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VOLUME AND ENERGY OF REEB VECTOR FIELDS

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Abstract

This paper contains a characterization of Reeb vector fields of K-contact forms in terms of J-holomorphic embeddings into the tangent unit sphere bundle. A consequence of this characterization is that these vector fields are critical points of a volume and an energy functionals defined on the set of unit vector fields. Reeb vector fields on closed, K-contact Einstein manifolds are absolute minimizers for the energy functional with a mean curvature correction. On odd-dimensional Einstein manifolds of positive sectional curvature, these unit vector fields are characterized by their minimizing property. It is also proved that any closed flat contact manifold admits a parallelization by three critical unit vector fields, one parallel (hence minimizing), the other two are Reeb vector fields of contact forms, not Killing and not minimizers of any of the volume or the energy functionals.

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1 Introduction

Many interesting geometric objects are characterized by their minimizing properties for appropriate functionals. For example, Einstein metrics are global minimizers of the total scalar curvature functional. Those minimizing objets often display great stability and high degree of organization. In this paper, we deal with volume and energy functionals which are defined on the space of unit vector fields on a riemannian manifold M. Looking at Reeb vector fields as maps from contact metric manifolds to their tangent unit sphere bundles endowed with the usual Sasaki metric, we prove the following:

Theorem A *The Reeb vector field of a contact form* α *is a J-holomorphic map if and only if* α *is a K-contact form (that is, the Reeb vector field is Killing).*

From an isometric embedding point of view, that is, considering the induced metric instead of the original contact metric on M, one has:

Theorem A' *The Reeb vector field of a K-contact form* α *is a contact invariant minimal embedding.*

A modified energy functional will also be considered and it will be shown here that Reeb vector fields of sasakian-Einstein structures are its absolute minimizers. The paper

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concludes with some examples of non-Killing critical unit vector fields arising from the flat contact metric geometry. The author would like to point out that Gil-Medrano's paper [6] contains a unified study of the energy and the volume functionals. As it turns out, each of the two functionals is the restriction of a general functional defined on the cross product of the space of all unit vector fields with that of all riemannian metrics on M.

2 Some contact metric geometry

A contact form on a $2n+1$-dimensional manifold M is a 1-form α such that the identity $\alpha \wedge (d\alpha)^n \neq 0$ holds everywhere on M. Given such a 1-form α, there is always a unique vector field Z determined by the system of partial differential equations $\alpha(Z) = 1$ and $d\alpha(Z,.) = 0$. This Z is called the characteristic vector field and the corresponding 1-dimensional foliation is called a contact flow. The $2n$-dimensional distribution

$$D(p) = \{X \in T_pM : \ \alpha(X) = 0\}$$

is called the contact distribution. On every contact manifold (M,α), there is a nonunique metric g and a $(1,1)$ type tensor field J such that Z and D are orthogonal, $JZ = 0$ and $-J^2$ is the identity when restricted to D. Moreover, the convention being that the exterior derivative of a 1-form α is given by

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]),$$

the tensors α, Z, g, and J are compatible in the sense that the following identities are satisfied.

$$d\alpha(X,Y) = 2g(X,JY), \quad \alpha(X) = g(Z,X)$$

for any tangent vectors X and Y. We will call the data (M,α,Z,g,J) a contact metric structure. The following identities on a contact metric structure are well known (see [1]). They involve the symmetric tensor $h = \frac{1}{2}L_Z J$ (L_Z is the usual Lie derivative of the tensor field J in the direction of the vector field Z), the Levi-Civita covariant derivative operator ∇ and the riemannian curvature tensor R.

$$\nabla_X Z = -JX - JhX \tag{2.1}$$

$$\frac{1}{2}(R(Z,X)Z - JR(Z,JX)Z) = h^2X + J^2X. \tag{2.2}$$

The derivation of these identities and many more may be found in [1].

Definition 2.1. A contact form α on M is said to be K-contact if there exists a contact metric structure (M,α,Z,g,J) where the Reeb vector field Z is Killing with respect to the metric g.

It is well known that this Killing condition is equivalent to the the tensor field h being identically zero (see [1]). When the identity

$$(\nabla_X J)Y = g(X,Y)Z - \alpha(Y)X$$

is satisfied on a contact metric structure, the structure is said to be sasakian, the contact form α is then called a sasakian form. Any sasakian structure is K-contact, but it is only in dimension 3 that any K-contact structure is automatically sasakian. In this paper we are concerned with closed Einstein manifolds; those riemannian manifolds whose metric g and Ricci tensor satisfies the identity

$$Ricci(X,Y) = Cg(X,Y)$$

for some constant C which is necessarily equal to $2n$ on a K-contact 2n+1-dimensional manifold.

