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The Corona Problem in Carleman Algebras on Non–Stein Domains in $\mathbb{C}^n$

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Abstract

New estimates are obtained for the $\bar{\partial}$–operator on non–Stein domains in $\mathbb{C}^n$ and the results are applied to the Corona problem in Carleman algebras on those domains.

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1 Introduction

Let Ω be an open subset of a complex manifold X, and let p be a non–negative function on Ω. Denote by $A_p(\Omega)$ the (Carleman) algebra of all holomorphic functions f in Ω such that for some positive constants c_1 and c_2

$$|f(z)| \leq c_1 \exp(c_2 p(z)),\ z \in \Omega. \tag{1.1}$$

In [3] where $X = \mathbb{C}^n$ and Ω is pseudoconvex, and in [2] where X is a complex manifold and Ω is a relatively compact Stein open subset, a condition is given on p such that a given finite set $f_1, \dots, f_N \in A_p(\Omega)$ generates $A_p(\Omega)$ if and only if

$$|f_1(z)| + |f_2(z)| + \cdots + |f_N(z)| \geq c_1 \exp(-c_2 p(z)),\ z \in \Omega \tag{1.2}$$

for some constants $c_1 > 0, c_2 > 0$.

Both in [2] and [3] Ω was Stein. As is always the case, it is natural to ask whether the condition of Steinness can be dropped. We show here that it can, if Ω is a domain in $\mathbb{C}^n$ and we modify the condition in [2] and [3] to the following Condition(H):

- p is a non-negative upper semicontinuous function on Ω;
- all polynomials belong to $A_p(\Omega)$; and
- there exist positive constants $K_1, \dots, K_4$ such that $z \in \Omega$ and $|z - \xi| \leq \exp(-K_1 p(z) - K_2) \Rightarrow \xi \in \Omega$ and $p(\xi) \leq K_3 p(z) + K_4$.

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The only difference between the condition in [3] and the Condition(H) here is the replacement of "plurisubharmonic" with "upper semicontinuous". Note that if Ω is an arbitrary domain in $\mathbb{C}^n$, and $d(z)$ denotes the distance from $z \in \Omega$ to the complement of Ω in $\mathbb{C}^n$, $p(z) = \log 1/d(z)$ satisfies Condition (H) on Ω.

If Ω is a domain in $\mathbb{C}^n$ and p satisfies Condition (H) on Ω, then we have (as in [3]) the following two lemmas.

Lemma 1.1. *If $f \in A_p(\Omega)$ it follows that $\frac{\partial f}{\partial z_j} \in A_p(\Omega)$, $1 \leq j \leq n$.*

Lemma 1.2. *If f is holomorphic in Ω, then $f \in A_p(\Omega)$ if and only if for some $K > 0$*

$$\int_\Omega |f|^2 \exp(-2Kp(z))\, d\lambda < \infty,$$

where $d\lambda$ denotes Lebesgue measure.

Our main Theorem is therefore the following

Theorem 1.3. *Let Ω be a domain in $\mathbb{C}^n$ and p a function on Ω satisfying Condition (H). Then a finite set of functions in $A_p(\Omega)$, $f_1, \ldots, f_N$ generates $A_p(\Omega)$ if and only if (1.2) is valid.*

To prove this theorem we follow the homological argument given in [3] almost word for word, using Lemmas 1.1 and 1.2 and $\mathcal{L}^p$–Carleman estimates for the $\bar{\partial}$–operator on Ω, which we establish in the next section.

2 $\mathcal{L}^p$–Carleman Estimates for the $\bar{\partial}$–operator

For $1 \leq p \leq \infty$, let $\mathcal{L}^p_{(r,q)}(U)$ denote the space of forms of type (r,q) with coefficients in $\mathcal{L}^p(U)$,

$$f = \sum_{|I|=r}{}' \sum_{|J|=q}{}' f_{I,J}\, dz^I \wedge d\bar{z}^J \tag{2.1}$$

where $\sum'$ means that the summation is performed only over strictly increasing multi–indices,

$$I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_q), dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_r}, d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

and U is open in $\mathbb{C}^n$.

