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BOUNDED AND COMPACT OPERATORS ON THE BERGMAN SPACE L_a^1 IN THE UNIT DISK OF $\mathbb{C}$

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Abstract

We characterize boundedness and compactness of the Toeplitz operator T_μ, on the Bergman space $L_a^1(\Delta)$, where the symbols, μ, are complex Borel measures on the unit disk of the complex plane, Δ. The case of Toeplitz operators whose symbols are anti-analytic integrable functions is settled. Our results are related to the reproducing kernel thesis. We also study the case of symbols which are positive measures and the case of radial symbols. Moreover, we give a characterization of compactness for general bounded operators on L_a^1.

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1 Introduction and Statement of results.

Let Δ denote the unit disk of $\mathbb{C}$, and let λ denote the Lebesgue area measure on Δ normalized so that $\lambda(\Delta) = 1$. For $0 < p \leq \infty$, the Bergman space L_a^p is the closed subspace of $L^p(\Delta, d\lambda)$ consisting of analytic functions on the unit disk Δ. When $p = 2$, there exists an orthogonal projector P, called the Bergman projector, from the Hilbert space $L^2(\Delta, d\lambda)$ onto its closed

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subspace L^2_a. The Bergman projection Pg of $g \in L^2(\Delta, d\lambda)$ is given by

$$(Pg)(w) = \langle g, K_w \rangle = \int_\Delta \frac{g(z)}{(1-\bar{z}w)^2}\, d\lambda(z),$$

where $w \in \Delta$, and $K_w(z) = \frac{1}{(1-z\overline{w})^2}$ is the Bergman kernel. The kernel

$$k_w(z) = \frac{1-|w|^2}{(1-z\overline{w})^2}$$

is called the normalized Bergman kernel and $\langle\ ,\ \rangle$ is the usual inner product in L^2. Given a complex Borel measure μ on Δ, the Bergman projection $P\mu$ of μ is defined by

$$(P\mu)(w) = \int_\Delta \frac{d\mu(z)}{(1-\bar{z}w)^2}, \qquad w \in \Delta.$$

The Toeplitz operator T_μ is densely defined on L^p_a by

$$(T_\mu h)(w) = \int_\Delta \frac{h(z)}{(1-w\bar{z})^2}\, d\mu(z),$$

for $h \in L^\infty_a$ (the space of bounded analytic functions in Δ) and $w \in \Delta$, that is

$$T_\mu h = P(h\mu).$$

Note that the previous formula makes sense and defines a function analytic on Δ, and that the operator T_μ is in general unbounded on L^p_a. For $\mu = f d\lambda$ with $f \in L^1(\Delta, d\lambda)$, we write $T_\mu = T_f$.

For $c > 0$, we let

$$\tilde{K}^{(c)}_\zeta(z) = \frac{1+c}{(1-z\bar{\zeta})^{2+c}}$$

and

$$\tilde{k}^{(c)}_\zeta(z) = \frac{\tilde{K}^{(c)}_\zeta(z)}{\|\tilde{K}^c_\zeta\|_1} = \frac{(1-|\zeta|^2)^c}{(1-z\bar{\zeta})^{2+c}}.$$

The study of boundedness and compactness of Toeplitz operators has generated many works over this last decade. See [15] and the references therein. Results are most often described in terms of the boundary behaviour of the so called Berezin transform. We recall that, for a bounded operator A on L^p_a, the Berezin transform of A is the function $\widetilde{A}$, defined by

$$\widetilde{A}(z) := \langle Ak_z, k_z \rangle.$$

When $p = 1$, a new phenomenon appears. For example, in [16], K. Zhu showed that a Toeplitz operator $T_{\bar{f}}$ associated to an antianalytic symbol $\bar{f}$ is bounded on L^1_a if and only if $f \in L^\infty \cap LB$, where LB is the logarithmic Bloch space defined below. At the same time, for $p > 1$, it is well known that $T_{\bar{f}}$ is bounded on L^p_a if and only if f is bounded. So the study of T_μ on L^1_a deserves a particular attention. The study of Toeplitz operators on L^1_a has been considered amongst others in [12, 13].

In [12], the authors introduced a technical condition in the study of T_μ on L^1_a. To be precise, they associate to every complex Borel measure μ on Δ the locally integrable function $R(\mu)$ defined on Δ by

$$R(\mu)(w) := (1-|w|^2)\int_\Delta \frac{d\mu(z)}{(z-w)(1-z\bar{w})^2}.$$

They say that μ satisfies condition (R) if the measure $|R(\mu)(w)|d\lambda(w)$ is a Carleson measure for Bergman spaces. See section 2 for the definition of a Carleson measure. We simply say that $f \in L^1$ satisfies condition (R) when the measure $d\mu = f d\lambda$ satisfies condition (R). We denote by B^∞ the Bloch space in Δ, that is the space of analytic functions g on Δ such that

$$\sup_{z\in\Delta}(1-|z|^2)|g'(z)| < \infty.$$

The space B^∞ is a Banach space under the norm

$$||g||_{B^\infty} = |g(0)| + \sup_{z\in\Delta}(1-|z|^2)|g'(z)|.$$

Next, the logarithmic Bloch space LB is the subspace of the Bloch space consisting of analytic functions g on Δ which satisfy the estimate

$$||g||_{LB} := |g(0)| + \sup_{z\in\Delta}(1-|z|^2)|g'(z)|log(\frac{2}{1-|z|^2}) < \infty.$$

In [12], Z. Wu, R. Zhao and N. Zorboska proved the following theorem:

Theorem 1.1. *Suppose that μ satisfies condition (R). Then the following two assertions are equivalent:*

(1) T_μ is bounded on L^1_a;

(2) $P(\bar{\mu})$ belongs to the logarithmic Bloch space LB.

Moreover, there exists a constant C such that for every complex Borel measure μ satisfying condition (R), the following estimate holds:

$$||P(\mu)||_{LB} \le C(||T_{\bar{\mu}}|| + Carl(R(\mu)),$$

where $Carl(R(\mu))$ denotes the Carleson constant of the Carleson measure $R(\mu)$.

The technical condition (R) is important in their argument. In the same paper [12], Z. Wu, R. Zhao and N. Zorboska also proved the following theorem:

Theorem 1.2. *Suppose that the complex Borel measure μ on Δ is such that the measure $|R(\bar{\mu})|d\lambda$ is a vanishing Carleson measure for Bergman spaces. Then the following two assertions are equivalent:*

(1) T_μ is compact on L^1_a;

(2) *$P(\bar{\mu})$ belongs to the subspace LB_0 of LB consisting of those analytic functions g which satisfy the estimate*

$$\lim_{|z|\to 1}(1-|z|^2)|g'(z)|log(\frac{2}{1-|z|^2})=0$$

See section 2 for the definition of a vanishing Carleson measure.

In [13], T. Yu obtained an interesting result on compactness of a general operator A on a certain weighted Bergman space $A^1(\psi)$. To state Yu's result, we need to recall briefly some definitions.

Let ϕ be a positive and continuous function on $(0,1)$ with

$$\lim_{r\to 1}\phi(r)=0.$$

The positive continuous function ϕ will be called normal if there exist $0<a<b$ and $r_0<1$ such that

$$\frac{\phi(r)}{(1-r^2)^a}\searrow 0 \quad \text{and} \quad \frac{\phi(r)}{(1-r^2)^b}\nearrow \infty \qquad (r_0\le r\to 1^-). \tag{1.1}$$

The pair of functions $\{\phi,\psi\}$ is called a normal pair if ϕ is normal, ψ is positive, continuous and integrable on $(0,1)$, and if for some b satisfying (1.1), there exists $c>b-1$ such that

$$\phi(r)\psi(r)=(1-r^2)^c,\ \ 0\le r<1. \tag{1.2}$$

Let $\{\phi,\psi\}$ be a normal pair and $H(\Delta)$ denote the space of analytic functions on the unit disk Δ. For $f\in H(\Delta)$, we define

$$\begin{aligned}\|f\|_\phi &= \sup_{z\in\Delta}|f(z)|\phi(|z|)=\sup_{0\le r<1}M_\infty(f,r)\phi(r),\\ \|f\|_\psi &= \int_\Delta |f(z)|\psi(|z|)\,d\nu(z)=2\int_0^1 rM_1(f,r)\psi(r)dr\end{aligned}$$

where

$$M_\infty(f,r)=\max_{|z|=r}|f(z)| \qquad \text{and} \qquad M_1(f,r)=\int_0^1 |f(re^{i\theta})|d\theta.$$

We define the following spaces of analytic functions.

$$\begin{aligned}A_\infty(\phi) &= \{f\in H(\Delta):\|f\|_\phi<\infty\},\\ A_0(\phi) &= \{f\in H(\Delta):\lim_{r\to 1^-}M_\infty(f,r)\phi(r)=0\},\\ A^1(\psi) &= \{f\in H(\Delta):\|f\|_\psi<\infty\}.\end{aligned}$$

Clearly $A_0(\phi)\subset A_\infty(\phi)$ so we may use the norm $\|f\|_\phi$ on $A_0(\phi)$. These three spaces are all norm linear spaces with the indicated norms. If $L^1(\psi)$ denotes the Banach space of measurable functions f on Δ such that $\|f\|_\psi=\int_\Delta |f|\,d\lambda_\psi<\infty$, where $d\lambda_\psi(z)=\psi(|z|)\,d\lambda(z)$ then $A^1(\psi)$ is the closed subspace of $L^1(\psi)$ consisting of all analytic functions. Also, $A_\infty(\phi)$ is a Banach space and $A_0(\phi)$ is a closed subspace of $A_\infty(\phi)$. Using the following pairing between $A^1(\psi)$ and $A_\infty(\phi)$,

$$[f,g]=\int_\Delta f(z)\overline{g(z)}(1-|z|^2)^c\,d\lambda(z), \tag{1.3}$$

A. L. Shields and D.L. Williams [9] showed that $(A_0(\phi))^* \cong A_1(\psi)$ and $(A^1(\psi))^* \cong A_\infty(\phi)$.

T. Yu [13] proved the following theorem:

Theorem 1.3. *Suppose that A is a bounded operator on $A^1(\psi)$. Let A^* be the adjoint of A with respect to the pairing in (1.3), K_w the reproducing kernel of $A^2(\psi)$ and k_w its normalization in $A^1(\psi)$. Then the following two assertions are equivalent:*

(1) A is compact on $A^1(\psi)$ and $A_0(\phi)$ is an invariant subspace of A^*;

(2) $\|Ak_w\|_\psi \to 0$ as $w \to \partial\Delta$.