3 The volume functional

Let us denote by $\mathcal{X}^1(M)$ the space of unit vector fields on a closed riemannian manifold (M,g) and by T^1M the tangent unit sphere bundle. We will denote by $\pi_*\colon TTM \to TM$ and $\kappa\colon TTM \to TM$ the projection and the Levi-Civita connection maps respectively. The volume $\mathcal{F}(V)$ of an element $V \in \mathcal{X}^1(M)$ is defined to be the volume of the submanifold $V(M) \subset T^1M$ where T^1M is endowed with the restriction of the usual Sasaki metric g_s (see reference [3]). We recall that g_s is defined by:

$$g_s(X,Y) = g(\pi_*X, \pi_*Y) + g(\kappa X, \kappa Y).$$

The metric g_V induced on M by the map $V\colon M \to T^1M$ is related to the metric g on M by the identity (see [7], [5])

$$g_V(X,Y) = g(\mathcal{L}_V X, Y),$$

where $\mathcal{L}_V = Id + (\nabla V)^t \circ \nabla V$ is a g-symmetric (1,1) tensor field ($(\nabla V)^t$ stands for the transpose of the operator ∇V).

The problem of determining the condition for an element of $\mathcal{X}^1(M)$ to be a critical point of the volume functional was tackled in [5] and [7] where necessary and sufficient conditions for criticality were derived for Killing vector fields. A geometric interpretation of these conditions is yet to be found. In this paper, we will exhibit some contact, non-Killing critical vector fields arising from the flat contact metric geometry setting. However, our critical non-Killing vector fields are not absolute minimizers for the volume.

Let $f(V) = \sqrt{det\ \mathcal{L}_V}$ where $det\ \mathcal{L}_V$ stands for the determinent of $\mathcal{L}_V$ and let Ω denotes the riemannian volume element on M. Then the volume functional on the space of unit vector fields is given by the expression

$$\mathcal{F}(V) = \int_M f(V)\Omega. \tag{3.1}$$

Identifying the tangent space to $\mathcal{X}^1(M)$ at V with the space of vector fields perpendicular to V, one can compute the first variation of $\mathcal{F}$ as follows. For an arbitrary vector field $A \in T_V\mathcal{X}^1(M)$, that is, for arbitrary A perpendicular to V, let $V(t)$ be a smooth path in $\mathcal{X}^1(M)$ such that $V(0) = V$ and $V'(0) = A$. Denoting $\mathcal{L}_{V(t)}$ simply by $\mathcal{L}(t)$, we observe that

$$\mathcal{L}'(t) = (\nabla V'(t))^t \circ \nabla V(t) + (\nabla V(t))^t \circ \nabla V'(t)$$

and therefore, $\mathcal{L}(t)$ satisfies a linear differential equation of the type

$$X'(t) = P(t)X(t), \tag{3.2}$$

with

$$P(t) = [(\nabla V'(t))^t \circ \nabla V(t) + (\nabla V(t))^t \circ \nabla V'(t)] \circ \mathcal{L}^{-1}(t).$$

It follows that

$$\frac{d}{dt} det\mathcal{L}(t) = tr(P(t)) det\mathcal{L}(t). \tag{3.3}$$

Now we can see that

$$(f \circ V)'(t) = \frac{1}{2}(det\mathcal{L}(t))^{-\frac{1}{2}} \frac{d}{dt}(det\mathcal{L}(t)). \tag{3.4}$$

Applying identity (3.3) and evaluating at $t = 0$, we obtain that the differential of $\mathcal{F}$ at V acting on the tangent vector A can be written as (see [5], [7])

$$T_V\mathcal{F}(A) = \int_M f(V) tr(\mathcal{L}_V^{-1} \circ (\nabla V)^t \circ \nabla A)\Omega. \tag{3.5}$$

The notation $tr(.)$ in this paper stands for the trace of an operator.

