauchy problem for some fractional abstract differential equation with non local conditions. *Nonlinear Anal.* **70** (2009), no.5, 1873-1876.

[11] J. Liang, J. H. Liu, and T. J. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. *Math. Comput. Model*. **49** (2009), no.3-4, 798-804.

[12] J. Liang, J. H. Liu, and T. J. Xiao, Nonlocal Cauchy problems governed by compact operator families. *Nonlinear Anal.* **57** (2004), no.2, 183-189.

[13] J. Liang and T. J. Xiao, Semilinear integrodifferential equations with nonlocal initial conditions. *Comput. Math. Appl.* **47** (2004), no.6-7, 863-875.

[14] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii, Measures of Noncompactness and Condensing Operators, *Birkhäuser, Boston-Basel-Berlin*, 1992.

[15] M. Kamenskii, V. Obukhovskii, and P. Zecca, Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, *de Gruyter Series in Nonlinear Analysis and Applications, 7, Walter de Gruyter, Berlin - New York*, 2001.

[16] F. Mainardi, P. Paradisi, and R. Gorenflo, Probability Distributions Generated by Fractional Diffusion Equations, *in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer, Dordrecht*, 2000.