T. Yu [13] also exhibited a compact operator A on $A^1(\psi)$ such that $\|Ak_w\|_\psi$ does not tend to zero as $w \to \partial\Delta$. Although Yu's Theorem does not give a complete characterization of compact operators on L_a^1, an application to a Toeplitz operator T_f with bounded symbol f gives that

$$T_f \text{ is compact on } L_a^1 \Longleftrightarrow \Big(||T_f\tilde{k}_z^{(c)}||_1 \to 0 \text{ as } |z| \to 1.\Big)$$

In this paper, our results are related to such reproducing kernel thesis. We recall that F. Nazarov proved that the reproducing kernel thesis is not valid for $p=2$, i.e. the following two assertions are not equivalent for general $f \in L^2(\Delta, d\lambda)$:

(1) T_f is bounded on L_a^2;

(2) $\sup_{\zeta\in\Delta} ||T_f k_\zeta||_2 < \infty$.

In [1], a set of symbols was constructed for which the above condition (2) is necessary and sufficient. Our main result for boundedness of operators on L_a^1 is the following.

Theorem 1.4. *Let A be a linear operator defined on L_a^∞ with values in the space of analytic functions on Δ and let $c > 0$. Then the implication $(1) \Rightarrow (2)$ holds for the following two assertions.*

(1) A extends to a bounded operator on L_a^1;

(2) the following estimate holds:

$$\sup_{\zeta\in\Delta} ||A\tilde{k}_\zeta^{(c)}||_1 < \infty.$$

The converse $(2) \Rightarrow (1)$ also holds in the following two cases.

(a) The operator A satisfies the following property:

$$\int_\Delta (A\tilde{k}_\zeta^{(c)})(z) g(\zeta) d\lambda(\zeta) = CAg(z)$$

for some absolute constant C and for all $z \in \Delta$ and g in the dense subspace $P_c(\mathcal{D})$ of L_a^1.

(b) $A = T_\mu$ where μ is a complex Borel measure on Δ.

Moreover, in such cases, if $C_1 = \sup_{\zeta\in\Delta} ||A\tilde{k}_\zeta^{(c)}||_1$, *there exists a constant C such that*

$$||A|| \le CC_1.$$

Here, $P_c(\mathcal{D})$ denote the space of weighted Bergman projections of functions in $\mathcal{D}(\Delta)$, the space of $\mathcal{C}^\infty$ functions with compact support in Δ.

Note that Theorem 1.4 gives a complete solution of the boundedness problem for Toeplitz operators with complex measures symbols without referring to the technical condition (R) in Theorem 1.1. Our general result for compact operators on L^1_a is the following:

Theorem 1.5. *Let A be a bounded operator on* L^1_a *and* $C > 0$. *The following two assertions are equivalent:*

(1) The operator A is compact on L^1_a;

(2) For every $\varepsilon > 0$, *there exists* $R \in (0,1)$ *such that*

$$\int_{R\le|z|<1} |(A\tilde{k}_\zeta^{(c)})(z)|d\lambda(z) < \varepsilon$$

for every $\zeta \in \Delta$.

We extend the results of T. Yu and Z. Wu et al. by proving the following characterization of compact Toeplitz operators T_μ whose symbols μ are such that $\bar{\mu}$ satisfies a "uniform" condition (R).

Theorem 1.6. *Let* $c > 0$. *Suppose that the complex measure* μ *is such that* $K_z\bar{\mu}$ *satisfies condition* (R) *for every* $z \in \Delta$ *with the following uniform condition:*

$$\forall r \in (0,1), \qquad \sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})) < \infty$$

(in particular if $|\mu|$ *is a Carleson measure for Bergman spaces.) Suppose further that* T_μ *is bounded on* L^1_a. *Then* T_μ *is compact on* L^1_a *if and only if* $\|T_\mu\tilde{k}_\zeta^{(c)}\|_1 \to 0$ *as* $\zeta \to \partial\Delta$.

In Theorem 1.6, $Carl(R(K_z\bar{\mu}))$ denotes the Carleson constant of the Carleson measure $|R(K_z\bar{\mu})|d\lambda$.

As it might not be seen at first sight, we would like to point out that our conditions in Theorem 1.6 are weaker in comparison to the conditions in Theorem 1.2. For example, if μ is a Carleson measure which is not a vanishing Carleson measure, our result still gives a compactness criterion for T_μ. We also study boundedness and compactness on L^1_a of Toeplitz operators associated with positive measures. In this case again, there is a difference with the case $p > 1$. We show that the Carleson measure property is no longer sufficient to characterize bounded Toeplitz operators with positive measures. This contradicts what is stated in [15, Exercise 6, Chap 7].

The paper is organized as follows. In section 2 we give the proof of Theorem 1.4 and we deduce a characterization of bounded Toeplitz operators whose symbols are anti-analytic functions on Δ. In section 3 we prove Theorem 1.5 and Theorem 1.6 and we deduce a characterization of compact Toeplitz operators with symbols that are anti-analytic functions. We also study the case of symbols which are positive Borel measures on Δ. In the final section, we study the case of radial symbols and we prove that for such symbols whose associated Toeplitz operator is bounded on L_a^1, the conclusion of Theorem 1.6 is true with no extra assumption on the symbol.

2 Bounded Toeplitz operators on L_a^1

2.1 Preliminary results.

In this subsection, we recall some definitions and we established some results that will be used later in this paper.

Definition 2.1. Let $p \in (0,\infty)$. A positive Borel measure μ on Δ is called a Carleson measure for the Bergman space L_a^p, or simply a Carleson measure, if there exists a constant $C > 0$ such that

$$\int_\Delta |f(z)|^p d\mu(z) \le C \int_\Delta |f(z)|^p d\lambda(z) \tag{2.1}$$

for all $f \in L_a^p$.

The infimum of all constants C which satisfy (2.1) is called the Carleson measure constant of μ and will be denoted by $Carl(\mu)$.

Definition 2.2. Let $p \in (0,\infty)$. A positive Borel measure μ on Δ is called a vanishing Carleson measure for the Bergman space L_a^p, or simply a vanishing Carleson measure, if for any sequence $\{f_n\}$ in L_a^p with $||f_n||_p \le 1$ and such that $f_n(z) \longrightarrow 0$ uniformly on compact subsets of Δ, we have

$$\lim_{n\to\infty} \int_\Delta |f_n(z)|^p d\mu(z) = 0.$$

We recall that the Bergman distance β on Δ is given by

$$\beta(z,w) = \log\left(\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}\right) \qquad (z,w \in \Delta).$$

For $r > 0$ and $z \in \Delta$, the set

$$D(z,r) := \{w \in \Delta : \beta(z,w) < r\}$$

is the Bergman ball centered at z with radius r, see [14] for more about the Bergman metric. The next theorem recalls a characterization of Carleson measures for Bergman spaces.

Theorem 2.3. *(cf. e.g. [14, Theorem 2.25]) Let μ be a positive Borel measure on Δ. The following four assertions are equivalent:*

(1) For some $p \in (0,\infty)$, μ is a Carleson measure for the Bergman space L_a^p.

(2) There exists a positive constant C such that

$$\int_\Delta \frac{(1-|z|^2)^2}{|1-w\bar{z}|^4} d\mu(w) \le C$$

for all $z \in \Delta$.

(3) There exists a positive constant C such that

$$\int_{D(z,r)} d\mu(w) \le C(1-|z|^2)^2$$

for all $z \in \Delta$.

(4) For all $p \in (0,\infty)$, μ is a Carleson measure for the Bergman space L^p_a.

Moreover, the Carleson measure constant $Carl(\mu)$ of μ is smaller than the constant C of assertion (2).

We recall the following Lemma for future reference (cf. Forelli-Rudin [3] or [5], Theorem 1.7):

Lemma 2.4. *For all $-1 < \alpha < \infty$ and all real β, let*

$$I_{\alpha,\beta}(z) := \int_\Delta \frac{(1-|w|^2)^\alpha}{|1-z\bar{w}|^{2+\alpha+\beta}} d\lambda(w) \qquad (z \in \Delta).$$

Then

(1) if $\beta < 0$, the function $I_{\alpha,\beta}$ is bounded;

(2) if $\beta = 0$, there exists a constant $C = C_{\alpha,\beta}$ such that for every $z \in \Delta$, the following estimate holds:

$$\frac{1}{C} log(\frac{2}{1-|z|^2}) \le I_{\alpha,\beta} \le C log(\frac{2}{1-|z|^2});$$

(3) if $\beta > 0$, there exists a constant $C = C_{\alpha,\beta}$ such that for every $z \in \Delta$, the following estimate holds:

$$\frac{1}{C}\frac{1}{(1-|z|^2)^\beta} \le I_{\alpha,\beta} \le C\frac{1}{(1-|z|^2)^\beta}.$$

Lemma 2.5. *If μ is a complex measure on Δ such that $|\mu|$ is a Carleson measure for Bergman spaces, then μ satisfies condition (R).*

Proof. We fix $r > 0$. The question is to prove that if $|\mu|$ is a Carleson measure for Bergman spaces, then

$$\sup_{z\in\Delta} \frac{1}{\lambda(D(z,r))} \int_{D(z,r)} |R(\mu)(w)| d\lambda(w) < \infty.$$

Applying Fubini's theorem we obtain

$$\frac{1}{\lambda(D(z,r))} \int_{D(z,r)} |R(\mu)(w)| d\lambda(w) \le \int_\Delta \frac{1}{\lambda(D(z,r))} \left(\int_{D(z,r)} \frac{1-|w|^2}{|u-w||1-u\overline{w}|^2} d\lambda(w) \right) d|\mu|(u).$$

Thus,

$$\frac{1}{\lambda(D(z,r))}\int_{D(z,r)}|R(\mu)(w)|d\lambda(w)\leq$$
$$C\int_{\Delta}\frac{1-|z|^2}{\lambda(D(z,r))|1-u\bar{z}|^2}\left(\int_{D(z,r)}\frac{1}{|u-w|}d\lambda(w)\right)d|\mu|(u),\quad (2.2)$$

since $1-|w|^2\approx 1-|z|^2$ and $|1-u\overline{w}|\approx|1-u\bar{z}|$ for $w\in D(z,r)$ and $u\in\Delta$. By making a change of variable $w=\varphi_z(w')$ we get

$$\begin{aligned}\lambda(D(z,r))^{-1}\int_{D(z,r)}\frac{1}{|u-w|}d\lambda(w) &= \frac{(1-|z|^2)^2}{\lambda(D(z,r))}\int_{D(0,r)}\frac{d\lambda(w')}{|u-\varphi_z(w')||1-w'\bar{z}|^4}\\ &\leq \frac{C}{|1-u\bar{z}|}\int_{D(0,r)}\frac{1}{|w'-\varphi_z(u)|}d\lambda(w')\\ &\leq \frac{C'}{|1-u\bar{z}|}.\end{aligned}$$

Here the first inequality is gotten from the fact that

$$\lambda(D(z,r)\approx(1-|z|^2)^2,\quad |1-w'\bar{z}|\geq 1-|w|$$

and the function $w\mapsto 1-|w|$ is bounded below by a positive constant on the set $D(0,r)$. This together with equation (2.2) shows that there exists a constant C depending on r only such that

$$\lambda(D(z,r))^{-1}\int_{D(z,r)}|R(\mu)(w)|d\lambda(w)\leq C(1-|z|^2)\int_{\Delta}\frac{1}{|1-u\bar{z}|^3}d|\mu|(u).\qquad (2.3)$$

Since $|\mu|$ is a Carleson measure for Bergman spaces, we have

$$\int_{\Delta}\frac{1}{|1-u\bar{z}|^3}d|\mu|(u)\leq Carl(\mu)\int_{\Delta}\frac{1}{|1-u\bar{z}|^3}d\lambda(u)\leq C'Carl(\mu)(1-|z|^2)^{-1}.$$

The latter inequality comes from an application of assertion (3). of Lemma 2.4. The conclusion follows. □

The existence of a lattice in the unit disk will be useful in our argument.