4 The energy functional

The energy $\mathcal{E}(V)$ of a unit vector field V on a closed n-dimensional riemannian manifold M is defined as the energy in the sense of [4] of the map

$$V\colon (M,g) \to (T^1M, g_s),$$

where g_s denotes also the restriction of the Sasaki metric on the tangent unit sphere bundle. The functional $\mathcal{E}(V)$ is given by the expression (see reference [22])

$$\mathcal{E}(V) = \frac{1}{2}\int_M \|\nabla V\|^2 \,\Omega + \frac{n}{2}\, Volume\ of\ M.$$

The first variation $T_V\mathcal{E}$ and the second variation $Hess_{\mathcal{E}}$ are given by the following formulas. For arbitrary vector fields A and B perpendicular to V,

$$T_V\mathcal{E}(A) = \int_M g(A, \nabla^*\nabla V)\ \Omega$$

$$Hess_{\mathcal{E}}(A,B) = \int_M g(\nabla^*\nabla A - \|\nabla V\|^2 A - R(A,\nabla V)\nabla V, B)\ \Omega,$$

where $-\nabla^*\nabla = tr(\nabla^2)$ is the trace of ∇^2 and R is the Riemann curvature tensor field.

5 The contact metric geometry of the unit tangent line bundle

The Liouville 1-form Θ is defined on T^*M by $\Theta_\mu(v) = \mu(\tau_* v)$ where $v \in T_\mu T^*M$ and $\tau_*\colon TT^*M \to TM$ is the differential of the projection map $\tau\colon T^*M \to M$. The 2-form $\Omega = -d\Theta$ is a symplectic form on T^*M. We refer to [12] for the basics of symplectic geometry. Given a riemannian metric g on M, the fundamental 1-form Θ pulls back to a 1-form $\tilde{\Theta}$ on TM, $\tilde{\Theta} = \flat^*\Theta$, where $\flat\colon TM \to T^*M$ is the usual musical isomorphism determined by the metric g. The 2-form $\tilde{\Omega} = -d\tilde{\Theta} = -\flat^* d\Theta$ is a symplectic form on TM (see [14], pages 246-247). The map

$$(\pi_*, \kappa)\colon TTM \to TM \oplus TM \tag{5.1}$$

is a vector bundle isomorphism along $\pi\colon TM \to M$, which determines an almost complex structure J_{TM} on TM such that, if $(\pi_*, \kappa)(x) = (u, v)$, then $(\pi_*, \kappa)(J_{TM}x) = (-v, u)$ ([3], [20]).

The tangent bundle of a riemannian manifold M carries a distinguished vector field U called the geodesic spray. U is determined by $\pi_* U(p, v) = v$, and $\kappa U(p, v) = 0$ for any $(p, v) \in TM$. With respect to the Sasaki metric g_s, the 1-form $\tilde{\Theta}$ satisfies the identity $\tilde{\Theta}(V) = g_s(U, V)$ for any section V of TTM. That is, $\tilde{\Theta}$ and U are (Sasaki) metric duals. The symplectic form $\tilde{\Omega} = -d\tilde{\Theta}$ is compatible with the pair (g_s, J_{TM}) in the sense that the identity

$$\tilde{\Omega}(X, Y) = g_s(X, J_{TM}Y) \tag{5.2}$$

holds for any pair (X, Y) of tangent vector fields on TM.

Letting $j\colon T^1M \to TM$ be the inclusion of the tangent unit sphere bundle as a hypersurface in TM, then the pulled back 1-form $\tilde{\alpha}_g = j^*\tilde{\Theta}$ is a contact form on T^1M (see reference [1]) whose characteristic vector field is tangent to the geodesic flow of M, its integral curves project to geodesics of M. The kernel of $\tilde{\alpha}_g$ has associated almost complex operator J_{T^1M} determined by the equations

$$J_{T^1M}(U) = 0 \text{ and } J_{T^1M}(X) = J_{TM}X$$

for any X tangent to T^1M satisfying the identity $g_s(X, U) = 0$.

Under the identification (5.1), a tangent vector at $(p, v) \in TM$ is a couple (u, w) where $u \in T_pM$ is nonzero and $w = \nabla_u W$ with W a vector field on M such that $W(p) = v$; or, $u = 0$ and w is just a tangent vector at p. A vector tangent at $(p, v) \in T^1M$ is a couple $(u, \nabla_u W)$ where as above, u is a tangent vector at p or $u = 0$ and $w \perp v$, but now W is a unit vector field such that $W(p) = v$. Note that in this case, one has automatically the identity $g(v, \nabla_u W) = 0$. Now, if (M, α, Z, g, J) is a contact metric structure, a vector tangent to $Z(M) \subset T^1M$ at (p, Z) is a couple $(u, \nabla_u Z)$ where u is a tangent vector at $p \in M$. We also point out that under the identification (5.1), the vector field U on T^1M is given by $U(p, v) = (v, 0)$. From now on, it will be understood that the tangent unit sphere bundle is endowed with the restriction of the Sasaki metric g_s.

