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How to Deal with Nonlinear Terms in the Non-Dissipative Case

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Abstract

In this work we suggest a way to estimate some nonlinear terms appearing in the study of semilinear viscoelastic problems. So far we know how to deal with these terms only when the energy is decreasing. In this case we can estimate parts of these nonlinearities by the initial energy. We solve this issue in the general case with the help of a new differential inequality.

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1 Introduction

We shall consider the following problem

$$\begin{cases} u_{tt} + |u|^p u = \Delta u - \int_0^t h(t-s)\Delta u(s)ds, \ in\ \Omega \times \mathbf{R}_+ \\ u = 0, \ on\ \Gamma \times \mathbf{R}_+ \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ in\ \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$ and $p > 0$. The functions $u_0(x)$ and $u_1(x)$ are given initial data and the (nonnegative) relaxation function $h(t)$ will be specified later on. The equation in (1) describes the equation of motion of a viscoelastic body with fading memory. In the last twenty five years or so, there has been an extensive development of the theory of viscoelasticity. This is mainly due to the growing interest in viscoelastic materials in industry. Indeed, viscoelastic material possess some very important properties. In particular, they are used to control and suppress or at least reduce vibrations in different structures.

Many papers appeared in the literature treating the well-posedness and asymptotic behavior of solutions. Researchers have focused in particular on enlarging the class of viscoelastic materials ensuring a certain decay and also on improving the decay rates (see [1-12,14-18,20-35] to cite but a few).

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In this work we do not intend to do neither of these and rather focus on the main contribution here which is concerned with the estimation of some nonlinear terms which arise while studying the asymptotic behavior of solutions. As far as we know, these terms are dealt with only in the dissipative case where we know from the beginning that the energy is decreasing and therefore bounded by its initial value. This is not valid in the non-dissipative case and we are lead to face a new differential inequality. We treat this problem with the help of a new differential inequality (new in the field of viscoelasticity) which may be found in [13].

The local existence can be proved using the Faedo Galerkin method (see for instance [4,5,6-8,14]).

Theorem: *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* $h(t)$ *is a nonnegative summable kernel. If* $0 < p < \frac{2}{n-2}$ *when* $n \geq 3$ *and* $p > 0$ *when* $n = 1, 2$, *then there exists a unique solution* u *to problem (1.1) such that*

$$u \in C\left([0,T]; H_0^1(\Omega) \cap L^p(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$$

for T *small enough.*

The plan of the paper is as follows: In the next section we prepare some material needed to prove our result. We introduce the different functionals we will use. The modified energy functional is defined in this section too. Section 3 is devoted to the statement and proof of our asymptotic behavior result. Section 4 contains some examples illustrating our results.

2 Preliminaries

We define the (classical) energy by

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|\nabla u\|_2^2\right) + \frac{1}{p+2}\|u\|_{p+2}^{p+2}$$

where $\|.\|_p$ denotes the norm in $L^p(\Omega)$ (the usual Lebesgue space). Then by the equation $(1.1)_1$ it is easy to see that

$$E'(t) = \int_\Omega \nabla u_t . \int_0^t h(t-s)\nabla u(s) ds dx.$$

Note that

$$2\int_\Omega \nabla u_t . \int_0^t h(t-s)\nabla u(s) ds dx = \int_\Omega (h' \Box \nabla u) dx - h(t)\|\nabla u\|_2^2$$
$$-\frac{d}{dt}\left\{\int_\Omega (h \Box \nabla u) dx - \left(\int_0^t h(s) ds\right)\|\nabla u\|_2^2\right\}$$

where

$$(h \Box v)(t) := \int_0^t h(t-s)|v(t) - v(s)|^2 ds.$$

Therefore, if we modify $E(t)$ to

$$\mathcal{E}(t) := \frac{1}{2}\left\{\|u_t\|_2^2 + \left(1 - \int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{2}{p+2}\|u\|_{p+2}^{p+2} + \int_\Omega (h\Box\nabla u)dx\right\}$$

we obtain

$$\mathcal{E}'(t) = \frac{1}{2}\int_\Omega \left((h'\Box\nabla u) - h(t)|\nabla u|^2\right)dx. \tag{2.1}$$