Theorem 2.6. (cf. e.g. Theorem 2.23 of [14]) *For every $r\in(0,1]$, there exist a positive integer N and a sequence $\{a_k\}$ of points in Δ with the following properties:*

(1) $\Delta=\cup_k D(a_k,r)$;

(2) the balls $D(a_k,\frac{r}{4})$ are mutually disjoint;

(3) each point $z\in\Delta$ belongs to at most N of the balls $D(a_k,4r)$.

Such a sequence $\{a_k\}$ is called an $r-$lattice.

From now on, c will denote a positive number. It is shown e.g. in [5], Lemma 1.17, that there exists a unique linear operator $\mathcal{D}$ on $H(\Delta)$ with the following properties.

- $\mathcal{D}$ is continuous on $H(\Delta)$ with respect to the topology of uniform convergence on compact sets of $\mathbb{C}$ contained in Δ;
- $\mathcal{D}_z[(1-z\bar{w})^{-2}] = (1-z\bar{w})^{-(2+c)}$ for every $w \in \Delta$;
- $\mathcal{D}$ is invertible on $H(\Delta)$.

We shall use the following Lemma.

Lemma 2.7. *For every $h \in L^1_a$, the function $\mathcal{D}h$ is given by*

$$\mathcal{D}h(z) = \int_\Delta \frac{h(w)}{(1-z\bar{w})^{2+c}} d\lambda(w) \qquad (z \in \Delta).$$

Moreover, there exists a constant C such that

$$\int_\Delta (1-|z|^2)^c \overline{\mathcal{D}h(z)} g(z) d\lambda(z) = C \int_\Delta \overline{h(z)} g(z) d\lambda(z),$$

for all $h \in L^1_a$ and $g \in L^\infty_a$.

Proof. The first assertion is proved in page 19 of [5], while the second assertion is proved in page 20 of [5] for $g \in L^\infty_a$ and either h or $(1-|z|^2)^c h(z)$ bounded. We give here a different proof.

We first prove the lemma for all $h \in L^2_a$ and $g \in L^\infty_a$. Let $\{a_k\}$ be a $r-$lattice as described in Theorem 2.6. By the atomic decomposition theorem (cf. e.g. Theorem 2.30 of [14]), for every $h \in L^2_a$, there exists a sequence $\{c_k\}$ of complex numbers belonging to the sequence space l^2 such that

$$h(z) = \sum_{k=1}^\infty c_k \frac{1-|a_k|^2}{(1-z\overline{a_k})^2} \qquad (z \in \Delta),$$

where the series converges in the norm topology of L^2_a. This series converges uniformly on compact sets of $\mathbb{C}$ contained in Δ to its sum $h(z)$. Next, the series $\sum_{k=1}^\infty c_k \frac{1-|a_k|^2}{(1-z\overline{a_k})^{2+c}}$ converges in the norm topology of the weighted Bergman space $L^2_a((1-|z|^2)^{2c} d\lambda(z))$, and thus it converges uniformly on compact sets of $\mathbb{C}$ contained in Δ to its sum.

We recall that $\mathcal{D}_z[(1-z\bar{w})^{-2}] = (1-z\bar{w})^{-(2+c)}$ for every $w \in \Delta$. Thus the partial sums

$$\sum_{k=1}^N c_k (1-|a_k|^2) \mathcal{D}_z[\frac{1}{(1-z\overline{a_k})^2}] = \mathcal{D}_z[\sum_{k=1}^N c_k \frac{1-|a_k|^2}{(1-z\overline{a_k})^2}]$$

converges uniformly on compact sets of $\mathbb{C}$ contained in Δ to the analytic function

$$\sum_{k=1}^\infty c_k \frac{1-|a_k|^2}{(1-z\overline{a_k})^{2+c}}$$

as $N \to \infty$. Since $\mathcal{D}$ is continuous in $H(\Delta)$, we conclude that

$$\mathcal{D}h(z) = \sum_{k=1}^\infty c_k \frac{1-|a_k|^2}{(1-z\overline{a_k})^{2+c}}. \tag{2.4}$$

Hence

$$\begin{aligned}\mathcal{D}h(z) &= \textstyle\sum_{k=1}^{\infty} c_k(1-|a_k|^2)\int_\Delta \frac{1}{(1-w\bar{a}_k)^2(1-z\bar{w})^{2+c}}d\lambda(w)\\ &= \textstyle\int_\Delta\{\sum_{k=1}^{\infty} c_k \frac{1-|a_k|^2}{(1-w\bar{a}_k)^2}\}\frac{1}{(1-z\bar{w})^{2+c}}d\lambda(w)\\ &= \textstyle\int_\Delta \frac{h(w)}{(1-z\bar{w})^{2+c}}d\lambda(w).\end{aligned}$$

Next the convergence in $L_a^2((1-|z|^2)^{2c}d\lambda(z))$ of the series in the right hand side of (2.4) implies that

$$\int_\Delta (1-|z|^2)^c\overline{\mathcal{D}h(z)}g(z)d\lambda(z) = \sum_{k=1}^{\infty} c_k(1-|a_k|^2)\int_\Delta \frac{(1-|z|^2)^c}{(1-\bar{z}a_k)^{2+c}}g(z)d\lambda(z).$$

Also, there exists a constant C such that for every $g \in L_a^\infty$ and for every positive integer k,

$$\int_\Delta \frac{(1-|z|^2)^c}{(1-\bar{z}a_k)^{2+c}}g(z)d\lambda(z) = Cg(a_k) = C\int_\Delta \frac{g(w)}{(1-a_k\bar{w})^2}d\lambda(w).$$

This implies,

$$\begin{aligned}\int_\Delta (1-|z|^2)^c\overline{\mathcal{D}h(z)}g(z)d\lambda(z) &= C\sum_{k=1}^{\infty} c_k(1-|a_k|^2)\int_\Delta \frac{g(w)}{(1-\bar{w}a_k)^2}d\lambda(w)\\ &= C\int_\Delta\{\sum_{k=1}^{\infty} c_k\frac{1-|a_k|^2}{(1-\bar{w}a_k)^2}\}g(w)d\lambda(w)\\ &= C\int_\Delta \bar{h}(w)g(w)d\lambda(w).\end{aligned}$$

We next consider the general case when $h \in L_a^1$. The announced conclusions follow from the density of L_a^2 in L_a^1 and from the existence of a constant C such that

$$\int_\Delta (1-|z|^2)^c|\mathcal{D}h(z)|d\lambda(z) \le C\int_\Delta |h(z)|d\lambda(z)$$

for all analytic functions h on Δ. For the latter result, cf. Theorem 2.19 of [14]. This finishes the proof of the lemma. □

For $c > 0$, we denote by P_c the orthogonal projector from $L^2((1-|z|^2)^c d\lambda(z))$ unto the weighted Bergman space $L_a^2((1-|z|^2)^c d\lambda(z))$. Then P_c is a weighted Bergman projector in Δ and for every $\phi \in L^2((1-|z|^2)^c d\lambda(z))$, we have that

$$P_c\phi(z) = (1+c)\int_\Delta \frac{(1-|\zeta|^2)^c}{(1-z\bar{\zeta})^{2+c}}\phi(\zeta)d\lambda(\zeta).$$

We denote by $\mathcal{D}(\Delta)$ the space of $\mathcal{C}^\infty$ functions with compact support in Δ. We shall need the following lemma.

Lemma 2.8. *The space $P_c(\mathcal{D}(\Delta))$ is a dense subspace of L_a^1.*

Proof. It is easy to check that $P_c(\mathcal{D}(\Delta)) \subset L_a^\infty \subset L_a^1$. Since the dual space of L_a^1 with respect to the usual duality pairing $\langle,\rangle$ in $L^2(\Delta, d\lambda)$ is the Bloch space B^∞, it suffices to show that every $h \in B^\infty$ such that

$$\int_\Delta P_c\phi(z)\bar{h}(z)d\lambda(z) = 0 \qquad \forall\phi \in \mathcal{D}(\Delta)$$

vanishes identically. An application of Fubini's Theorem and Lemma 2.7 gives

$$\begin{aligned} 0 &= \int_\Delta P_c\phi(z)\bar{h}(z)d\lambda(z) = \int_\Delta \left(\int_\Delta \frac{(1-|\zeta|^2)^c}{(1-z\bar{\zeta})^{2+c}}\phi(\zeta)d\lambda(\zeta)\right)\bar{h}(z)d\lambda(z) \\ &= \int_\Delta \phi(\zeta)\overline{\mathcal{D}h(\zeta)}(1-|\zeta|^2)^c d\lambda(\zeta). \end{aligned}$$

It is easy to conclude that $\mathcal{D}h \equiv 0$ on Δ. Using the invertibility of $\mathcal{D}$ on $H(\Delta)$, we obtain that $h \equiv 0$ on Δ. □

The following three lemmas are proved in [14].

Lemma 2.9. *Suppose $p > 0$, $c > 0$ and $\alpha > -1$. Let $d\lambda_\alpha(z) = (1-|z|^2)^\alpha d\lambda(z)$. There exist constants A and B such that*

$$A\int_\Delta |f(z)|^p d\lambda_\alpha(z) \leq \int_\Delta |(1-|z|^2)^c\mathcal{D}^c f(z)|^p d\lambda_\alpha(z) \leq B\int_\Delta |f(z)|^p d\lambda_\alpha(z) \quad (2.5)$$

for all holomorphic functions f in Δ.