6 Reeb vector fields of K-contact forms

In the case of the three-dimensional sphere, the Hopf vector fields are absolute minima for the volume and the energy functionals (see [2] and [8]). However, in higher dimensions, these vector fields are not minima anymore, but they are still critical. Their criticality is a special case of a more general result whose proof is contained in the author's paper [17]. To state it, we need the following definition.

Definition 6.1. A submanifold M in a contact manifold (N,α,J,Z) is said to be contact invariant if the characteristic vector field Z is tangent to M and JX is tangent to M whenever X is.

Clearly, a contact invariant submanifold inherits a contact metric structure from the ambient manifold. In this context, the inclusion of a contact invariant submanifold is a J-holomorphic map in the sense of [9] where such maps are shown to be harmonic.

Theorem 6.2. *Let (α,Z,g,J) be contact metric structure tensors on a manifold M. Then $Z(M)$ is a contact invariant submanifold and $Z\colon M \to T^1M$ is a J-holomorphic embedding if and only if α is a K-contact form.*

Proof. The geodesic spray U is clearly tangent to $Z(M)$. Let $(u,\nabla_u Z)$ be a tangent vector at $(p,Z) \in Z(M)$ such that $G((u,\nabla_u Z),(Z,0)) = 0$; that is, a tangent vector in the kernel of the contact form $\tilde{\alpha}_g$ on T^1M. Then

$$J_{T^1M}(u,\nabla_u Z) = (-\nabla_u Z,u) = (Ju+Jhu,u).$$

Therefore, $J_{T^1M}(u,\nabla_u Z)$ will be tangent to $Z(M)$ if and only if $\nabla_{Ju+Jhu}Z = u$. But it is easily seen that $\nabla_{Ju+Jhu}Z = u-h^2u$, hence $Z(M)$ will be contact invariant if and only if $h=0$, that is, α is a K-contact form.

Also, observe that $Z_*(Z) = (Z,0) = U(p,Z)$, which means that as a map between two contact manifolds, Z exchanges the two Reeb vector fields involved. Let $v \in T_pM$ be a tangent vector such that $\alpha(v) = 0$. On one hand,

$$J_{T^1M}Z_*(v) = J_{T^1M}(\pi_*Z_*(v),\kappa Z_*(v)) = J_{T^1M}(v,\nabla_v Z) = (Jv+Jhv,v).$$

On the other hand,

$$Z_*(Jv) = (Jv,\nabla_{Jv}Z) = (Jv,v-hv).$$

The two identities above show that Z is a J-holomorphic map if and only if $h=0$, that is, α is a K-contact form. □

From Theorem 6.2 above, we derive the following corollary.

Corollary 6.3. *Let (M,α,Z,g,J) be a K-contact structure. Then the Reeb vector field Z is a minimal embedding and a harmonic map into T^1M.*

7 Reeb vector fields of sasakian forms on Einstein Manifolds

It was shown in the previous section that Reeb vector fields of K-contact forms are critical for the energy and volume functionals, but in general they are not minimizing. In this section, we consider the following modified functional introduced by Brito [2].

$$\mathcal{D}(V) = \int_M (\|\nabla V\|^2 + (n-1)(n-3)\|H_{V^\perp}\|^2)\Omega \tag{7.1}$$

where $H_{V^\perp}$ is the mean curvature vector field of the distribution orthogonal to V and n is the dimension of M. When $n=3$, the functionals $\mathcal{E}$ and $\mathcal{D}$ coincide up to renormalizations.