The norm of the (r,q)–form in (2.1) is defined by

$$\|f\|_{\mathcal{L}^p_{(r,q)}(U)} = \left\{ \sum_I{}' \sum_J{}' \|f_{I,J}\|^p_{\mathcal{L}^p(U)} \right\}^{1/p}, \quad 1 \leq p < \infty.$$

Let $B_q(\xi, z)$ be the Bochner–Martinelli–Koppelman kernel of degree $(0,q)$ in z and degree $(n, n-q-1)$ in ξ, so that, with $\beta = |\xi - z|^2$,

$$B_q(\xi, z) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \binom{n-1}{q} \beta^{-n} \partial_\xi \beta \wedge \left(\bar{\partial}_\xi \partial_\xi \beta\right)^{n-q-1} \wedge \left(\bar{\partial}_z \partial_\xi \beta\right)^q \tag{2.2}$$

for $0 \le q \le n$.

An upper semicontinuous function φ is said to be admissible in an open set U in $\mathbb{C}^n$, if for every coefficient $b_q(\xi,z)$ of $B_q(\xi,z)$, $0 \le q \le n$,

$$\int_U \left|b_q(\xi,z)\right| e^{-\varphi(z)}\, d\lambda(z) \le C, \quad \int_U \left|b_q(\xi,z)\right| e^{-\varphi(\xi)}\, d\lambda(\xi) \le C \tag{2.3}$$

where $C > 0$ is a constant and λ is Lebesgue measure.

For an upper semicontinuous φ, we define $\mathcal{L}^p(U,\varphi)$ where U is open in $\mathbb{C}^n$ by

$$\mathcal{L}^p(U,\varphi) := \left\{ g \text{ is measurable on } U : \int_U |g|^p e^{-\varphi}\, d\lambda < \infty \right\}, \tag{2.4}$$

$1 \le p < \infty$, and

$$\|g\|_{\mathcal{L}^p(U,\varphi)} = \left\{ \int_U |g|^p e^{-\varphi}\, d\lambda \right\}^{1/p}.$$

$\mathcal{L}^p_{(r,q)}(U,\varphi)$ is the space of (r,q)–forms with coefficients in $\mathcal{L}^p(U,\varphi)$, and if f is as in (2.1),

$$\|f\|_{\mathcal{L}^p_{(r,q)}(U,\varphi)} = \left\{ \sum_I{}' \sum_J{}' \|f_{I,J}\|^p_{\mathcal{L}^p(U,\varphi)} \right\}^{1/p}$$

$1 \le p < \infty$.

Our second main result is

Theorem 2.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $f \in \mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)$ be $\bar\partial$–closed, $1 < p < \infty$ and φ an upper semicontinuous function admissible in Ω. Then there is $u \in \mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ such that $\bar\partial u = f$ and*

$$\|u\|_{\mathcal{L}^p_{(0,q)}(\Omega,\varphi)} \le \delta \|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)},$$

where δ is independent of f.

To prove Theorem 2.1 we need a lemma about Sobolev Space estimates for the $\bar\partial$–operator on bounded domains in $\mathbb{C}^n$ with boundaries of Lebesgue measure zero. Accordingly, let $W^{1,1}(U)$ be the space of functions which together with their distributional derivatives of order one are in $\mathcal{L}^1(U)$, with the usual norm, and $W^{1,1}_{(r,q)}(U)$ is the space of (r,q)–forms with coefficients in $W^{1,1}(U)$. We then have

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with boundary of Lebesgue measure zero. Let $f \in W^{1,1}_{(0,q+1)}(\Omega)$ be $\bar\partial$–closed. Then there is a $u \in W^{1,1}_{(0,q)}(\Omega)$ such that $\bar\partial u = f$.*

To prove Lemma 2.2 we need the Bochner–Martinelli–Koppelman formula:

Theorem 2.3. *Let Ω be any bounded domain in $\mathbb{C}^n$ with C^1 boundary. For $f \in C^1_{(0,q)}\left(\overline{\Omega}\right)$, $0 \le q \le n$, we have*

$$f(z) = \int_{\partial\Omega} B_q(\cdot,z) \wedge f + \int_\Omega B_q(\cdot,z) \wedge \bar\partial_\xi f + \bar\partial_z \int_\Omega B_{q-1}(\cdot,z) \wedge f,\ z \in \Omega \tag{2.5}$$

where $B_q(\xi,z)$ is as in (2.2). (For the proof see [1] page 266).