Lemma 2.10. *Suppose $p > 0$ and $\alpha > -1$. For $F \in L^p_a(d\lambda_\alpha)$, we have*

$$|F(z)| \leq \frac{||F||_{p,\alpha}}{(1-|z|^2)^{(2+\alpha)/p}} \quad (2.6)$$

for all $z \in \Delta$.

Lemma 2.11 (Theorem 3.9 in [14])**.** *For any z and w in Δ we have*

$$\beta(z,w) = \sup\{|f(z)-f(w)| : \quad f \in B^\infty; \quad ||f||_{B^\infty} \leq 1\}.$$

2.2 Proof of Theorem 1.4

Suppose (1) holds. Then

$$||A\tilde{k}^{(c)}_\zeta||_1 \leq ||A||\,||\tilde{k}^{(c)}_\zeta||_1$$

and since

$$||\tilde{k}^{(c)}_\zeta||_1 = \int_\Delta \frac{(1-|\zeta|^2)^c}{|1-w.\bar{\zeta}|^{2+c}}d\lambda(w)$$

is bounded in ζ by Lemma 2.4, this gives (2).

Suppose that (2) is satisfied.

Case (a): By our assumption on A, we have

$$\begin{aligned} \int_\Delta |Ag(z)|d\lambda(z) &\leq C^{-1}\int_\Delta\left(\int_\Delta |A\tilde{k}^{(c)}_\zeta(z)||g(\zeta)|d\lambda(\zeta)\right)d\lambda(z) \\ &= C^{-1}\int_\Delta\left(\int_\Delta |A\tilde{k}^{(c)}_\zeta(z)|d\lambda(z)\right)|g(\zeta)|d\lambda(\zeta) \\ &= C^{-1}\sup_{\zeta\in\Delta}||A\tilde{k}^{(c)}_\zeta||_1||g||_1. \end{aligned}$$

This shows the implication $(2) \Longrightarrow (1)$ for the case (a).

Case (b): Let μ be a complex Borel measure on Δ. From case (a), it is enough to prove that if $z \in \Delta$ and g in the dense subspace $P_c(\mathcal{D}(\Delta))$ of L_a^1, then

$$\int_\Delta (T_\mu \tilde{k}_\zeta^{(c)})(z) g(\zeta) d\lambda(\zeta) = \frac{1}{1+c} T_\mu g(z). \tag{2.7}$$

Let $h \in L_a^1(\Delta, d\lambda(\zeta))$ and $g = P_c\phi$ with $\phi \in \mathcal{D}(\Delta)$. Then

$$\int_\Delta \bar{h}(\zeta) g(\zeta)(1-|\zeta|^2)^c d\lambda(\zeta) = \int_\Delta \bar{h}(\zeta)\phi(\zeta))(1-|\zeta|^2)^c d\lambda(\zeta). \tag{2.8}$$

Fix $z \in \Delta$ and take

$$h_z(\zeta) := \overline{(T_\mu \tilde{K}_\zeta^{(c)})(z)} = (1+c)\int_\Delta \frac{1}{(1-w\bar{z})^2(1-\zeta\bar{w})^{2+c}} d\bar{\mu}(w).$$

It is clear that the function h_z is analytic and for every $\zeta \in \Delta$, and the function $z \mapsto h_z(\zeta)$ is antianalytic. By the mean value property, there exists a constant C_z such that

$$|h_z(\zeta)| \le C_z ||T_\mu \tilde{K}_\zeta^{(c)}||_1$$

and hence

$$\int_\Delta |h_z(\zeta)|(1-|\zeta|^2)^c d\lambda(\zeta) \le C_z \sup_{\zeta\in\Delta} ||T_\mu \tilde{k}_\zeta^{(c)}||_1 < \infty.$$

In the latter inequality, we applied assertion (2).

For every ϕ in the space $\mathcal{D}(\Delta)$, we have

$$\int_\Delta \left(\int_\Delta \frac{(1-|\zeta|^2)^c}{|1-w\bar{\zeta}|^{2+c}} |\phi(\zeta)| d\lambda(\zeta) \right) \frac{d|\mu|(w)}{|1-z\bar{w}|^2} \quad \le \quad \frac{C(\phi)}{(1-|z|^2)^2} \int_\Delta d|\mu|(w) < \infty$$

for every $z \in \Delta$. By identity (2.8) and Fubini's Theorem, we obtain that for every $g = P_c\phi$ in the dense subspace $P_c(\mathcal{D}(\Delta))$ of L_a^1 and for every $z \in \Delta$,

$$\begin{aligned}
\int_\Delta (T_\mu \tilde{k}_\zeta^{(c)})(z) g(\zeta) d\lambda(\zeta) &= \int_\Delta (T_\mu \tilde{k}_\zeta^{(c)})(z)\phi(\zeta) d\lambda(\zeta) \\
&= \int_\Delta \left(\int_\Delta \frac{1}{(1-z\bar{w})^2} \frac{(1-|\zeta|^2)^c}{(1-w\bar{\zeta})^{2+c}} d\mu(w) \right) \phi(\zeta) d\lambda(\zeta) \\
&= \int_\Delta \left(\int_\Delta \frac{(1-|\zeta|^2)^c}{(1-w\bar{\zeta})^{2+c}} \phi(\zeta) d\lambda(\zeta) \right) \frac{1}{(1-z\bar{w})^2} d\mu(w) \\
&= \frac{1}{1+c} \int_\Delta \frac{g(w)}{(1-z\bar{w})^2} d\mu(w) = \frac{1}{1+c} T_\mu g(z).
\end{aligned}$$

This proves identity (2.7) and so the implication $(2) \Rightarrow (1)$ is proved for case (b). □

One would like to know whether $(2) \Longrightarrow (1)$ for general operators A in the above Theorem. The next lemma shows that our necessary condition, when $A = T_\mu$, in Theorem 1.4 is remarkably strong.

Lemma 2.12. *Let $c > 0$ and μ a complex Borel measure in Δ. Then there exists a constant C such that*

$$\left|\int_\Delta \frac{(1-|a|^2)^c}{(1-\overline{a}w)^{2+c}} \frac{d\mu(w)}{(1-\overline{w}z)^{2+c}}\right| \leq \frac{C}{(1-|z|^2)^{2+c}} \|T_\mu \tilde{k}_a^{(c)}\|_1 \tag{2.9}$$

for all $a, z \in \Delta$.

Proof. Let $a \in \Delta$. For $z \in \Delta$, we have by (2.6), (2.5) that

$$\begin{aligned}\left|\int_\Delta \frac{(1-|a|^2)^c}{(1-\overline{a}w)^{2+c}} \frac{d\mu(w)}{(1-\overline{w}z)^{2+c}}\right| &\leq \frac{1}{(1-|z|^2)^{2+c}} \int_\Delta \left|\int_\Delta \frac{(1-|a|^2)^c}{(1-\overline{a}w)^{2+c}} \frac{d\mu(w)}{(1-\overline{w}\zeta)^{2+c}}\right| d\lambda_c(\zeta) \\ &\leq \frac{B}{(1-|z|^2)^{2+c}} \int_\Delta \left|\int_\Delta \frac{(1-|a|^2)^c}{(1-\overline{a}w)^{2+c}} \frac{d\mu(w)}{(1-\overline{w}\zeta)^2}\right| d\lambda(\zeta) \\ &= \frac{B}{(1-|z|^2)^{2+c}} \|T_\mu \tilde{k}_a^{(c)}\|_1. \quad \square\end{aligned}$$

If we take $a = z$ in (2.9), we easily find that for any $a \in \Delta$,

$$\left|\int_\Delta \frac{(1-|a|^2)^{2+2c}}{|1-\overline{a}w|^{4+2c}} d\mu(w)\right| \leq C \sup_{a\in\Delta} \|T_\mu \tilde{k}_a^{(c)}\|_1.$$

Hence for positive measures, this clearly shows that our necessary condition implies that μ must be a Carleson measure. The following result shows that the converse of this is not true and gives several characterizations of boundedness of Toeplitz operators with positive measures.

Theorem 2.13. *Let μ be a positive measure in the unit disk Δ. The following propositions are equivalent:*

(i) T_μ *is bounded on* L_a^1.

(ii) For every strictly positive c, there is a constant A such that

$$\sup_{a\in\Delta} \|T_\mu \tilde{k}_a^{(c)}\|_1 \leq A.$$

(iii) There is a constant A such that

$$\sup_{a\in\Delta} \|T_\mu \tilde{k}_a^{(1)}\|_1 \leq A.$$

(iv) μ *is a Carleson measure for Bergman spaces and* $P(\mu) \in LB$.

We remark that $(i) \Leftrightarrow (iv)$ has already appeared in Wang and Liu [11] but we obtain this result independently and our proof is different from theirs.

Proof. It is clear that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$. We will show that $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$.

$(iii) \Rightarrow (iv)$: From the observation after Lemma 2.12, (iii) implies that μ is a Carleson measure. In this case, we have the following lemma.

Lemma 2.14. *Let μ be a positive measure in the unit disk Δ. Suppose that μ is a Carleson measure for Bergman spaces. Then the following operator*

$$S_\mu(h)(z) = (1-|z|^2)\int_\Delta \frac{h(z)-h(\zeta)}{(1-z\overline{\zeta})^3}d\mu(\zeta)$$

is bounded from B^∞ to L^∞.