Lemma 7.1. *Let Z be the Reeb vector field of a contact form on a $2n+1$-dimensional contact metric manifold M. Then the mean curvature vector field $H_{Z^\perp}$ of the contact distribution is trivial.*

Proof. We shall denote by (g, α, Z, J) the contact metric structure tensors. Choose a local orthonormal basis Z, E_i, JE_i, $i = 1, 2, ..., n$ consisting of eigenvectors for the symmetric tensor $h = \frac{1}{2} L_Z J$ with eigenvalues λ_i and $-\lambda_i$ respectively. Then, using identity (2.1),

$$\begin{aligned} g(H_{Z^\perp}, Z) &= \frac{1}{2n} \sum_{i=1}^{n} g(\nabla_{E_i} E_i + \nabla_{JE_i} JE_i, Z) \\ &= \frac{1}{2n} \sum_{i=1}^{n} \lambda_i (g(E_i, JE_i) - g(JE_i, J^2 E_i)) = 0. \end{aligned}$$

□

Brito [2] has shown that Hopf vector fields are absolute minimizers for the functional $\mathcal{D}$ and that in dimension 3, they are the unique minimizers. We would like to point out that this seemingly "Hopf vector fields" phenomena is largely K-contact Einstein geometric. For general sasakian-Einstein manifolds, we can only prove the following.

Proposition 7.2. *Let M be a closed connected sasakian-Einstein (2n+1)-dimensional manifold with Reeb vector field Z. Then Z is an absolute minimum for the functional $\mathcal{D}$.*

Proof. If M is a (2n+1)-dimensional sasakian-Einstein manifold, then $Ricci(V, V) = 2n$ for any unit vector field V. Therefore, from Brito's lower bound

$$\mathcal{D}(V) \geq \int_M Ricci(V, V)\Omega, \tag{7.2}$$

we see that on a sasakian-Einstein $2n+1$-manifold M,

$$\mathcal{D}(V) \geq 2nVol(M). \tag{7.3}$$

If Z is the characteristic vector field of a sasakian form, then using the same orthonormal basis Z, E_i, JE_i, $i = 1, 2, ..., n$ as in the above lemma, we see that

$$\|\nabla Z\|^2 = 2n.$$

This together with Lemma 7.1 shows that $\mathcal{D}(Z) = 2nVol(M)$ and hence Z is an absolute minimizer by estimate (7.3). □

We will introduce a functional for which uniqueness of minimizers can be extended to all odd dimensions. Let's point out first that for a unit vector field V and a local orthonormal frame field $\{V, E_1, ..., E_m\}$ and up to a constant factor, the mean curvature vector field $H_{V^\perp}$ is given by

$$H_{V^\perp} = \sum_{i=1}^{m} g(\nabla_{E_i} V, E_i)$$

which is nothing other than the divergence $div(V)$ of V. In view of this observation, we consider the following modification of Brito's energy functional, which we denote by Emc.

$$Emc(V) = \int_M (\|\nabla V\|^2 + (div(V))^2) dM.$$

Proposition 7.3. *Let (M,α,Z,J,g) be structure tensors of a closed, K-contact Einstein 2n+1-dimensional manifold. Then Z is an absolute minimum for the functional Emc.*

Proof. For any vector field X on a riemannian manifold (M,g), the following identity holds: (see [14], page 170).

$$\|\nabla X\|^2 + (div\, X)^2 = Ricci(X,X) + \frac{1}{2}\|L_X g\|^2 - div(\nabla_X X) + div((div\, X)X). \tag{7.4}$$

It follows that, on a closed Einstein manifold (M,g), the functional Emc satisfies the following estimate:

$$Emc(V) = \int_M \|\nabla V\|^2 + (div\, V)^2 dM \geq \int_M Ricci(V,V) dM = C\, Volume(M), \tag{7.5}$$

where C is the constant Ricci curvature of (M,g). In particular, on a K-contact Einstein manifold (M^{2n+1},α,Z,J,g), one has $C = 2n$, $div\, Z = 0$ and $\|\nabla Z\|^2 = 2n$. Therefore $Emc(Z) = 2n\, Volume(M)$ is the absolute minimum value for Emc. □

If one assumes positivity of the sectional curvature, then the above proposition has a weak converse, more precisely:

Theorem 7.4. *Let (M,g) be a closed Einstein manifold with positive sectional curvature. If Z is a unit vector field minimizing Emc, then Z is the Reeb vector field of a K-contact form on M not necessarily with g as contact metric. In particular, M is odd-dimensional.*

Proof. From identity (7.4), one has the following which is valid on any Einstein manifold.

$$Emc(V) = \int_M \|\nabla V\|^2 + (div\, V)^2 dM = C\, Volume(M) + \frac{1}{2}\int_M \|L_V g\|^2 dM. \tag{7.6}$$

Suppose $Emc(Z) = C\, Volume(M)$, then $L_Z g = 0$ and Z is a Killing unit vector field, hence geodesic and divergence free. That is, one has the following identities for Z:

$$\nabla_Z Z = 0, \quad g(\nabla_X Z, Y) + g(X, \nabla_Y Z) = 0, \quad div(Z) = 0. \tag{7.7}$$

