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EQUIVALENCE FOR DIFFERENTIAL EQUATIONS

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Abstract

We shall study the equivalence problem for ordinary differential equations with respect to the affine transformation group $A(2,\mathbb{R})$.

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1 Introduction

Two differential equations are called equivalent if one can be transformed into another by a certain change of variables. In particular this change of variables transforms solutions of one equation into solutions of another. In general, the problem of equivalence of differential equations consists in determining whether two equations are equivalent up to a given class of transformations. W. Kryńsky [12] and B. Dubrov [7] examine differential equations up to contact transformations. Sophus Lie was the first to use some approach to the problem of equivalence of differential equations (description of invariants, computation of symmetry group).

In [9] Fels considers the problem of equivalence between two systems of second-order differential equations

$$\frac{d^2x^i}{dt^2} = f^i(t,x^j,\frac{dx^j}{dt}) \quad (1 \leq i,\ j \leq n) \tag{1.1}$$

$$\frac{d^2y^i}{ds^2} = g^i(s,y^j,\frac{dy^j}{ds}) \quad (1 \leq i,\ j \leq n) \tag{1.2}$$

under the pseudo-group of smooth invertible local point transformations

$$\psi\left(t,x^j\right) = \left(s,y^i\right) = \left(\phi\left(t,x^j\right),\varphi_i\left(t,x^j\right)\right)$$

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This notion of two systems being equivalent defines an equivalence relation on the set of differential equations on the form (1.1). Fels was able to cast the question of equivalence between (1.1) and (1.2) into a question about the equivalence of associated exterior differential systems on the jet space $J^1(\mathbb{R},\mathbb{R}^n)$, at which point the Cartan's method [4, 8, 10, 11, 15, 17] be may applied. The problem of equivalence for $n = 1$ was originally solved by Cartan in [4]. Chern in [6] investigated the two equivalence problems for systems under the restricted pseudo-groups of smooth invertible local transformations which preserve the independent variable as given by

$$\psi\left(t,x^j\right) = \left(t,\varphi_i\left(x^j\right)\right) \text{ and } \psi\left(t,x^j\right) = \left(t,\varphi_i\left(t,x^j\right)\right)$$

In [17] the authors reconsider an example in the plan brought up by E. Cartan, make the situation precise and explore a case that Cartan did not consider.

In the previous papers [1, 2] the relationship between differential equations, Pfaffian systems and geometric structures are studied. We have seen that every differential equation can be expressed as a Pfaffian system satisfying the structure equation and that the integration of a given equation is deeply related to the structure equation. We shall show it by means of interesting examples. My contribution here is the study of equivalence problem for the family of ordinary equations with respect to the affine transformation group $A(2,\mathbb{R})$ (section 4). There exist 19 different types of first order ordinary differential equations which admit at least 1- dimensional Lie groups in $A(2,\mathbb{R})$. All the equations which belong to the above types can be integrated by quadrature.

In the paper, by the word differentiable we mean always differentiable of classe C^∞.

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2 Basic Definition, Examples

2.1 Pfaffian systems

In this subsection we will review some basic concepts and facts about Pfaffian systems theory [3]. Let M be an differentiable manifold. $F(M)$ denotes the ring of real-valued differentiable functions on M and $\Lambda^1(M)$ the $F(M)$-module of all 1-forms (Pfaffian forms) on M.

A $F(M)$-submodule Σ of $\Lambda^1(M)$ is called a *Pfaffian system of rank n on M* if Σ is generated by n linearly independent Pfaffian forms $\theta^1,\dots,\theta^n$. A submanifold N of M is said an *integral manifold* of Σ if $i^*(\theta) = 0$ for all $\theta \in \Sigma$, where i denotes the immersion $N \hookrightarrow M$. A differentiable function is said a *first integral of* Σ if the exterior derivate df belongs to Σ. By the symbol $\Sigma = \langle\theta^1,\dots,\theta^n\rangle$ we mean that the Pfaffian system Σ is generated by linearly independent Pfaffian forms $\theta^1\dots,\theta^n$ defined on M.