Proof of Lemma 2.14

Using the Carleson condition, Lemma 2.11, and a change of variable $\zeta = \varphi_z(w)$, we have

$$\begin{aligned}
|S_\mu(h)(z)| &\leq (1-|z|^2)\int_\Delta \frac{|h(z)-h(\zeta)|}{|1-z\overline{\zeta}|^3}d\mu(\zeta)\\
&\leq Carl(\mu)(1-|z|^2)\int_\Delta \frac{|h(z)-h(\zeta)|}{|1-z\overline{\zeta}|^3}d\lambda(\zeta)\\
&\leq Carl(\mu)(1-|z|^2)||h||_{B^\infty}\int_\Delta \frac{\beta(z,\zeta)}{|1-z\overline{\zeta}|^3}d\lambda(\zeta)\\
&= Carl(\mu)||h||_{B^\infty}\int_\Delta \frac{\beta(w,0)}{|1-z\overline{w}|}d\lambda(w)\\
&\leq C||h||_{B^\infty}. \qquad \square
\end{aligned}$$

We are now ready to prove the second part of (iv). For $h \in B^\infty$ and $z \in \Delta$, we have

$$\begin{aligned}
\left\langle T_\mu \tilde{k}_z^1, h\right\rangle &= \int_\Delta T_\mu \tilde{k}_z^1(w)\overline{h(w)}d\lambda(w)\\
&= \int_\Delta \left(\int_\Delta \frac{(1-|z|^2)}{(1-\overline{z}\zeta)^3}\frac{d\mu(\zeta)}{(1-w\overline{\zeta})^2}\right)\overline{h(w)}d\lambda(w)\\
&= (1-|z|^2)\int_\Delta \frac{1}{(1-\overline{z}\zeta)^3}\left(\overline{\int_\Delta \frac{h(w)d\lambda(w)}{(1-\overline{w}\zeta)^2}}\right)d\mu(\zeta)\\
&= (1-|z|^2)\overline{Q_\mu(h)(z)}
\end{aligned}$$

where $Q_\mu(h)(z) = \int_\Delta \frac{h(\zeta)}{(1-z\overline{\zeta})^3}d\mu(\zeta)$. It is then easy to obtain the identity,

$$(1-|z|^2)\overline{h(z)\,Q_\mu(1)(z)} = \left\langle T_\mu \tilde{k}_z^1, h\right\rangle + \overline{S_\mu(h)(z)}, \tag{2.10}$$

for $z \in \Delta$ and $h \in B^\infty$. So that using Lemma 2.14 we get

$$(1-|z|^2)|h(z)\,Q_\mu(1)(z)| \leq C||h||_{B^\infty} \tag{2.11}$$

for $z \in \Delta$ and $h \in B^\infty$. Taking the supremum over all $h \in B^\infty$ with $||h||_{B^\infty} \leq 1$ and $h(0) = 0$, and applying Lemma 2.11, we obtain that

$$(1-|z|^2)|\,Q_\mu(1)(z)|\log\frac{2}{1-|z|^2} \leq C \tag{2.12}$$

for all $z \in \Delta$. On the other hand, observe that

$$2Q_\mu(1)(z) \quad = \quad 2P(\mu)(z) + zP(\mu)'(z) \tag{2.13}$$

and that, since μ is a Carleson measure, then $||P(\mu)||_{B^\infty} \leq CCarl(\mu)$ and hence

$$(1-|z|^2)|P(\mu)(z)|\log\frac{2}{1-|z|^2} \quad \leq \quad CCarl(\mu). \tag{2.14}$$

Therefore, by (2.12), (2.13) and (2.14) we have

$$\sup_{z\in\Delta}(1-|z|^2)|P(\mu)'(z)|\log\frac{2}{1-|z|^2} < \infty.$$

So $P(\mu) \in LB$.

$(iv) \Longrightarrow (i)$:

We may use Lemma 2.5 and Theorem 1.1 as well to conclude. However we include here a direct proof since all the ingredients are contained in the previous implication. Indeed, for $g \in L^\infty_a$ and $h \in B^\infty$, we have

$$\begin{aligned}\langle T_\mu g, h\rangle &= \int_\Delta T_\mu g(w)\overline{h(w)}d\lambda(w)\\ &= C\int_\Delta g(w)(1-|w|^2)\overline{Q_\mu(h)(w)}d\lambda(w).\end{aligned}$$

It is then enough to show that $(1-|w|^2)Q_\mu(h)(w)$ is bounded whenever $h \in B^\infty$. Observe that

$$(1-|w|^2)Q_\mu(h)(w) = (1-|w|^2)h(w)Q_\mu(1)(w) - S_\mu(h)(w).$$

Using the fact $|h(w)| \leq C||h||_{B^\infty}\log\frac{2}{1-|w|^2}$, the result follows by applying Lemma 2.14, (2.14) and (2.13). □

Remark 2.15. In contrast to what is stated in Exercise 6 of Chapter 7 in [15], the property $P(\mu) \in LB$ is not superfluous in assertion *iv*) of the above Theorem. In fact, if this were the case, every Carleson measure μ for Bergman spaces would satisfy $P(\mu) \in LB$. In particular, for every bounded non-negative function f on Δ, we would have $P(f) \in LB$. This would imply that for every bounded function f on Δ, we have $P(f) \in LB$ (to get this property, write the real part and the imaginary part of f as the differences of their positive and negative parts). We are led to the false conclusion that the Bloch space B^∞ is contained in LB.

We next state the following characterization of bounded Toeplitz operators with antianalytic symbols.

Theorem 2.16. *Let $f \in L^1_a$. The following three assertions are equivalent:*

(1) $T_{\bar f}$ is bounded on L^1_a;

(2) For all $c > 0$,

$$\sup_{z\in\Delta}||T_{\bar f}\tilde{k}_z^{(c)}||_1 < \infty;$$

(3) f belongs to $L_a^\infty \cap LB$.

Proof. The equivalence $(1) \Leftrightarrow (2)$ is given by Theorem 1.4, since $\bar{f}d\lambda$ is a complex Borel measure on Δ. The equivalence $(1) \Leftrightarrow (3)$ was proved by K. Zhu [16]. □

In the proof of Theorem 1.6, we shall need the following lemma.

Lemma 2.17. *Suppose that μ is a complex Borel measure on Δ such that T_μ is bounded on L_a^1. Then for every $z \in \Delta$, the Toeplitz operator $T_{K_z\bar{\mu}}$ is bounded on the Bloch space B^∞.*

We suppose further that the measure $K_z\bar{\mu}$ satisfies condition (R) for every $z \in \Delta$ with the following uniform condition:

$$\forall r \in (0,1), \qquad \sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})) < \infty$$

(this is the case when $|\mu|$ is a Carleson measure for Bergman spaces). Then for every $r \in (0,1)$, there exists a constant $C = C(r)$ such that

$$\sup_{z\in r\Delta} ||P(K_z\bar{\mu})||_{LB} \le C(||T_\mu|| + \sup_{z\in r\Delta} Carl(R(K_z\bar{\mu}))),$$

where $||T_\mu||$ denotes the norm operator of T_μ on L_a^1 and $Carl(R(K_z\bar{\mu}))$ denotes the Carleson constant of the Carleson measure $|R(K_z\bar{\mu})|d\lambda$.

Proof. By the duality between B^∞ and L_a^1 with respect to the usual pairing in $L^2(\Delta, d\lambda)$, if T_μ is bounded on L_a^1, the adjoint operator of T_μ is $T_{\bar{\mu}}$ and is bounded on B^∞. It is easy to check that for every $g \in B^\infty$ and for every $z \in \Delta$, the function K_zg belongs to B^∞ and there exists a constant $C(z)$ such that $||K_zg||_{B^\infty} \le C(z)||g||_{B^\infty}$. Hence, for all $g \in B^\infty$ and $h \in L_a^2$ we have, for $z \in \Delta$, that

$$\begin{aligned} |\langle T_{K_z\bar{\mu}}g, h\rangle| &= |\langle K_zg, T_\mu h\rangle| \le \|K_zg\|_{B^\infty}\|T_\mu h\|_1 \\ &\le C(z)||g||_{B^\infty}||T_\mu||||h||_1. \end{aligned}$$

For every $r \in (0,1)$, there exists a constant $C(r)$ such that

$$\sup_{z\in r\Delta} ||K_zg||_{B^\infty} \le C(r)||g||_{B^\infty}.$$

Hence for all $g \in B^\infty$ and $h \in L_a^2$ we have, for $z \in r\Delta$, that

$$|\langle T_{K_z\bar{\mu}}g, h\rangle| \quad \le \quad C(r)\|g\|_{B^\infty}||T_\mu||||h||_1.$$

If we denote by $||T_{K_z\bar{\mu}}||'$ the operator norm of $T_{K_z\bar{\mu}}$ on B^∞, we obtain

$$||T_{K_z\bar{\mu}}||' \le C(r)||T_\mu||.$$

Since the measure $K_z\bar{\mu}$ satisfies condition (R) for every $z \in \Delta$, the conclusion follows from the inequality

$$||P(K_z\bar{\mu})||_{LB} \le C(||T_{K_z\bar{\mu}}||' + Carl(K_z\bar{\mu}))$$

which is given by Theorem 1.1. □

3 Compactness of Toeplitz operators

In this section, we give a general criterion of compactness on L^1_a and a proof of Theorem 1.6. We obtain as corollaries a compactness characterization of Toeplitz operators with positive measures or with antianalytic symbols. The following theorem will be useful; for its proof, the reader can consult [3, page 74].

Theorem 3.1. *Let $\mathcal{F}$ denote a bounded subset of $L^1(\Delta, d\lambda)$. The following two assertions are equivalent:*

(1) The closure $\mathcal{F}$ in $L^1(\Delta, d\lambda)$ is compact in $L^1(\Delta, d\lambda)$;

(2) (a) For all $\varepsilon > 0$ and $R \in (0,1)$, there exists $\delta \in (0, 1-R)$ such that

$$\int_{|z|<R} |\phi(z+h) - \phi(z)| d\lambda(z) < \varepsilon$$

for all $\phi \in \mathcal{F}$ and all $h \in \mathbb{C}$ such that $|h| < \delta$, and

(b) For every $\varepsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R\leq|z|<1} |\phi(z)| d\lambda(z) < \varepsilon$$

for every $\phi \in \mathcal{F}$.

3.1 Proof of Theorem 1.5

Let $\mathcal{F} := \{Ag : g \in L^1_a, ||g||_1 \leq 1\}$. Since A is bounded on L^1_a, the set $\mathcal{F}$ is a bounded subset of L^1_a and hence a bounded subset of $L^1(\Delta, d\lambda)$. Moreover, the compactness of $\mathcal{F}$ in L^1_a is equivalent to the compactness of $\mathcal{F}$ in $L^1(\Delta, d\lambda)$. According to Theorem 3.1, it suffices to show that the following two properties are equivalent:

(1) For every $\varepsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R\leq|z|<1} |(A\tilde{k}_\zeta^{(c)})(z)| d\lambda(z) < \varepsilon,$$

for all $\zeta \in \Delta$.

(2) (a) For all $\varepsilon > 0$ and $R \in (0,1)$, there exists $\delta \in (0, 1-R)$ such that

$$\int_{|z|<R} |\phi(z+h) - \phi(z)| d\lambda(z) < \varepsilon$$

for all $\phi \in \mathcal{F}$ and all $h \in \mathbb{C}$ such that $|h| < \delta$ and
(b) For every $\varepsilon > 0$, there exists $R \in (0,1)$ such that $\int_{R\leq|z|<1} |\phi(z)| d\lambda(z) < \varepsilon$ for every $\phi \in \mathcal{F}$.