Define a 1-form α by

$$\alpha(X) = g(Z,X).$$

Then a simple calculation shows that :

$$\begin{aligned} d\alpha(X,Y) &= X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \\ &= g(\nabla_X Z, Y) - g(\nabla_Y Z, X), \end{aligned}$$

that is, using the second of identities (7.7),

$$d\alpha(X,Y) = 2g(\nabla_X Z, Y) = -2g(X, \nabla_Y Z). \tag{7.8}$$

Notice that, from the definition of α and the first of identities (7.7), one has

$$\alpha(Z) = 1 \text{ and } d\alpha(Z,Y) = 0$$

for any Y. In particular, Z is in the kernel of $d\alpha$. Let X be any unit vector field perpendicular to Z. Then

$$d\alpha(X,\nabla_X Z) = 2g(\nabla_X Z,\nabla_X Z) > 0$$

because $g(\nabla_X Z,\nabla_X Z)$ is equal to the sectional curvature of the plane spanned by X and Z, which is positive by assumption. Therefore, the kernel of $d\alpha$ is one dimensional. The above observations show that α is a contact form with characterisitic vector field Z and M is odd-dimensional. Since Z is Killing, it follows from Theorem 1 in [18] that the contact form α admits a K-contact metric (not necessarily the same as g). □

A characterization of Reeb vector fields of sasakian forms on Einstein manifolds is obtained if one assummes constant positive curvature 1.

Theorem 7.5. *Let (M,g) be a closed manifold with constant curvature 1. Then a unit vector field Z minimizes Emc if and only if Z is the Reeb vector field of a sasakian form with contact metric g.*

Proof. If Z is the characteristic vector field of a sasakian, hence K-contact form on (M,g), then by Proposition 7.2, Z is a minimizer of Emc.

Conversely, suppose that Z is a unit vector field minimizing Emc. As in the proof of Theorem 7.4, Z is a unit, Killing, geodesic and divergence free vector field. Moreover, the 1-form defined for any X by $\alpha(X) = g(Z,X)$ is a contact form. Let us define a (1,1) tensor field J by

$$JX = -\nabla_X Z.$$

Clearly, one has $d\alpha(X,Y) = 2g(X,JY)$ by identity (7.8) and also, $JZ = 0$ since Z is geodesic.

Let now X be a unit vector field orthogonal to Z. The following identity

$$g(JX,JX) = g(\nabla_X Z,\nabla_X Z) = 1, \tag{7.9}$$

holds, because $g(\nabla_X Z,\nabla_X Z)$ is precisely the sectional curvature of the plane spanned by $\{X,Z\}$, which is equal to 1 by assumption.

Also, in view of identity (7.9),

$$g(J^2X,X) = -g(\nabla_{JX}Z,X) = g(JX,\nabla_X Z) = -g(JX,JX) = -1.$$

Therefore, one has

$$J^2X = -X + W,$$

where W is some vector field orthogonal to both Z and X. But

$$\begin{aligned}
g(W,W) &= g(J^2X+X, J^2X+X)\\
&= g(J^2X,J^2X) + g(X,X) + 2g(J^2X,X)\\
&= g(\nabla_{\nabla_X Z}Z,\nabla_{\nabla_X Z}Z) + g(X,X) + 2g(J^2X,X)\\
&= 1+1-2 = 0.
\end{aligned}$$

It follows that $W = 0$ and consequently, $J^2X = -X$, making J into a partial complex structure adapted to α and g, that is to say, (α,Z,J,g) are structure tensors of a K-contact metric structure on M. If M is 3-dimensional, then this K-contact structure is automatically sasakian. If M is 5-dimensional or higher, then we can apply a result of Olszak ([13]) which says that in dimension 5 and higher, a contact metric structure of constant curvature c is sasakian with constant curvature $c = 1$. □

There exists another notion of contact riemannian structure which is slightly weaker than the K-contact one. An R-contact metric structure is given by a contact form α with Reeb vector field Z and a riemannian metric g satisfying the following identities ([16]):

$$\alpha(X) = g(Z,X), \quad L_Z g = 0.$$

In this situation, the contact form α is then called an R-contact form and the metric, an R-contact metric. Given an R-contact metric structure, one can always construct a K-contact metric associated with the same contact form ([18]). In the R-contact metric setting, Proposition 7.3 and Theorem 7.4 are special cases of the following more general statement:

Theorem 7.6. *Let (M,g) be a closed, Einstein manifold with positive sectional curvature. Then, a unit vector field Z minimizes Emc on M if and only if Z is the Reeb vector field of an R-contact form α with R-contact metric g.*

Proof. Recall the estimate from (7.5):

$$Emc(V) \geq C\, Volume(M)$$

which is valid on any Einstein manifold with Einstein constant C. If Z is an R-contact vector field, then it is a minimizer of Emc. This follows immediately from identity (7.4) which, for a unit Killing vector field, becomes:

$$\|\nabla Z\|^2 = Ricci(Z,Z)$$

and thus one has

$$Emc(Z) = \int_M \|\nabla Z\|^2 \, dM = \int_M Ricci(Z,Z)\, dM = C\, Volume(M).$$

Conversely, suppose Z is a unit vector field minimizing Emc, then as in the proof of Theorem 7.4, Z is the Reeb vector field of a contact form α which is defined by

$$\alpha(X) = g(Z,X).$$

Since Z is Killing for the metric g, it follows that (α, Z, g) are structure tensors of an R-contact metric structure. □

8 Non-Killing critical unit vector fields

If the curvature tensor of a riemannian manifold (M,g) is identically zero, then identity (2.2), with X a unit tangent horizontal vector, leads to the identity

$$0 = g(hX,hX) - g(X,X) = g(hX,hX) - 1.$$

Therefore, the eigenvalues of h are ± 1 and the contact distribution D decomposes into the positive and negative eigenbundles as $D = [+1] \oplus [-1]$.

Throughout this section, α, g, J, and Z will denote structure tensors of a flat contact metric, necessarily three-dimensional manifold M.

It is proven in [15] that M admits a parallelization by three unit vector fields Z, X, and JX where Z and X are characteristic vector fields of contact forms sharing the same flat contact metric g. The three vector fields Z, X and JX are respectively eigenvector fields corresponding to the eigenvalues 0, -1 and 1 of the tensor field h. Concerning these particular vector fields Z, X and JX, the following identities were established in [15].

$$\nabla_Z X = 0 = \nabla_X Z, \quad \nabla_{JX} JX = 0 \tag{8.1}$$

$$\nabla_{JX} Z = 2X, \quad \nabla_{JX} X = -2Z. \tag{8.2}$$

Moreover, the vector field JX has been shown to be parallel. Still in the flat contact metric geometry setting, using identities (8.1), (8.2) and notations of Section 3, we see that:

$$\mathcal{L}_Z Z = Z, \quad \mathcal{L}_Z X = X, \quad \mathcal{L}_Z JX = 5JX. \tag{8.3}$$

From identities (8.3), we deduce that

$$\mathcal{L}_Z^{-1} Z = Z, \quad \mathcal{L}_Z^{-1} X = X, \quad \mathcal{L}_Z^{-1} JX = \frac{1}{5} JX. \tag{8.4}$$

Also, a short calculation shows that $f(Z) = \sqrt{5} = f(X)$ and $f(JX) = 1$. Hence

$$\mathcal{F}(Z) = \sqrt{5} Volume\ of\ M = \mathcal{F}(X), \ \ and \ \ \mathcal{F}(JX) = Volume\ of\ M. \tag{8.5}$$

Next, we compute the expression of $T_Z \mathcal{F}(A)$ for any horizontal vector field A. Using the symmetry of $\mathcal{L}_Z^{-1}$ and identities (8.1), (8.2) and (8.4), we see that:

$$g(\mathcal{L}_Z^{-1} \circ (\nabla Z)^t (\nabla_Z A), Z) = g((\nabla Z)^t (\nabla_Z A), Z) = g(\nabla_Z A, \nabla_Z Z) = 0 \tag{8.6}$$

$$g(\mathcal{L}_Z^{-1} \circ (\nabla Z)^t (\nabla_X A), X) = g(\nabla_X A, \nabla_X Z) = 0. \tag{8.7}$$