For each Pfaffian systems Σ on M, we can construct the dual system, that is, the differentiable subbundle $D = D(\Sigma)$ of the tangent bundle $T(M)$ on M such that the fiber dimension of D is equal to $\dim(M) - n$. Let $\Gamma(D)$ be the sheaf of germs of local vector fields which belong to D and $\Gamma(D)_x, x \in M$, the stalk of $\Gamma(D)$ at x. For $x \in M$, we defined the subspaces $Ch(D)_x$ of $T_x(M)$ by

$$Ch(D)_x = \{X_x \in D_x; [X'_x, \Gamma(D)_x] \subseteq \Gamma(D)_x\}$$

where X denotes a vector field and X'_x the germ at x determined by X. We suppose that $\dim Ch(D)_x$ is constant on M. Thus, we obtain the subbundle $Ch(D)$ of $T(M)$ called the *Cauchy characteristic* of D. The distribution characteristic of D is the module spanned by all vector fields Y that belongs to D such that $[Y,D] \subseteq D$. The dual system $Ch(\Sigma)$ of $Ch(D)$ is called the *Cauchy characteristic system* of Σ. The following theorem is due to E. Cartan.

Theorem 2.1. *Let $\Sigma = \langle\theta^1,\dots,\theta^n\rangle$ be a Pfaffian system.*

1. If Σ is completely integrable, i.e. $d\theta^i = 0(\mathrm{mod}\,\theta^1,\dots,\theta^n)$, $(1 \le i \le n)$, then $Ch(\Sigma) = \Sigma$.

2. If Σ is not completely integrable, then there exist linearly independent Pfaffian forms $w^1,\dots,w^m$ satisfying the following conditions:

(i) $\theta^1,\dots,\theta^n,w^1,\dots,w^m$ are also linearly independent;

(ii) $(\theta^1,\dots,\theta^n,w^1,\dots,w^m)$ forms a (local) generator of $Ch(\Sigma)$;

(iii) $d\theta^i = \sum_{j,k=1}^{m} C^i_{jk} w^i \wedge w^k \ \left(\mathrm{mod}\,\theta^1,\dots,\theta^n\right)$, where C^i_{jk} denotes a differentiable function ($1\le i \le n$, $1\le j,k \le m$).

3. $Ch(\Sigma)$ is completely integrable.

4. Let $x^1,\dots,x^{n+m}$ be independent first integrals of $Ch(\Sigma)$. Then there exist linearly independent Pfaffian forms $\eta^i = \sum_{j=1}^{n+m} A^i_j(x^1,\dots,x^{n+m})dx^j$, $i=1,\dots,n$, such that $(\eta^1,\dots,\eta^n)$ forms a (local) generator of Σ.

By making use of Property 2, we can construct the Cauchy characteristic system $Ch(\Sigma)$.

Definition 2.2. A system $(w^1,\dots,w^m)$ of linearly independent Pfaffian forms on M will be said to be a solvable system of $\Sigma = \langle\theta^1,\dots,\theta^n\rangle$ if it satisfies the following conditions:
(i) $(w^1,\dots,w^m)$ forms a generator of $Ch(\Sigma)$.
(ii) $dw^1 = 0$ *and* $dw^p = 0\,\mathrm{mod}(w^1,\dots,w^{p-1})$ for all $p = 2,\dots,m$.

If we can find a solvable system of Σ, then m independent first integrals of $Ch(\Sigma)$ are given par quadrature.