Proof of Lemma 2.2. With Ω and f as in Lemma 2.2, if

$$u(z) = \int_\Omega B_q(\cdot, z) \wedge f,\ z \in \Omega, \tag{2.6}$$

then $\bar\partial u = f$:

Let $f = \sum_J' f_J \, d\bar z^J$ be defined as zero outside Ω and regularize f coefficientwise: $f_m = \sum_J (f_J)_m \, d\bar z^J$,

$$\text{where } (f_J)_m^{(z)} = \int_{\mathbb{C}^n} f_J(z - \xi/m)\psi(\xi)\, d\lambda(\xi)$$
$$= m^{2n} \int_{\mathbb{C}^n} f_J(\xi)\psi(m(z-\xi))\, d\lambda(\xi)$$

and $\psi \in C_0^\infty(\mathbb{C}^n), \int \psi \, d\lambda = 1, \psi \geq 0, \operatorname{supp}\psi = \{z \in \mathbb{C}^n : |z| \leq 1\}$, and λ is Lebesgue measure. Then $\|f_m\|_{\mathcal{L}^1_{(0,q+1)}(\mathbb{C}^n)} \leq \|f\|_{\mathcal{L}^p_{(0,q+1)}(\mathbb{C}^n)}, f_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega)$ as $m \to \infty$ and f_m is $\bar\partial$–closed in $\mathbb{C}^n$.

$$\text{Now let } u_m(z) = \int_{\mathbb{C}^n} B_q(\cdot, z) \wedge f_m. \tag{2.7}$$

Then from Theorem 2.3, we have $\bar\partial u_m = f_m$, and since $f_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega)$, we have $u_m \to u$ in $\mathcal{L}^1_{(0,q)}(\Omega)$, and $\bar\partial u = f$. □

Proof of Theorem 2.1. We first assume that Ω is bounded. It is clear that there is a sequence $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots$ of bounded domains, each with boundary of Lebesgue measure zero, such that $\bigcup_{v=1}^\infty \Omega_v = \Omega$. We construct a sequence of $(0,q)$–forms $\{u_v\}_{v=1}^\infty$ with $u_v \in \mathcal{L}^p_{(0,q)}(\Omega, \varphi), \bar\partial u_v = f$ in Ω_v and

$$\|u_v\|_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K \|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)},$$

where K is the same for all $v, 1 < p < \infty$. Let us regularize f as above. For v fixed, if m is sufficiently large, $f_m \in W^{1,1}_{(0,q+1)}(\Omega_v)$ and $\bar\partial f_m = 0$ in Ω_v. For such an m (sufficiently large) define

$$g_m := \begin{cases} f_m \text{ in } \Omega_v \\ 0 \text{ outside } \Omega_v. \end{cases}$$

Then from Lemma 2.2, if

$$u_{v,m} = \int_{\Omega_v} B_q(\cdot, z) \wedge g_m,$$
$$\bar\partial u_{v,m} = g_m \text{ in } \Omega_v$$

and since φ is admissible on Ω_v

$$\|u_{v,m}\|_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K \|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)}.$$

Now it is clear that as $m \to \infty, g_m \to f$ in $\mathcal{L}^1_{(0,q+1)}(\Omega_v)$ and $u_{v,m} \to$ some u_v in $\mathcal{L}^1_{(0,q)}(\Omega_v)$, $\bar\partial u_v = f$ and

$$\|u_v\|_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K \|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)}. \tag{2.8}$$

Define u_v as zero outside Ω_v, then since $\mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ is reflexive, for $1<p<\infty$, by the Banach–Alaoglu Theorem, there is u in $\mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ with

$$\|u\|_{\mathcal{L}^p_{(0,q)}(\Omega,\varphi)} \leq \|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)}, \tag{2.9}$$

$(1<p<\infty)$, and a subsequence $\{u_{v_\lambda}\}$ of $\{u_v\}$ such that $u_{v_\lambda} \to u$ weakly in $\mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ as $\lambda\to\infty$. In particular, $u_{v_\lambda}\to u$ in the sense of distributions, as $\lambda\to\infty$. Therefore $\overline{\partial}u = f$.