The implication (2) $\Rightarrow$ (1) is obtained by taking $\phi = A\tilde{k}_{\zeta}^{(c)}$ in part (b) of assertion (2). We next prove the implication (1) $\Rightarrow$ (2). We first point out that part (a) of assertion (2) is valid for every bounded subset $\mathcal{F}$ of L_a^1. In fact, the closed subdisk $\omega = \{z \in \Delta : |z| \leq \frac{1+R}{2}\}$ is a compact subset of Δ and hence on this set, the Bergman distance β on Δ is equivalent to the Euclidean distance. On the other hand, it is well known (cf. e.g. [2], Proposition 5.5, page 67) that, if ϕ analytic on Δ, and $z, \zeta \in \Delta$ such that $\beta(z,\zeta) < \delta$, for some $\delta \in (0,1)$, then

$$|\phi(z) - \phi(\zeta)| \leq C\delta \int_{\beta(z,w)<1} |\phi(w)| \frac{d\lambda(w)}{(1-|w|^2)^2}.$$

We recall that the measure $\frac{d\lambda(w)}{(1-|w|^2)^2}$ is invariant under automorphisms of Δ. On ω, there exist two constants A and B such that $A|z-\zeta| \leq \beta(z,\zeta) \leq B|z-\zeta|$ for all $z,\zeta \in \omega$. We suppose that $\delta < \frac{A(1-R)}{2}$. Now, for all $h \in \mathbb{C}$ such that $|h| < \frac{\delta}{A}$ and all $z \in \mathbb{C}$ such that $|z| < R$, it is easy to check that z and $z+h$ both lie in ω. Moreover, if ϕ is analytic on Δ then for every $h \in \mathbb{C}$ such that $|h| < \frac{\delta}{B}$ and every z such that $|z| < R$, we have

$$|\phi(z+h) - \phi(z)| \leq C(R)\delta||\phi||_1.$$

We set $C = \sup_{\phi \in \mathcal{F}} ||\phi||_1$. Then

$$\int_{|z|<R} |\phi(z+h) - \phi(z)| d\lambda(z) \leq CC(R)\delta R^2.$$

Part (a) of assertion (2) follows when we take $\delta < \frac{\varepsilon}{CC(R)R^2}$.

We next prove that assertion (1) implies part (b) of assertion (2). Let $\phi = Ag \in \mathcal{F}$. By the atomic decomposition theorem (cf. e.g. Theorem 2.30 of [14]), for every $g \in L_a^1$, there exists a sequence $\{c_k\}$ of complex numbers belonging to the sequence space l^1 such that

$$g(z) = \sum_{k=1}^{\infty} c_k \tilde{k}_{a_k}^{(c)}(z) \qquad (z \in \Delta).$$

This series converges to g in the norm topology of L_a^1. Moreover, there exists a constant C such that for every $g \in L_a^1$, we have

$$\sum_{k=1}^{\infty} |c_k| \leq C||g||_1.$$

Here, the sequence $\{a_k\}$ is again an $r-$lattice as in Theorem 2.6. Since A is bounded on L_a^1, we get

$$\begin{aligned}
\int_{R\leq|z|<1} |Ag(\zeta)| d\lambda(\zeta) &= \int_{R\leq|z|<1} |A(\sum_{k=1}^{\infty} c_k \tilde{k}_{a_k}^{(c)})(\zeta)| d\lambda(\zeta) \\
&= \int_{R\leq|z|<1} |\sum_{k=1}^{\infty} c_k A(\tilde{k}_{a_k}^{(c)})(\zeta)| d\lambda(\zeta) \\
&\leq \int_{R\leq|z|<1} \sum_{k=1}^{\infty} |c_k||A(\tilde{k}_{a_k}^{(c)})(\zeta)| d\lambda(\zeta) \\
&= \sum_{k=1}^{\infty} |c_k| \int_{R\leq|z|<1} |A(\tilde{k}_{a_k}^{(c)})(\zeta)| d\lambda(\zeta).
\end{aligned}$$

Assertion (1) implies that

$$\int_{R\leq|z|<1}|Ag(\zeta)|d\lambda(\zeta)\leq\varepsilon\sum_{k=1}^{\infty}|c_k|\leq C\varepsilon||g||_1\leq C\varepsilon,$$

because $||g||_1\leq 1$. □

To prove Theorem 1.6 We will make use of the following well known formula for functions in $L^2_a(\Delta)$ (for example, see [10], Lemma 2.2), which we state below.

Lemma 3.2. *If F and G are in $L^2_a(\Delta)$ then*

$$\begin{aligned}\langle F,G\rangle &= 3\int_{\Delta}(1-|z|^2)^2F(z)\overline{G(z)}\,d\lambda(z)+(1/2)\int_{\Delta}(1-|z|^2)^2F'(z)\overline{G'(z)}\,d\lambda(z)\\ &+ (1/3)\int_{\Delta}(1-|z|^2)^3F'(z)\overline{G'(z)}\,d\lambda(z).\end{aligned}$$

3.2 Proof of Theorem 1.6

Let $r\in(0,1)$. Let $c>0$ and $\xi\in\Delta$.

First step. We will first prove that for fixed r,

$$A(\xi,r):=\int_{|z|<r}|T_\mu\tilde{k}^{(c)}_\xi(z)|d\lambda(z)\longrightarrow 0 \text{ as } |\xi|\longrightarrow 1. \tag{3.1}$$

We have

$$A(\xi,r)=\int_{|z|<r}\left|\int_{\Delta}\frac{K_w(z)(1-|\xi|^2)^c}{(1-\overline{\xi}w)^{2+c}}\,d\mu(w)\right|d\lambda(z). \tag{3.2}$$

We first study the inner integral. We observe that

$$\overline{\int_{\Delta}\frac{K_w(z)}{(1-\overline{\xi}w)^{2+c}}\,d\mu(w)}=\frac{1}{1+c}\langle T_{\overline{\mu}}K_z,\tilde{K}^{(c)}_\xi\rangle,$$

where $\tilde{K}^{(c)}_\xi(w)=\frac{1+c}{(1-\overline{\xi}w)^{2+c}}$. Lemma 3.2 implies

$$\langle T_{\overline{\mu}}K_z,\tilde{K}^{(c)}_\xi\rangle=J_1+J_2+J_3$$

where

$$\begin{aligned}J_1 &= \frac{3}{1+c}\int_{\Delta}(1-|w|^2)^2T_{\overline{\mu}}K_z(w)\overline{\tilde{K}^{(c)}_\xi(w)}\,d\lambda(w)\\ J_2 &= \frac{1}{2(1+c)}\int_{\Delta}(1-|w|^2)^2(T_{\overline{\mu}}K_z)'(w)\overline{(\tilde{K}^{(c)}_\xi)'(w)}\,d\lambda(w)\\ J_3 &= \frac{1}{3(1+c)}\int_{\Delta}(1-|w|^2)^3(T_{\overline{\mu}}K_z)'(w)\overline{(\tilde{K}^{(c)}_\xi)'(w)}\,d\lambda(w).\end{aligned}$$

Now, since T_μ is bounded on L^1_a, Lemma 2.17 implies that there exists a constant $C(r)$ such that

$$\sup_{|z|<r}||P(K_z\overline{\mu})||_{LB}\leq C(r)(||T_\mu||+\sup_{z\in r\Delta}Carl(R(K_z\overline{\mu}))). \tag{3.3}$$

Estimates of J_1. We have

$$\begin{aligned}|J_1| &\le 3\int_\Delta (1-|w|^2)^2|P(K_z\bar{\mu})(w)|\frac{1}{|1-\bar{\xi}w|^{2+c}}\,d\lambda(w)\\ &\le 3\{\int_\Delta (1-|w|^2)^2|P(K_z\bar{\mu})(w)-P(K_z\bar{\mu})(0)|\frac{1}{|1-\bar{\xi}w|^{2+c}}d\lambda(w)\\ &\qquad\qquad +|P(K_z\bar{\mu})(0)|\int_\Delta \frac{(1-|w|^2)^2}{|1-\bar{\xi}w|^{2+c}}d\lambda(w)\}\\ &\le C\|P(K_z\bar{\mu})\|_{B^\infty}\left\{\int_\Delta \frac{(1-|w|^2)^2\beta(0,w)}{|1-\bar{\xi}w|^{2+c}}\,d\lambda(w)+\int_\Delta \frac{(1-|w|^2)^2}{|1-\bar{\xi}w|^{2+c}}d\lambda(w)\right\}.\end{aligned}$$

It is easy to check that for every $\nu>0$, there exists a constant $C(\nu)$ such that

$$\beta(0,w)\le C(\nu)(1-|w|^2)^{-\nu},\ w\in\Delta.$$

Hence,

$$|J_1|\le C(\nu)||P(K_z\bar{\mu})||_{B^\infty}\int_\Delta \frac{(1-|w|^2)^{2-\nu}}{|1-\bar{\xi}w|^{2+c}}d\lambda(w).$$

Since $||g||_{B^\infty}\le \frac{||g||_{LB}}{log2}$ for every $g\in B^\infty$, we obtain by (3.3) that

$$\begin{aligned}|J_1| &\le C'(\nu)||P(K_z\bar{\mu})||_{LB}\int_\Delta \frac{(1-|w|^2)^{2-\nu}}{|1-\bar{\xi}w|^{2+c}}d\lambda(w)\\ &\le C(r,\nu)(||T_\mu||+\sup_{z\in r\Delta}Carl(R(K_z\bar{\mu})))\int_\Delta \frac{(1-|w|^2)^{2-\nu}}{|1-\bar{\xi}w|^{2+c}}d\lambda(w).\end{aligned}$$

Applying Lemma 2.4, we have the following conclusion for $|J_1|$:

(1) If $c<2$, we take ν such that $\nu<2-c$ and get

$$|J_1|\le C(r,\nu)(||T_\mu||+\sup_{z\in r\Delta}Carl(R(K_z\bar{\mu})));$$

1. If $c=2$, we take $\nu\in(0,1)$ and get

$$|J_1|\le C(r,\nu)(||T_\mu||+\sup_{z\in r\Delta}Carl(R(K_z\bar{\mu})))\frac{1}{(1-|\xi|^2)^\nu};$$

(3) If $c>2$, we take $\nu\in(0,1)$ and get

$$|J_1|\le C(r,\nu)(||T_\mu||+\sup_{z\in r\Delta}Carl(R(K_z\bar{\mu})))\frac{1}{(1-|\xi|^2)^{c-2+\nu}}.$$