2.2 Examples

In the subsection we shall consider by means of examples [2] the relation between the differential equations, Pfaffian systems and structure equation.

a) Consider the Pfaffian system $\Sigma = \langle\theta\rangle$, $\theta = dz + pdx + p^2dy$, on $\mathbb{R}^4 = \{(x,y,z,p)\}$. We have $d\theta = dp \wedge (dx + 2pdy)$ and

$$w^1 = dp,\ w^2 = dx + 2pdy,\ w^3 = \theta$$

determine the Cauchy characteristic system of Σ. We can find by quadrature three independent first integrals as follows:

$$u_1 = z + xp + yp^2,\ u_2 = x + 2yp,\ u_3 = p$$

and θ itself is expressed as $\theta = du_1 - u_2 du_3$. The system (w_1, w_2, w_3) is a solvable system of Σ.

b) We consider an absolute parallelism $w^1, w^2, w^3, w^4, w^5, w^6$ on $\mathbb{R}^6$ satisfying the equations

$$\begin{aligned} dw^1 &= 0,\ dw^2 = 0 \operatorname{mod}(w^1, w^2), \\ dw^3 &= w^1 \wedge w^4 + w^2 \wedge w^5 \ \operatorname{mod}(w^3), \\ dw^4 &= 0 \operatorname{mod}(w^3, w^4, w^5), \\ dw^5 &= w^2 \wedge w^6 \operatorname{mod}(w^3, w^4, w^5). \end{aligned} \tag{2.1}$$

Let x and y be two independent first integrals of the completely Pfaffian integrable system $w^1 = w^2 = 0$; the form w^3 is expressed as

$$w^3 = a(dz - pdx - qdy), \quad a \neq 0.$$

The functions x, y, z, p and q are independent first integrals of the completely integrable Pfaffian system $w^1 = w^2 = w^3 = w^4 = w^5 = 0$.Therefore w^4 and w^5 can be written by means of the exterior derivatives dx, dy, dz, dp, dq and the formulas

$$\begin{aligned} dp - rdx - sdy &= a_1 w^4 + a_2 w^5 + a_3 w^3 \\ dq - s'dx - tdy &= a_4 w^4 + a_5 w^5 + a_6 w^3 \end{aligned}$$

determine the functions r, s, s', t and a_i' s of the variables x, y, z, p, q and another u. From the equation $dw^3 = w^1 \wedge w^4 + w^2 \wedge w^5 (\operatorname{mod} w^3)$, one can verify that the function s coincides with s'. Moreover, the equations $dw^4 = 0$, $dw^5 = w^2 \wedge w^6 \operatorname{mod}(w^3, w^4, w^5)$ imply

$$rank\Big(\frac{\partial r}{\partial u}, \frac{\partial s}{\partial u}, \frac{\partial t}{\partial u}\Big) = 1.$$

Therefore the functions

$$r = r(x, y, z, p, q, u) \ , \ s = s(x, y, z, p, q, u) \ , \ t = t(x, y, z, p, q, u)$$

determine a system of second-order partial differential equations. This family of systems of differential equations determined by an absolute parallelism satisfying (2.1) is the main subject of Cartan's researches in his paper [4, 5]. For example, take the system of differential equations

$$\frac{\partial^2 z}{\partial x^2} = 0 \ , \ \frac{\partial^2 z}{\partial x \partial y} = z - x\frac{\partial z}{\partial x} \tag{2.2}$$

Putting on $\mathbb{R}^6 = \{(x, y, z, p, q, t)\}$, $w^1 = dx$, $w^2 = dy$, $w^3 = dz - pdx - qdy$, $w^4 = dp - (z - xp)dy$, $w^5 = dq - (z - xp)dx - tdy$ and $w^6 = dt - (q - x(z - xp))dx$ we have the structure equations

$$\begin{aligned}
dw^1 &= 0,\ dw^2 = 0,\\
dw^3 &= w^1 \wedge w^4 + w^2 \wedge w^5,\\
dw^4 &= w^2 \wedge w^3 - xw^2 \wedge w^4,\\
dw^5 &= w^2 \wedge w^6 + w^1 \wedge w^3 - xw^1 \wedge w^4,\\
dw^6 &= w^1 \wedge w^5 - xw^1 \wedge w^3 - x^2 w^1 \wedge w^4 + Kw^1 \wedge w^2,
\end{aligned}$$