If Ω is not bounded, we can find a sequence of bounded domains $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots$ exhausting Ω and a sequences of $(0,q)$–forms $\{u_v\}_{v=1}^{\infty}$ as above, such that $\overline{\partial}u_v = f$ on Ω_v and

$$\|u_v\|_{\mathcal{L}^p_{(0,q)}(\Omega_v,\varphi)} \leq K\|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)} \tag{2.10}$$

and K is the same for all v.

Treating the sequence in (2.10) as the sequence in (2.8) was treated, we get an $(0,q)$–form $u\in\mathcal{L}^p_{(0,q)}(\Omega,\varphi)$ with $\overline{\partial}u = f$ and

$$\|u\|_{\mathcal{L}^p_{(0,q)}(\Omega,\varphi)} \leq K\|f\|_{\mathcal{L}^p_{(0,q+1)}(\Omega,\varphi)}.$$

□

3 Proof of Theorem 1.3

The format of the proof is the same as that in [2]: Because of (1.1) and (1.2), where $|f|^2 = |f_1|^2+\cdots+|f_N|^2$, for each $V_j = \frac{\overline{f}_j}{|f|^2}$ there is $K>0$ such that

$$\int_\Omega |V_j|^2 \exp(-2Kp)\,d\lambda < \infty \tag{3.1}$$

and it is clear that

$$\sum_{j=1}^{N} V_j f_j = 1. \tag{3.2}$$

For non–negative integers s and r let L^s_r denote the set of all differential forms h of type $(0,r)$ with values in $\Lambda^s\mathbb{C}^N$, such that for some $K>0$

$$\int_\Omega |h|^2 \exp(-2Kp)\,d\lambda < \infty. \tag{3.3}$$

This means that for each multi–index $I=(i_1,\ldots,i_s)$ of length $|I|=s$ with indices between 1 and N inclusively, h has component h_I which is a differential form of type $(0,r)$ such that h_I is skew symmetric in I and

$$\int_\Omega |h_I|^2 \exp(-2Kp)\,d\lambda < \infty. \tag{3.4}$$

As in [3], $\overline{\partial}$ is an unbounded operator from L^s_r to L^s_{r+1} and the interior product P_f by $(f_1,\ldots,f_N)$ maps L^{s+1}_r into L^s_r.

$$(P_I(h))_I = \sum_{j=1}^{N} h_{I_j} f_j, \quad |I| = s. \tag{3.5}$$

If we define $P_f L_r^0 = 0$, then clearly $P_f^2 = 0$ and P_f commutes with $\overline{\partial}$, so we have a double complex. We now have (as in [3]) the following

Theorem 3.1. *For every $g \in L_r^s$ with $\overline{\partial} g = P_f g = 0$ one can find $h \in L_r^{s+1}$ so that $\overline{\partial} h = 0$ and $P_f h = g$.*

Now from (3.2) $P_f \overline{\partial} V = \overline{\partial} P_f V = \overline{\partial}(1) = 0$, where $V = (V_1, \dots, V_N)$, therefore by Theorem 3.1 there exist $w \in L_1^2$ with $P_f w = \overline{\partial} V$ and $\overline{\partial} w = 0$. Let $k \in L_0^2$ solve $\overline{\partial} k = w$ and set

$$h = V - P_f k \in L_0^1. \tag{3.6}$$

Then $\overline{\partial} h = \overline{\partial} V - P_f w = 0$ and

$$P_f(h) = P_f V = 1 \tag{3.7}$$

i.e. there exist $h_1, \dots, h_N \in A_p(\Omega)$ such that

$$\sum_{j=1}^{N} h_j f_j = 1. \tag{3.8}$$

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