We assume that the kernel is such that

$$1 - \int_0^{+\infty} h(s)ds = 1 - \kappa > 0.$$

Next, we define the standard functionals

$$\Phi_1(t) := \int_\Omega u_t u dx$$

and

$$\Phi_2(t) := -\int_\Omega u_t \int_0^t h(t-s)(u(t) - u(s))dsdx.$$

The next functionals have been introduced by the present author in [34]

$$\Phi_3(t) := \int_0^t H_\gamma(t-s)\|\nabla u(s)\|_2^2 ds,\ \Phi_4(t) := \int_0^t \Psi_\gamma(t-s)\|\nabla u(s)\|_2^2 ds$$

where

$$H_\gamma(t) := \gamma(t)^{-1}\int_t^\infty h(s)\gamma(s)ds,\ \Psi_\gamma(t) := \gamma(t)^{-1}\int_t^\infty \xi(s)\gamma(s)ds$$

and $\gamma(t)$ and $\xi(t)$ are two functions which will be precised later (see **(H2)**, **(H3)** and Examples at the end of the paper). The modified energy we will work with is

$$L(t) := \mathcal{E}(t) + \sum_{i=1}^4 \lambda_i \Phi_i(t) \tag{2.2}$$

for some $\lambda_i > 0$, $i = 1, 2, 3, 4$ to be determined.

The first result tells us that $L(t)$ and $\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)$ are equivalent.

Proposition 1: There exist $\rho_i > 0$, $i = 1, 2$ such that

$$\rho_1[\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)] \le L(t) \le \rho_2[\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t)]$$

for all $t \ge 0$ and small λ_i, $i = 1, 2$.

Proof. By the inequalities

$$\Phi_1(t) = \int_\Omega u_t u dx \le \frac{1}{2}\|u_t\|_2^2 + \frac{C_p}{2}\|\nabla u\|_2^2,$$

and

$$\begin{aligned}
\Phi_2(t) &\le \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{1}{2}\int_\Omega \left(\int_0^t h(t-s)(u(t)-u(s))ds\right)^2 dx \\
&\le \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{C_p}{2}\int_\Omega \left(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx \\
&\le \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{C_p}{2}\int_\Omega \left(\int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}|\nabla u(t)-\nabla u(s)|ds\right)^2 dx \\
&\le \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{C_p}{2}\int_\Omega \left(\int_0^t h(s)ds\right)\left(\int_0^t h(t-s)|\nabla u(t)-\nabla u(s)|^2 ds\right) dx \\
&\le \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{C_p\kappa}{2}\int_\Omega (h\Box\nabla u)dx
\end{aligned}$$

where C_p is the Poincaré constant, we have

$$\begin{aligned}
L(t) &\le \tfrac{1}{2}(1+\lambda_1+\lambda_2)\|u_t\|_2^2 + \tfrac{1}{2}\left(1-\int_0^t h(s)ds + \lambda_1 C_p\right)\|\nabla u\|_2^2 \\
&+ \tfrac{1}{p+2}\|u\|_{p+2}^{p+2} + \tfrac{1}{2}\left(1+\lambda_2 C_p\kappa\right)\int_\Omega (h\Box\nabla u)dx + \lambda_3\Phi_3(t) + \lambda_4\Phi_4(t).
\end{aligned}$$

On the other hand

$$\begin{aligned}
2L(t) &\ge (1-\lambda_1-\lambda_2)\|u_t\|_2^2 + \left(1-\lambda_2 C_p\kappa\right)\int_\Omega (h\Box\nabla u)dx + \tfrac{2}{p+2}\|u\|_{p+2}^{p+2} \\
&+ [1-\kappa-\lambda_1 C_p]\|\nabla u\|_2^2 + 2\lambda_3\Phi_3(t) + 2\lambda_4\Phi_4(t).
\end{aligned}$$

Therefore, $\rho_1[\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)] \le L(t) \le \rho_2[\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)]$ for some constant $\rho_i > 0$, $i = 1,2$ and small λ_i, $i = 1,2$ such that $\lambda_1 < \min\left\{1,(1-\kappa)/C_p\right\}$ and $\lambda_2 < \min\left\{\frac{1}{C_p\kappa}, 1-\lambda_1\right\}$.

The following inequality will be used repeatedly in the sequel.

Lemma 1: We have

$$ab \le \delta a^2 + \frac{b^2}{4\delta},\ a,b \in \mathbf{R},\ \delta > 0.$$

The next result will be used later to estimate

$$\int_\Omega \nabla u.\int_0^t h(t-s)\nabla u(s)dsdx.$$

Lemma 2: We have for continuous functions h and v on $(0,\infty)$

$$\begin{aligned}
v(t)\int_0^t h(t-s)v(s)ds &= \tfrac{1}{2}\left(\int_0^t h(s)ds\right)v^2(t) + \tfrac{1}{2}\int_0^t h(t-s)v^2(s)ds \\
&- \tfrac{1}{2}(h\Box v)(t),\ t \ge 0.
\end{aligned}$$

Proof. It suffices to develop the last term in the right hand side of the identity itself. Indeed, we have

$$(h\Box v)(t) = \int_0^t h(t-s)\,|v(t)-v(s)|^2\,ds$$
$$= \int_0^t h(t-s)\left[v^2(t) - 2v(t)v(s) + v^2(s)\right]ds$$
$$= \left(\int_0^t h(s)ds\right)v^2(t) - 2v(t)\int_0^t h(t-s)v(s)ds + \int_0^t h(t-s)v^2(s)ds.$$

The proof is complete.