Estimates for J_2 and J_3. Also, using (3.3) we have

$$\begin{aligned} 2|J_2| &\leq \frac{1}{1+c}\int_\Delta (1-|w|^2)^2|(P(K_z\bar{\mu})'(w))||(\tilde{K}_\xi^c)'(w)|d\lambda(w) \\ &\leq (2+c)\int_\Delta |(P(K_z\bar{\mu})'(w))|\log\left(\frac{2}{1-|w|^2}\right)\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{(1-|w|^2)^2}{|1-\bar{\xi}w|^{3+c}}d\lambda(w) \\ &\leq (2+c)\|P(K_z\bar{\mu})\|_{LB}\int_\Delta \frac{1}{\log(\frac{2}{1-|w|^2})}\frac{(1-|w|^2)}{|1-\bar{\xi}w|^{3+c}}d\lambda(w) \\ &\leq (2+c)C(r)(||T_\mu||+\sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})))\int_\Delta \frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1-|w|^2}{|1-\bar{\xi}w|^{3+c}}d\lambda(w). \end{aligned}$$

Given $\varepsilon>0$, there exists $s\in(0,1)$ such that $\frac{1}{\varepsilon}<log(\frac{2}{1-|w|^2})$ whenever $s<|w|<1$. We fix such an s. Then

$$\begin{aligned} \int_\Delta \frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1-|w|^2}{|1-\bar{\xi}w|^{3+c}}d\lambda(w) &= \left\{\int_{s\bar{\Delta}}+\int_{\Delta\setminus s\bar{\Delta}}\right\}\frac{1}{\log(\frac{2}{1-|w|^2})}\frac{1-|w|^2}{|1-\bar{\xi}w|^{3+c}}d\lambda(w) \\ &\leq C_s+\frac{C\varepsilon}{(1-|\xi|^2)^c} \end{aligned}$$

with $C_s=(log2)^{-1}(1-s)^{-3-c}$. This implies,

$$|J_2|\leq\frac{1}{2}(2+c)C(r)(||T_\mu||+\sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})))\{C_s+\frac{C\varepsilon}{(1-|\xi|^2)^c}\}.$$

In a similar manner, we obtain

$$|J_3|\leq\frac{1}{3}(2+c)C(r)(||T_\mu||+\sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})))\{C_s+\frac{C\varepsilon}{(1-|\xi|^2)^c}\}.$$

Conclusion. Since

$$A(\xi,r)\leq(|J_1|+|J_2|+|J_3|)(1-|\xi|)^c,$$

given $\varepsilon>0$, we can fix $s\in(0,1)$ such that

(1) if $c<2$, then for ν positive such that $\nu<2-c$, we have

$$A(\xi,r)\leq C(c,r,\nu)(||T_\mu||+\sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})))[1+C_s+\frac{C\varepsilon}{(1-|\xi|^2)^c}](1-|\xi|)^c;$$

(2) if $c=2$, then for $\nu\in(0,1)$, we have

$$A(\xi,r)\leq C(c,r,\nu)(||T_\mu||+\sup_{z\in r\Delta} Carl(R(K_z\bar{\mu}))) \left[\frac{1}{(1-|\xi|^2)^\nu}+C_s+\frac{C\varepsilon}{(1-|\xi|^2)^c}\right](1-|\xi|)^c;$$

(3) if $c > 2$, then for $\nu \in (0,1)$, we have

$$A(\xi,r) \leq C(c,r,\nu)(||T_\mu|| + \sup_{z\in r\Delta} Carl(R(K_z\bar{\mu})))$$
$$\left[\frac{1}{(1-|\xi|^2)^{c-2+\nu}} + C_s + \frac{C\varepsilon}{(1-|\xi|^2)^c}\right](1-|\xi|)^c.$$

Combining these estimates we have $A(\xi,r) \longrightarrow 0$ when $|\xi| \to 1^-$. This gives (3.1).

Second step. Now, take $\psi(r) = 1$, $\phi(r) = (1-r^2)^c$, $c > 0$. Then by Theorem 1.3, it suffices to prove that $A_0(\phi)$ is an invariant subspace of the adjoint operator $T_\mu^\star$ of T_μ with respect to the duality pairing $[,]$ defined in (1.3). We just suppose T_μ is bounded on L_a^1. Then $T_\mu^\star$ is bounded on $A_\infty(\phi)$. Since the weighted Bergman kernel $\tilde{K}_\xi^{(c)}(z) = \frac{1+c}{(1-\bar{\xi}z)^{2+c}}$ reproduces $A_\infty(\varphi)-$functions in the sense that for every $h \in A_\infty(\phi)$,

$$h(\xi) = [h, \tilde{K}_\xi^{(c)}], \qquad (\xi \in \Delta).$$

We have that, for every $h \in A_\infty(\phi)$ and for every $\xi \in \Delta$,

$$\begin{aligned} T_\mu^\star h(\xi) &= [T_\mu^\star h, \tilde{K}_\xi^{(c)}] = [h, T_\mu \tilde{K}_\xi^{(c)}] \\ &= (1+c)\int_\Delta \overline{\left(\int_\Delta \frac{K_w(z)}{(1-\bar{\xi}w)^{2+c}} d\mu(w)\right)} h(z)(1-|z|^2)^c d\lambda(z). \end{aligned}$$

We need to show that $T_\mu^\star h \in A_0(\phi)$ if $h \in A_0(\varphi)$. We fix $\varepsilon > 0$ arbitrary. There exists $r = r(\varepsilon) \in (0,1)$ such that

$$(1-|z|^2)^c|h(z)| < \varepsilon \quad \text{whenever} \quad r < |z| < 1. \tag{3.4}$$

We write

$$\frac{1}{1+c} T_\mu^\star h(\xi)(1-|\xi|^2)^c = I + II$$

where

$$\begin{aligned} I &= \int_{r\leq|z|<1} \overline{\left(\int_\Delta \frac{K_w(z)(1-|\xi|^2)^c}{(1-\bar{\xi}w)^{2+c}} d\mu(w)\right)} h(z)(1-|z|^2)^c d\lambda(z) \\ &= \int_{r\leq|z|<1} \overline{T_\mu \tilde{k}_\xi^c(z)} h(z)(1-|z|^2)^c d\lambda(z) \end{aligned}$$

and

$$II = \int_{|z|<r} \overline{\left(\int_\Delta \frac{K_w(z)(1-|\xi|^2)^c}{(1-\bar{\xi}w)^{2+c}} d\mu(w)\right)} h(z)(1-|z|^2)^c d\lambda(z). \tag{3.5}$$

Concerning I, we deduce from (3.4) that

$$|I| \leq \int_{r\leq|z|<1} |T_\mu \tilde{k}_\xi^c(z)||h(z)|(1-|z|^2)^c d\lambda(z) \leq C\varepsilon, \tag{3.6}$$

with $C = \sup_{\xi\in\Delta} \|T_\mu \tilde{k}_\xi^c\|_1 < \infty$, since T_μ is bounded on L_a^1.

Now for II, we observe that

$$|II| \leq A(\xi, r)||h||_{A_\infty(\phi)}$$

Combining these estimates when $|\xi| \to 1^-$, with (3.6) and (3.1) easily implies the desired conclusion. □

Remark 3.3. If $\sup_{\xi\in\Delta}\|T_\mu \tilde{k}_\xi^c\|_1 < \infty$, it follows from the proof of Theorem 1.6, that the following two assertions are equivalent.

1. $A_0(\phi)$ is an invariant subspace of the adjoint operator $T_\mu^\star$ of T_μ with respect to the duality pairing $[,]$ defined in (1.3).
2. For fixed $r \in (0,1)$, the following estimate holds.

$$A(\xi, r) := \int_{|z|<r} |T_\mu \tilde{k}_\xi^{(c)}(z)| d\lambda(z) \longrightarrow 0 \text{ as } |\xi| \longrightarrow 1.$$

Corollary 3.4. *Let μ be a positive measure on Δ such that the Toeplitz operator T_μ is bounded on L^1_a and let $c > 0$. The following assertions are equivalent:*

(1) The Toeplitz operator T_μ is compact on L^1_a;

(2) $\|T_{\bar{f}} \tilde{k}_\zeta^{(1)}\|_1 \to 0$ as $\zeta \to \partial\Delta$;

(3) For every $\varepsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R\leq|z|<1} |(T_\mu \tilde{k}_\zeta^{(c)})(z)| d\lambda(z) < \varepsilon$$

for every $\zeta \in \Delta$.

Proof. The Toeplitz operator T_μ is bounded on L^1_a. It follows from observation after Lemma 2.12 that μ is a Carleson measure for Bergman spaces. Thus, the proof of the equivalence $(1) \Leftrightarrow (2)$ follows from a direct application of Theorem 1.6. The equivalence $(1) \Leftrightarrow (3)$ is a direct application of Theorem 1.5. □

Corollary 3.5. *Let $f \in L^1_a$ be such that $T_{\bar{f}}$ is a bounded operator on L^1_a. Then the following assertions are equivalent:*

(1) The Toeplitz operator $T_{\bar{f}}$ is compact on L^1_a;

(2) $\|T_{\bar{f}} \tilde{k}_\xi^{(c)}\|_1 \to 0$ as $\xi \to \partial\Delta$ for every $c > 0$;

(3) For every $c > 0$ and for every $\varepsilon > 0$, there exists $R \in (0,1)$ such that

$$\int_{R\leq|z|<1} |(T_{\bar{f}} \tilde{k}_\zeta^{(c)})(z)| d\lambda(z) < \varepsilon$$

for every $\zeta \in \Delta$.

(4) $\|T_{\bar{f}} \tilde{k}_\xi^{(1)}\|_1 \to 0$ as $\xi \to \partial\Delta$;

(5) f vanishes identically.

Let us mention that, using duality, property (1) is equivalent to the property "f is a compact multiplier of B^∞". The latter was shown in [7] to be equivalent to property (5).