In order to complete our computation of $T_Z \mathcal{F}(A)$, we need the fact that any smooth horizontal vector field A decomposes as $A = aX + bJX$ for some smooth functions a and b on M. Again, using identities (8.1), (8.2), (8.4) and symmetry of $\mathcal{L}_Z^{-1}$, we find that

$$\begin{aligned} g(\mathcal{L}_Z^{-1} \circ (\nabla Z)^t (\nabla_{JX} A), JX) &= \frac{1}{5} g(\nabla_{JX} A, \nabla_{JX} Z) \\ &= \frac{2}{5} g(\nabla_{JX} A, X) \\ &= \frac{2}{5} g(da(JX)X + a\nabla_{JX} X + db(JX)JX, X) \\ &= \frac{2}{5} da(JX). \end{aligned}$$

That is

$$g(\mathcal{L}_Z^{-1} \circ (\nabla Z)^t (\nabla_{JX} A), JX) = \frac{2}{5} da(JX) \tag{8.8}$$

From identities (8.6), (8.7) and (8.8), we deduce that

$$tr(\mathcal{L}_Z^{-1} \circ (\nabla Z)^t \circ \nabla A) = \frac{2}{5} da(JX). \tag{8.9}$$

Therefore, using the fact that JX is Killing, hence divergence free, we obtain

$$T_Z\mathcal{F}(A) = \frac{2}{\sqrt{5}} \int_M da(JX)\Omega = \frac{2}{\sqrt{5}} \int_M L_{JX}(a\,\Omega) \tag{8.10}$$

$$= \frac{2}{\sqrt{5}} \int_M d(i_{JX}(a\,\Omega)) = 0. \tag{8.11}$$

(The symbol L_{JX} above stands for the Lie derivative, not to be confused with $\mathcal{L}_{JX}$.)

Remark: Since Z and X play interchangeable roles in the flat contact metric setting of this section, one also has that $T_X\mathcal{F}(A) = 0$ for any A perpendicular to X.

Theorem 8.1. *Every closed, flat contact metric manifold has a parallelization by three critical unit vector fields. One of them is parallel minimizing, the other two are Reeb vector fields, non-Killing and do not minimize the volume functional.*

Proof. Since JX is parallel, it is obviously a minimizer for the volume functional. Identity (8.11) and the above remark show that Z and X are critical unit vector fields. Using identities (8.3), a direct computation of $L_Z g$ and $L_X g$ shows that none of Z and X is Killing. Finally, identities (8.5) show that, even though they are critical, neither Z, nor X is minimizing for the volume functional $\mathcal{F}$. □

Using identities (8.1)and (8.2), we easily see that

$$\nabla^*\nabla Z = 4Z.$$

Therefore, for any vector field A perpendicular to Z, one obtains that

$$T_Z\mathcal{E}(A) = \int_M g(A, 4Z)\,\Omega = 0. \tag{8.12}$$

Identity (8.12) above shows that Z is a critical vector field for the energy functional. We also have the following fact.

$$Hess_{\mathcal{E}}(JX, JX) = -\int_M 4\,\Omega = -4\,Volume\ of\ M\ < 0. \tag{8.13}$$

Identities (8.12) and (8.13) imply the following result.

Theorem 8.2. *The Reeb vector field in a flat contact metric structure on a closed manifold is critical unstable for the energy functional.*

9 More examples through deformation of contact Metrics

Given a contact metric structure with structures tensors $\alpha, g, Z, J, h = \frac{1}{2}L_Z J$ and a positive function l on a manifold M, we consider the following deformation

$$g_l = lg + (1-l)\alpha \otimes \alpha.$$

Denoting by η, S, G the g_l-induced contact metric structure tensors on the unit tangent sphere bundle T^1M, the following theorem is proved in [19].

Theorem 9.1. *The vector field*

$$Z\colon M \to (T^1M, \eta, S, G)$$

is a contact invariant embedding if and only if $dl(Z) = 0$ *and* $h^2X = (\frac{1}{l^2} - 1)X$ *for any vector field* X *such that* $\alpha(X) = 0$.

Corollary 9.2. *Let* (M, α, Z, J, g_l) *be a deformed 3-dimensional contact metric structure with* $dl(Z) = 0$ *and* $h^2X = (\frac{1}{l^2} - 1)X$ *for any* X *such that* $\alpha(X) = 0$. *Then the vector field*

$$Z\colon M \to (T^1M, \eta, S, G)$$

is a minimal, contact invariant embedding.

Proof. The corollary follows from the theorem above and Theorem 3.1 in [19]. □

Examples of 3-dimensional contact manifolds whose tensor h^2 is transversally multiplication by a basic function can be found in [19], Proposition 5.1. These are obtained by deforming the standard flat contact metric structure on the torus T^3.

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