The next lemma is well-known as the Sobolev-Poincaré inequality.

Lemma 3: Assume that $2 \le q < +\infty$ if $n = 1,2$ or $2 \le q < \frac{2n}{n-2}$ if $n \ge 3$. The there exists a positive constant $C_e = C_e(\Omega, q)$ such that

$$\|u\|_q \le C_e \|\nabla u\|_2$$

for $u \in H_0^1(\Omega)$.

We end this section by the following lemma (see [13]) which is the key tool in the present contribution.

Lemma 4: Let $\chi(t)$, $\alpha(t), \beta(t) \in C[t_0, \infty)$ and $\alpha(t) \ge 0$, for all $t \ge t_0$. Suppose that there exists a positive function $\mu(t) \in C^1[t_0, \infty)$ such that

$$\frac{\alpha(t)}{\mu^p(t)} + \beta(t) \le \frac{1}{\mu(t)}\left(\chi(t) - \frac{\mu'(t)}{\mu(t)}\right),$$

then a nonnegative solution to the following inequality

$$v'(t) \le -\chi(t)v(t) + \alpha(t)v^p(t) + \beta(t),\ p > 1$$

such that $\mu(t_0)v(t_0) < 1$, satisfies the estimate

$$v(t) < \frac{1}{\mu(t)},\ \forall t \ge t_0.$$

3 Asymptotic Behavior

In this section we state and prove our result. For every measurable set $\mathcal{A} \subset \mathbf{R}^+$, we define the probability measure $\hat{h}$ by

$$\hat{h}(\mathcal{A}) := \frac{1}{\kappa}\int_{\mathcal{A}} h(s)ds. \tag{3.1}$$

The non-decreasingness set and the non-decreasingness rate of h are defined by

$$Q_h := \{s \in \mathbf{R}^+ : h'(s) \ge 0\} \tag{3.2}$$

and

$$\mathcal{R}_h := \hat{h}(Q_h),$$

respectively.

Our assumptions on the kernel $h(t)$ are the following

(H1) $h(t) \geq 0$ for all $t \geq 0$ and $0 < \kappa = \int_0^{+\infty} h(s)ds < 1$.

(H2) h is absolutely continuous and of bounded variation on $(0,\infty)$ and $h'(t) \leq \xi(t)$ for some non-negative summable function $\xi(t)$ ($= \max\{0, h'(t)\}$ where $h'(t)$ exists) and almost all $t > 0$.

(H3) There exists a non-decreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \eta(t)$ is a non-increasing function and $\int_0^{+\infty} h(s)\gamma(s)ds < +\infty$.

Note that the assumption **(H3)** is satisfied by a large class of functions like the polynomials and exponential functions. Let $t_* > 0$ be a number such that $\int_0^{t_*} h(s)ds = h_* > 0$ and

$$I(u_0,u_1) = \frac{1+\lambda}{2}\|u_1\|_2^2 + \frac{1+\lambda C_p}{2}\|\nabla u_0\|_2^2 + \frac{1}{p+2}\|u_0\|_{p+2}^{p+2}$$

where $\lambda = \kappa^2/2C_pBV[h,\mathcal{A}]$, BV is the total variation and $\mathcal{A}$ is the set on which h' is negative.

Theorem 1: Assume that the hypotheses **(H1)-(H3)** hold, $2 \leq q < +\infty$ if $n = 1,2$ or $2 \leq q < \frac{2n}{n-2}$ if $n \geq 3$, $\mathcal{R}_h < 1/4$ and $\int_{Q_h} \xi(s)ds$ is small enough. Then, $E(t) \leq C/\mu(t)$, $t \geq 0$ for some positive constants C in case

(a) $\lim_{t\to\infty} \eta(t) = \bar{\eta} \neq 0$ and $B \leq \mu^p(t)\left[A - \frac{\mu'(t)}{\mu(t)}\right]$, $t \geq 0$ or

(b) $\lim_{t\to\infty} \eta(t) = 0$ and $B \leq \mu^p(t)\left[D\eta(t) - \frac{\mu'(t)}{\mu(t)}\right]$, $t \geq 0$

for some positive constants A, B and D to be determined provided that $I(u_0,u_1)\mu(0) < 1$.