Proof. The proof goes along the following implications: $(1) \Leftrightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2)$ and $(1) \Leftrightarrow (3)$. From Theorem 2.16, $T_{\overline{f}}$ bounded on L_a^1 implies that f is bounded. Hence we can apply Theorem 1.6, thus we have $(1) \Leftrightarrow (2)$. Theorem 1.5 gives $(1) \Leftrightarrow (3)$. Taking $c = 1$, we have $(2) \Rightarrow (4)$. Suppose (4) holds, using (2.9) and Lemma 2.7, we have, taking $z = 0$, that

$$(1-|a|^2)|\mathcal{D}f(a)| \longrightarrow 0 \quad \text{as} \quad |a| \longrightarrow 1. \tag{3.7}$$

On the other hand, observing that in this case $\mathcal{D} = I + \frac{1}{2}z\frac{\partial}{\partial z}$ (where I is the identity), we have for some absolute constant C and for all $a, z \in \Delta$

$$\left|\frac{f(a)}{(1-\bar{z}a)^4}\right| \leq C\left|\mathcal{D}\left\{\frac{f}{(1-\bar{z}\,\cdot)^3}\right\}(a)\right| + C\left|\frac{\mathcal{D}f(a)}{(1-\bar{z}a)^3}\right|. \tag{3.8}$$

Multiplying (3.8) by $(1-|a|^2)(1-|z|^2)^3$ and then using again (2.9), Lemma 2.7 and (3.7), we have, taking $z = a$, that $|f(a)| \longrightarrow 0$ as $|a|$ tends to 1. Hence (5) holds. The implication $(5) \Rightarrow (2)$ is obvious. This finishes the proof of the Corollary. □

Remark 3.6. For $f \in L_a^1$, let us compare Theorem 1.2 and Corollary 3.5. Looking at the hypotheses of Theorem 1.2, it follows from an application of Cauchy's integral formula that

$$R(f)(w) = -\frac{1}{w}\{f(0)(1-|w|^2) - f(w)\}.$$

The property "$|R(f)|d\lambda$ is a vanishing Carleson measure for Bergman spaces" is actually equivalent to the property "f vanishes identically". So for Toeplitz operators with antianalytic symbols, Theorem 1.2 does not bring new information while Corollary 3.5 does. This shows again that we have improved Theorem 1.2.

4 The case of radial symbols

In this section, we are interested in the case of Toeplitz operators associated with radial symbols $f(w) = f(|w|)$. We get the following proposition:

Proposition 4.1. *Let f be an integrable radial function on Δ. Then $R(f)$ is given by*

$$R(f)(w) = \frac{2\bar{w}(2-|w|^2)}{1-|w|^2}\int_{|w|}^1 f(r)rdr - \frac{2(1-|w|^2)}{w}\int_0^{|w|} f(r)rdr.$$

Moreover, the associated Toeplitz operator T_f is bounded (respectively compact) on L_a^1 if $|R(f)|d\lambda$ is a Carleson measure (resp. a vanishing Carleson measure) for Bergman spaces.

Proof. In this case, the Bergman projection Pf of f is constant and identically equal to $\int_\Delta f(\rho)\rho d\rho$. So the second assertion is a consequence of Theorem 1.1 (resp. Theorem 1.2).

We give the proof of the announced expression of $R(f)$. First,

$$\begin{aligned} R(f)(w) &= \tfrac{1-|w|^2}{\pi}\int_\Delta \tfrac{f(r)}{(re^{i\theta}-w)(1-re^{i\theta}\bar{w})^2}rdrd\theta \\ &= \tfrac{1-|w|^2}{\pi i}\int_0^1 f(r)\{\int_{|z|=r}\tfrac{1}{(z-w)z(1-z\bar{w})^2}dz\}rdr. \end{aligned}$$

Next, since $\frac{1}{(z-w)z} = -\frac{1}{w}(\frac{1}{z} - \frac{1}{z-w})$ and since the function $z \mapsto \frac{1}{(1-z\bar{w})^2}$ is analytic on Δ, an application of the Cauchy integral formula gives

$$\begin{aligned} \int_{|z|=r} \frac{1}{(z-w)z(1-z\bar{w})^2} dz &= -\frac{1}{w}\{\int_{|z|=r} \frac{1}{z}\frac{1}{(1-z\bar{w})^2} dz - \int_{|z|=r} \frac{1}{z-w}\frac{1}{(1-z\bar{w})^2} dz\} \\ &= \begin{cases} -\frac{2\pi i}{w}\{1 - \frac{1}{(1-|w|^2)^2}\} \quad = \frac{2\pi i \bar{w}(2-|w|^2)}{(1-|w|^2)^2} & \text{if} \quad |w| < r \\ -\frac{2\pi i}{w} & \text{if} \quad |w| > r \end{cases}. \end{aligned}$$

Finally, we obtain

$$R(f)(w) = \frac{2\bar{w}(2-|w|^2)}{1-|w|^2}\int_{|w|}^{1} f(r)rdr - \frac{2(1-|w|^2)}{w}\int_0^{|w|} f(r)rdr.$$

□

For radial symbols f, we also give the following expressions of $T_f\tilde{k}_z^{(c)}$ and $T_f g$, where $c > 0$, $z \in \Delta$ and $g \in L_a^\infty$.

Lemma 4.2. *Let f be an integrable radial function on Δ. Then*

$$T_f g(\zeta) = 2\int_0^1 f(\rho)\{g(\zeta\rho^2) + \zeta\rho^2 g'(\zeta\rho^2)\}\rho d\rho \qquad (\zeta \in \Delta),$$

for every $g \in L_a^\infty$. In particular, for all $c > 0$ and $z \in \Delta$, we have

$$T_f\tilde{k}_z^{(c)}(\zeta) = 2(1-|z|^2)^c \int_0^1 f(\rho)\{\frac{1}{(1-\zeta\rho^2\bar{z})^{2+c}} + \frac{2+c}{(1-\zeta\rho^2\bar{z})^{3+c}}\}\rho d\rho \qquad (\zeta \in \Delta).$$

Proof. We start with the formula,

$$T_f g(\zeta) = \frac{1}{\pi}\int_0^1 f(\rho)(\int_0^{2\pi} \frac{g(\rho e^{i\phi})}{(1-\zeta\rho e^{-i\phi})^2} d\phi)\rho d\rho.$$

We denote by $I(r,\zeta)$ the inner integral. Then

$$\begin{aligned} I(r,\zeta) &= \frac{1}{i}\int_{|w|=\rho} \frac{g(w)}{(1-\frac{\zeta\rho^2}{w})^2}\frac{dw}{w} \\ &= \frac{1}{i}\int_{|w|=\rho} \frac{wg(w)}{(w-\zeta\rho^2)^2} dw \\ &= 2\pi[wg(w)]'|_{w=\zeta\rho^2} = 2\pi\{g(\zeta\rho^2) + \zeta\rho^2 g'(\zeta\rho^2)\}. \end{aligned}$$

For the latest but one equality, we applied the Cauchy integral to the analytic function $wg(w)$ with the observation that $|\zeta\rho^2| < \rho$. The desired conclusion for $T_f g(\zeta)$ follows at once. We deduce the expression of $T_f\tilde{k}_z^{(c)}(\zeta)$ as the particular case where $g = \tilde{k}_z^{(c)}(\zeta)$. □

Theorem 1.4 can be expressed in the following explicit form for radial symbols.

Corollary 4.3. *Let f be an integrable radial function on Δ and let $c > 0$. Then the following two properties are equivalent:*

1. *The Toeplitz operator T_f is bounded on L_a^1;*

2. *The following estimate holds:*

$$\sup_{z\in\Delta}(1-|z|^2)^c\int_\Delta|\int_0^1 f(\rho)\{\frac{1}{(1-\zeta\rho^2\bar{z})^{2+c}}+\frac{2+c}{(1-\zeta\rho^2\bar{z})^{3+c}}\}\rho d\rho|d\lambda(\zeta)<\infty.$$

We also characterise compactness with radial symbols.

Theorem 4.4. *Let f be an integrable radial function on Δ and let $c>0$. Then the following two properties are equivalent:*

(1) The Toeplitz operator T_f is compact on L_a^1;

(2) The following estimate holds:

$$\lim_{z\to\partial\Delta}(1-|z|^2)^c\int_\Delta|\int_0^1 f(\rho)\{\frac{1}{(1-\zeta\rho^2\bar{z})^{2+c}}+\frac{2+c}{(1-\zeta\rho^2\bar{z})^{3+c}}\}\rho d\rho|d\lambda(\zeta)=0.$$

Proof. From Remark 3.3 and Theorem 1.3, we observe that all we have to show is that for fixed $r\in(0,1)$,

$$A(\xi,r)\to 0 \text{ as } \xi\to\partial\Delta,$$

where $A(\xi,r)$ is given by (3.2), that is

$$A(\xi,r)=\int_{|z|<r}\left|\int_\Delta\frac{f(w)K_w(z)(1-|\xi|^2)^c}{(1-\bar{\xi}w)^{2+c}}\lambda(w)\right|d\lambda(z).$$

We study the inner integral when f is radial.

$$\int_\Delta\overline{\frac{f(w)K_w(z)(1-|\xi|^2)^c}{(1-\bar{\xi}w)^{2+c}}}d\lambda(w)=$$
$$\frac{(1-|\xi|^2)^c}{2\pi}\int_0^1\overline{f}(\rho)\left(\int_0^{2\pi}\frac{1}{(1-\bar{z}\rho e^{i\theta})^2}\frac{1}{(1-\xi\rho e^{-i\theta})^{2+c}}d\theta\right)\rho d\rho.$$

We call $I(\rho)$ the integral with respect to $d\theta$. Then

$$I(\rho)=\sum_{m=0}^\infty\frac{(m+1)\Gamma(m+2+c)}{\Gamma(2+c)\Gamma(m+1)}(\bar{z}\zeta\rho^2)^m.$$

This implies,

$$\begin{aligned}
|I(\rho)| &\leq \sum_{m=0}^\infty\frac{(m+1)\Gamma(m+2+c)}{\Gamma(2+c)\Gamma(m+1)}|\bar{z}\zeta\rho^2|^m\\
&\leq \sum_{m=0}^\infty\frac{(m+1)\Gamma(m+2+c)}{\Gamma(2+c)\Gamma(m+1)}|z|^m\\
&= 1+\sum_{m=1}^\infty\frac{(m+1)\Gamma(m+2+c)}{\Gamma(2+c)m\Gamma(m)}|z|^m\\
&\leq 1+2\sum_{m=1}^\infty\frac{\Gamma(m+2+c)}{\Gamma(2+c)\Gamma(m)}|z|^m\\
&= 1+2|z|\sum_{n=0}^\infty\frac{\Gamma(n+3+c)}{\Gamma(2+c)\Gamma(n+1)}|z|^n=1+\frac{2|z|}{(1-|z|)^{3+c}}\frac{\Gamma(3+c)}{\Gamma(2+c)}.
\end{aligned}$$

So there exists a constant $C(r)$ such that

$$\begin{aligned} A(\xi, r) &\leq C(r)(1-|\xi|^2)^c \int_0^1 |f(\rho)|\rho d\rho \\ &\leq C'(r)(1-|\xi|^2)^c. \end{aligned}$$

This shows that $A(\xi, r) \to 0$ as $\xi \to \partial\Delta$. □

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