Proof. A differentiation of $\Phi_1(t)$ with respect to t along trajectories of (1.1) gives

$$\Phi_1'(t) := \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega \nabla u. \int_0^t h(t-s)\nabla u(s)dsdx - \|u\|_{p+2}^{p+2},\ t \geq 0$$

and by Lemma 2 we obtain

$$\begin{aligned} \Phi_1'(t) \leq \|u_t\|_2^2 - (1-\tfrac{\kappa}{2})\|\nabla u\|_2^2 + \tfrac{1}{2}\int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\ -\tfrac{1}{2}\int_\Omega (h\Box\nabla u)dx - \|u\|_{p+2}^{p+2},\ t \geq 0. \end{aligned} \tag{3.3}$$

For $\Phi_2(t)$ we have

$$\begin{aligned} \Phi_2'(t) = -\int_\Omega u_{tt}\int_0^t h(t-s)(u(t)-u(s))dsdx \\ -\int_\Omega u_t\left[\int_0^t h'(t-s)(u(t)-u(s))ds + u_t\int_0^t h(s)ds\right]dx \end{aligned}$$

or

$$\Phi_2'(t) = -\int_\Omega \left[\left(1 - \int_0^t h(s)ds\right) \Delta u - |u|^p u + \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\, ds \right]$$
$$\times \int_0^t h(t-s)(u(t) - u(s))\, ds dx - \left(\int_0^t h(s)ds\right) \|u_t\|_2^2$$
$$- \int_\Omega u_t \int_0^t h'(t-s)(u(t) - u(s))\, dsdx,\ t \geq 0.$$

Therefore,

$$\begin{aligned}\Phi_2'(t) &= \left(1 - \int_0^t h(s)ds\right) \int_\Omega \nabla u. \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, dsdx \\ &+ \int_\Omega \left| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right|^2 dx - \left(\int_0^t h(s)ds\right) \|u_t\|_2^2 \\ &- \int_\Omega u_t \int_0^t h'(t-s)(u(t) - u(s))\, dsdx \\ &+ \int_\Omega |u|^p u \int_0^t h(t-s)(u(t) - u(s))\, dsdx.\end{aligned} \tag{3.4}$$

The last term in (3.4) used to be estimated using the bound $E(0)$ of $E(t)$. This holds in the dissipative case. That is, when $E'(t) \leq 0$, which is clearly not the case here. We have

$$\begin{aligned}&\int_\Omega |u|^p u \int_0^t h(t-s)(u(t) - u(s))\, dsdx \\ &\leq \delta \int_\Omega |u|^{2(p+1)}\, dx + \tfrac{C_p}{4\delta} \left(\int_0^t h(s)ds\right) \int_\Omega (h \square \nabla u) dx \\ &\leq \delta C_e \|\nabla u\|_2^{2(p+1)} + \tfrac{C_p}{4\delta} \left(\int_0^t h(s)ds\right) \int_\Omega (h \square \nabla u) dx \\ &\leq \tfrac{2\delta C_e}{1-\kappa} \mathcal{E}^{p+1}(t) + \tfrac{C_p}{4\delta} \left(\int_0^t h(s)ds\right) \int_\Omega (h \square \nabla u) dx.\end{aligned} \tag{3.5}$$

For all measurable sets $\mathcal{A}$ and Q such that $\mathcal{A} = \mathbf{R}^+ \backslash Q$, we may estimate the first term in the right hand side of (3.4) as follows

$$\begin{aligned}&\int_\Omega \nabla u. \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, dsdx \\ &= \int_\Omega \nabla u. \int_{\mathcal{A}_t} h(t-s)(\nabla u(t) - \nabla u(s))\, dsdx \\ &+ \int_\Omega \nabla u. \int_{Q_t} h(t-s)(\nabla u(t) - \nabla u(s))\, dsdx \\ &\leq \int_\Omega \nabla u. \int_{\mathcal{A}_t} h(t-s)(\nabla u(t) - \nabla u(s))\, dsdx \\ &+ \left(\int_{Q_t} h(t-s)ds\right) \|\nabla u\|_2^2 - \int_\Omega \nabla u. \int_{Q_t} h(t-s) \nabla u(s) dsdx\end{aligned} \tag{3.6}$$

where we have adopted the notation: $\mathcal{B}_t := \mathcal{B} \cap [0,t]$. Using Lemma 2, it is easy to see that for $\delta_1 > 0$

$$\begin{aligned}&\int_\Omega \nabla u.\int_{\mathcal{A}_t} h(t-s)(\nabla u(t)-\nabla u(s))\,dsdx\\ &\le \delta_1 \|\nabla u\|_2^2 + \tfrac{\kappa}{4\delta_1}\int_\Omega\int_{\mathcal{A}_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx,\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&\int_\Omega \nabla u.\int_{Q_t} h(t-s)\nabla u(s)dsdx\\ &\le \tfrac{1}{2}\left(\int_{Q_t} h(t-s)ds\right)\|\nabla u\|_2^2 + \tfrac{1}{2}\int_{Q_t} h(t-s)\,\|\nabla u(s)\|_2^2\,ds.\end{aligned}\tag{3.8}$$

These relations (3.7) and (3.8) together with (3.6) imply that

$$\begin{aligned}&\int_\Omega \nabla u.\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,dsdx\\ &\le \left(\delta_1 + \tfrac{3}{2}\int_{Q_t} h(t-s)ds\right)\|\nabla u\|_2^2 + \tfrac{\kappa}{4\delta_1}\int_\Omega\int_{\mathcal{A}_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx\\ &\quad+\tfrac{1}{2}\int_{Q_t} h(t-s)\,\|\nabla u(s)\|_2^2\,ds\end{aligned}\tag{3.9}$$

where $\hat{h}$ is defined in (3.1). For the second term in the right hand side of (3.4) we have

$$\begin{aligned}&\int_\Omega \left|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right|^2 dx\\ &\le (1+\tfrac{1}{\delta_2})\kappa\int_\Omega\int_{\mathcal{A}_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx\\ &+(1+\delta_2)\left(\int_{Q_t} h(t-s)ds\right)\int_\Omega\int_{Q_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx,\ \delta_2>0.\end{aligned}\tag{3.10}$$

Finally we may write

$$\begin{aligned}&\int_\Omega u_t\int_0^t h'(t-s)(u(t)-u(s))\,dsdx\\ &\le \delta_3\|u_t\|_2^2 - \tfrac{C_p}{4\delta_3}BV[h,\mathcal{A}]\int_\Omega\int_{\mathcal{A}_t} h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx\\ &\quad+\tfrac{C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\int_\Omega\int_{Q_t}\xi(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx\\ &\le \delta_3\|u_t\|_2^2 - \tfrac{C_p}{4\delta_3}BV[h,\mathcal{A}]\int_\Omega\int_{\mathcal{A}_t} h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx\\ &+\tfrac{C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\left[\le \tfrac{3}{2}\left(\int_{Q_{nt}}\xi(t-s)ds\right)\|\nabla u\|_2^2 + 3\int_{Q_{nt}}\xi(t-s)\,\|\nabla u(s)\|_2^2\,ds\right],\end{aligned}\tag{3.11}$$

for $\delta_3 > 0$. Having in mind the relations (3.5), (3.9)-(3.11) we infer from (3.4) that

$$
\begin{aligned}
\Phi_2'(t) \le & \left\{(1-h_*)\left[\delta_1 + \tfrac{3}{2}\int_{Q_t} h(t-s)ds\right] + \tfrac{3C_p}{8\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)^2\right\}\|\nabla u\|_2^2 \\
& +(\delta_3 - h_*)\|u_t\|_2^2 + \kappa\left[1+\tfrac{1-h_*}{4\delta_1}+\tfrac{1}{\delta_2}\right]\int_\Omega\int_{\mathcal{A}_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
& +\tfrac{1}{2}(1-h_*)\int_{Q_t} h(t-s)\|\nabla u(s)\|_2^2\,ds + \tfrac{C_p}{4\delta}\left(\int_0^t h(s)ds\right)\int_\Omega (h\Box\nabla u)dx \\
& +\tfrac{2\delta C_e}{1-\kappa}\mathcal{E}^{p+1}(t) - \tfrac{C_p}{4\delta_3}BV[h,\mathcal{A}]\int_\Omega\int_{\mathcal{A}_t} h'(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
& +(1+\delta_2)\left(\int_{Q_t} h(t-s)ds\right)\int_\Omega\int_{Q_t} h(t-s)|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
& +\tfrac{3C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)\int_{Q_t}\xi(t-s)\|\nabla u\|_2^2\,ds.
\end{aligned}
\tag{3.12}
$$

In virtue of the fact that $\gamma'(t)/\gamma(t) = \eta(t)$ is a non-increasing function, we have

$$
\begin{aligned}
\Phi_3'(t) &= H_\gamma(0)\|\nabla u\|_2^2 + \int_0^t H_\gamma'(t-s)\|\nabla u(s)\|_2^2\,ds \\
&= H_\gamma(0)\|\nabla u\|_2^2 - \int_0^t \tfrac{\gamma'(t-s)}{\gamma(t-s)} H_\gamma(t-s)\|\nabla u(s)\|_2^2\,ds - \int_0^t h(t-s)\|\nabla u(s)\|_2^2\,ds \\
&\le H_\gamma(0)\|\nabla u\|_2^2 - \eta(t)\Phi_3(t) - \int_0^t h(t-s)\|\nabla u(s)\|_2^2\,ds
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\Phi_4'(t) &= \Psi_\gamma(0)\|\nabla u\|_2^2 + \int_0^t \Psi_\gamma'(t-s)\|\nabla u(s)\|_2^2\,ds \\
&= \Psi_\gamma(0)\|\nabla u\|_2^2 - \int_0^t \tfrac{\gamma'(t-s)}{\gamma(t-s)}\Psi_\gamma(t-s)\|\nabla u(s)\|_2^2\,ds - \int_0^t \xi(t-s)\|\nabla u(s)\|_2^2\,ds \\
&\le \Psi_\gamma(0)\|\nabla u\|_2^2 - \eta(t)\Phi_4(t) - \int_0^t \xi(t-s)\|\nabla u(s)\|_2^2\,ds.
\end{aligned}
$$

Taking into account the relations (2.1), (3.3), (3.12), (3.13), we see that

$$
\begin{aligned}
L'(t) \le\ & \tfrac{1}{2}\int_\Omega (h'\Box\nabla u)dx - \tfrac{C_p}{4\delta_3}\lambda_2 BV[h,\mathcal{A}]\int_\Omega\int_{\mathcal{A}_t} h'(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
&+\left\{\lambda_2(1-h_*)\left[\delta_1+\tfrac{3}{2}\int_{Q_t} h(t-s)ds\right]+\tfrac{3\lambda_2 C_p}{8\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right)^2+\lambda_3 H_\gamma(0)\right. \\
&\left.-\lambda_1(1-\tfrac{\kappa}{2})\right\}\|\nabla u\|_2^2+\left(\tfrac{\lambda_1}{2}-\lambda_3\right)\int_0^t h(t-s)\,\|\nabla u(s)\|_2^2\,ds \\
&+\left[\tfrac{\lambda_2 C_p}{4\delta}\left(\int_0^t h(s)ds\right)-\tfrac{\lambda_1}{2}\right]\int_\Omega(h\Box\nabla u)dx+[\lambda_1+(\delta_3-h_*)\lambda_2]\,\|u_t\|_2^2 \\
&-\lambda_3\eta(t)\Phi_3(t)+\lambda_2\kappa\left(1+\tfrac{1-h_*}{4\delta_1}+\tfrac{1}{\delta_2}\right)\int_\Omega\int_{\mathcal{A}_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \qquad (3.14)\\
&+(1+\delta_2)\lambda_2\int_{Q_t} h(t-s)ds\int_\Omega\int_{Q_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
&+\tfrac{\lambda_2}{2}(1-h_*)\int_{Q_t} h(t-s)\,\|\nabla u(s)\|_2^2\,ds+\tfrac{3\lambda_2 C_p}{4\delta_3}\left(\int_{Q_t}\xi(t-s)ds\right) \\
&\times\int_{Q_t}\xi(t-s)\,\|\nabla u(s)\|_2^2\,ds+\lambda_4\Psi_\gamma(0)\,\|\nabla u\|_2^2-\lambda_4\eta(t)\Phi_4(t) \\
&-\lambda_4\int_0^t\xi(t-s)\,\|\nabla u(s)\|_2^2\,ds+\tfrac{2\delta C_e\lambda_2}{1-\kappa}\mathcal{E}^{p+1}(t)-\lambda_1\,\|u\|_{p+2}^{p+2}.
\end{aligned}
$$

Let us introduce the sets

$$\mathcal{A}_n := \{s\in\mathbf{R}^+ : nh'(s)+h(s)\le 0\},\ n\in\mathbf{N},$$

$$\tilde{A}_{nt} := \{s\in\mathbf{R}^+ : 0\le s\le t,\ nh'(t-s)+h(t-s)\le 0\},\ n\in\mathbf{N},$$

$$\tilde{Q}_{ht} := \{s\in\mathbf{R}^+ : 0\le s\le t,\ 0\le h'(t-s)\le\xi(t-s)\}$$

and observe that

$$\bigcup_n \mathcal{A}_n = \mathbf{R}^+\backslash\{Q_h\cup\mathcal{N}_h\}$$

where

$$Q_h := \{s\in\mathbf{R}^+ : 0\le h'(s)\le\xi(s)\}$$

and $\mathcal{N}_h$ is the nullset where h' is not defined. Furthermore, if we denote $Q_n := \mathbf{R}^+\backslash\mathcal{A}_n$, then $\lim_{n\to\infty}\hat{h}(Q_n)=\hat{h}(Q_h)$ because $Q_{n+1}\subset Q_n$ for all n and $\bigcap_n Q_n = Q_h\cup\mathcal{N}_h$. In (3.14), we take

$\mathcal{A}_t := \tilde{A}_{nt}$ and $Q_t := \tilde{Q}_{nt}$ (the complement in $[0,t]$). It follows that

$$
\begin{aligned}
L'(t) \le{}& \tfrac{1}{4}\left(1-\tfrac{\lambda_2 C_p}{\delta_3}BV[h,\mathcal{A}]\right)\int_\Omega\int_{\mathcal{A}_{nt}} h'(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
&+[\lambda_1+(\delta_3-h_*)\lambda_2]\,\|u_t\|_2^2+\Bigg\{\lambda_2(1-h_*)\left(\delta_1+\tfrac{3}{2}\int_{\tilde{Q}_{nt}} h(t-s)ds\right)+\lambda_3 H_\gamma(0) \\
&+\lambda_4\Psi_\gamma(0)+\tfrac{3\lambda_2 C_p}{8\delta_3}\left(\int_{\tilde{Q}_{nt}}\xi(t-s)ds\right)^2-\lambda_1(1-\tfrac{\kappa}{2})\Bigg\}\|\nabla u\|_2^2+\left(\tfrac{(1-\varepsilon)\lambda_2}{2}-\lambda_3\right) \\
&\times\int_0^t h(t-s)\,\|\nabla u(s)\|_2^2\,ds+\left[\tfrac{\lambda_2\kappa C_p}{4\delta}+\lambda_2\kappa\left(1+\tfrac{1-h_*}{4\delta_1}+\tfrac{1}{\delta_2}\right)-\tfrac{1}{4n}\right] \\
&\times\int_\Omega\int_{\mathcal{A}_{nt}} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx-\lambda_3\eta(t)\Phi_3(t)-\lambda_4\eta(t)\Phi_4(t) \\
&+\left[\tfrac{\lambda_2\kappa C_p}{4\delta}+(1+\delta_2)\lambda_2\int_{\tilde{Q}_{nt}} h(t-s)ds-\tfrac{\lambda_1}{2}\right]\int_\Omega\int_{\tilde{Q}_{nt}} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\
&+\tfrac{3\lambda_2 C_p}{4\delta_3}\left(\int_{\tilde{Q}_{nt}}\xi(t-s)ds\right)\int_{\tilde{Q}_{nt}}\xi(t-s)\,\|\nabla u(s)\|_2^2\,ds-\lambda_4\int_0^t\xi(t-s)\,\|\nabla u(s)\|_2^2\,ds \\
&+\tfrac{2\delta C_e\lambda_2}{1-\kappa}\mathcal{E}^{p+1}(t)-\lambda_1\,\|u\|_{p+2}^{p+2}.
\end{aligned}
\tag{3.15}
$$

Let $\lambda_1=(h_*-\varepsilon)\lambda_2$ for some $\varepsilon>0$. If $\hat{h}(Q)<1/4$, then $\frac{3(1-h_*)}{2}\int_{\tilde{Q}_{nt}} h(t-s)ds<\frac{\delta h_*(2-\kappa)}{2}$ with $\delta=\frac{3(1-h_*)\kappa}{4h_*(2-\kappa)}+\beta$ where β is a small positive constant and n large enough. Further we may select λ_3 and $H_\gamma(0)$ such that

$$\frac{(1-\varepsilon)\lambda_2}{2}H_\gamma(0)<\lambda_3 H_\gamma(0)<\lambda_2\frac{(1-\delta)h_*(2-\kappa)}{2}.$$

Note that this is possible if $H_\gamma(0)$ is small enough and (t_* is so large that) $h_*>7\kappa/(8-\kappa)$ eventhough

$$H_\gamma(0)=\gamma(0)^{-1}\int_0^\infty h(s)\gamma(s)ds\ge\int_0^\infty h(s)ds=\kappa.$$

It is clear that

$$(1+\delta_2)\lambda_2\int_{\tilde{Q}_{nt}} h(t-s)ds-\frac{\lambda_1}{2}\le 0$$

for small ε, δ_2, large n and if $\hat{h}(Q_h)<1/4$. Select $\lambda_2\le\delta_3/C_p BV[h,\mathcal{A}]$ so that

$$\lambda_2\kappa\left(1+\frac{1-h_*}{4\delta_1}+\frac{1}{\delta_2}+\frac{C_p}{4\delta_4}\right)<\frac{1}{4n}. \tag{3.16}$$

Furthermore, we select λ_4 large enough so that

$$\lambda_4>\frac{3\lambda_2 C_p}{4\delta_3}\int_{Q_h}\xi(s)ds.$$

Therefore, if $\delta_3 = \varepsilon/2$, $\varepsilon, \beta, \delta_i$, $i = 1, 2$, $\int_{Q_h} \xi(s)ds$ are sufficiently small and δ large enough, then

$$L'(t) \leq -C_1\mathcal{E}(t) - \lambda_3\eta(t)\Phi_3(t) - \lambda_4\eta(t)\Phi_4(t) + \frac{2\delta C_e\lambda_2}{1-\kappa}\mathcal{E}^{p+1}(t)$$

for some $C_1 > 0$.

If $\lim_{t\to\infty}\eta(t) = \bar{\eta} \neq 0$, then $\eta(t) \geq \bar{\eta}$ and there exist $C_2 > 0$ such that

$$L'(t) \leq -C_2L(t) + \frac{2\delta C_e\lambda_2}{(1-\kappa)\rho_1^{p+1}}L^{p+1}(t)$$

(by Proposition 1). This relation is of the same form as the one in Lemma 4 with $L(t)$, C_2, $\frac{2\delta C_e\lambda_2}{(1-\kappa)\rho_1^{p+1}}$ and 0 instead of $v(t)$, $\chi(t)$, $\alpha(t)$ and $\beta(t)$, respectively. Observe that we have here $p+1$ instead of p. Note also that the condition of Lemma 4 is fulfilled if $L(0) < I(u_0, u_1)$. Therefore

$$E(t) \leq C/\mu(t),\ t \geq 0 \tag{3.17}$$

for some positive constant C.

If $\lim_{t\to\infty}\eta(t) = 0$, there exist $\hat{t} \geq t_*$ such that $\eta(t) \leq C_1$, $\forall t \geq \hat{t}$. We deduce that

$$L'(t) \leq -C_3\eta(t)L(t) + \frac{2\delta C_e\lambda_2}{1-\kappa}\mathcal{E}^{p+1}(t)$$

and thereafter as in the previous argument the relation (20) holds again with $\chi(t) = C_3\eta(t)$, $\alpha(t) = \frac{2\delta C_e\lambda_2}{(1-\kappa)\rho_1^{p+1}}$ and $\beta(t) = 0$.

Remark 1: The smallness of the integral of ξ over Q has been discussed in [34]. It is difficult to be determined exactly. Some simpler situations where more reasonable kernels may be considered are, for instance, the exponentially decaying kernels (satisfying $h'(t) \leq -Ch(t)$ on $\mathcal{A}$) or $h'(t) \leq -\omega(t)h(t)$, for all $t \in \mathcal{A}$ where $\omega(t)$ is a continuous function such that $\inf_{t\geq 0}\omega(t) = \omega > 0$. In these cases the bound $1/4n$ in (3.17) may be very large.

4 Examples

The class of functions $\gamma(t)$ include polynomials and exponentials. Indeed, if we consider $\gamma(t) = (1+t)^\alpha$, $\alpha > 0$ we are lead to $\eta(t) = \gamma'(t)/\gamma(t) = \alpha(1+t)^{-1}$ and if we consider $\gamma(t) = e^{\alpha t}$, $\alpha > 0$ then we find $\eta(t) = \gamma'(t)/\gamma(t) = \alpha$.

Example 1:

For part 1 in the theorem it is easy to check that $\mu(t) = \mu_0 e^{\sigma t}$ with $\mu_0^p \geq \frac{\alpha}{C_2-\sigma}$, $\alpha = \frac{2\delta C_e\lambda_2}{(1-\kappa)\rho_1^{p+1}}$, $\sigma < C_2$ satisfies the hypotheses of Lemma 5. Therefore the decay rate in case $\gamma(t)$ is of an exponential type (for instance, see first paragraph above in this section) is also of exponential type.

Example 2:

To illustrate the second part of our theorem we consider $\eta(t) = \gamma_0(1+t)^{-1}$ (which results in case $\gamma(t) = \gamma_0(1+t)^\alpha$, see first paragraph above in this section). The decay rate is also polynomial, that is $\mu(t) = \mu_0(1+t)^\sigma$ with $\mu_0^p \geq \frac{\alpha}{\gamma_0-\sigma}$, $\gamma_0 < \sigma < 1/p$.

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