where $K = t - xq + x^2(z - xp)$. The absolute parallelism satisfies the equations (2.1). It is easy to see that the system $\left(w^2, w^3, w^4, w^5, w^6\right)$ forms a solvable system of $\Sigma = \langle w^3, w^4, w^5 \rangle$. Five independent first integrals of the solvable system are given by quadrature as follows:

$$u_1 = y,\ u_2 = z - xp,\ u_3 = p,\ u_4 = q - x(z - xp),\ u_5 = K\ ,$$

and we have [5]

$$\begin{aligned}
w^3 - xw^4 &= du_2 - u_4 du_1,\\
w^4 &= du_3 - u_2 du_1,\\
w^5 - xw^3 &= du_4 - u_5 du_1.
\end{aligned}$$

By the expression, the general integral surface of (2.2) is given by the formulas:

$$p = f(y),\ z - xp = f'(y),\ q - x(z - xp) = f''(y),\ t - x(q - x(z - xp)) = f'''(y)$$

where f is a differentiable function and f', $f''$and f''' denote its derivatives.

3 Equivalence

In the paper [2] , we have seen that every differential equation can be expressed as a Pfaffian system satisfying the structure equation and the integration of a given equation is deeply related to the structure equation. In this section, we shall consider the equivalence problem for Pfaffian systems and hence for differential equations under the action of Lie groups.

Let M be a differentiable manifold, G be a Lie group acting on M on the left. For a Pfaffian system Σ on M we set

$$g^*\Sigma = \left\{L_g^*\theta \in \Lambda^1 M;\ \theta \in \Sigma\right\},$$

and

$$G(\Sigma) = \{g \in G;\ g^*\Sigma = \Sigma\ \},$$

where L_g denotes the left action of $g \in G$ on M; $g^*\Sigma$ is a Pfaffian system on M and $G(\Sigma)$ is a subgroup of G .

Two Pfaffian systems Σ_1 and Σ_2 on M are *equivalent under the action of* G if there is an element $g \in G$ such that $g^*\Sigma_1 = \Sigma_2$.

Let F be a family of Pfaffian systems on M. The problems (Lie programme) to be solved are as follows:

1) Determine the condition for the equivalence of the elements of F.

2) Classify the Pfaffian systems in F under the action of G.

3) For each $\Sigma \in F$, determine the structure of the subgroup $G(\Sigma)$.

4) Research the relation between the integration of a Pfaffian system Σ and the structure of the subgroup $G(\Sigma)$, *i.e.*, reduce the integration of a Pfaffian system Σ to auxiliary systems obtained via the knowledge of the structure of $G(\Sigma)$.

, In Section 4, we consider Problem 2, in the particular case where $G = A(2,\mathbb{R})$.
For a Lie subgroup G' of G, we set

$$F(G') = \left\{\Sigma \in F;\ G(\Sigma) = G'\right\}. \tag{3.1}$$

For every $\Sigma \in F(G')$, G' is the largest subgroup of G which leaves Σ invariant. It is possible that $F(G')$ is an empty set.

Proposition 3.1. *Let Σ, Σ_1, Σ_2 be Pffafian systems $\in F$ and let G' be a subgroup of G.*

1. For any $g \in G$, $G(g^*\Sigma) = g^{-1}G(\Sigma)g$.

2. If $gG(\Sigma_2)g^{-1} = G(\Sigma_1) = G'$ for an element $g \in G$, then Σ_1 and $g^*\Sigma_2$ are in $F(G')$.

3. The normalizer $N(G' : G)$ of G' in G acts on $F(G')$.

4. If $\Sigma_1, \Sigma_2 \in F(G')$ and $g^*\Sigma_2 = \Sigma_1$ for an element $g \in G$, then $\Sigma_1$and Σ_2 lie in the same orbit determined by the action of $N(G' : G)$ on $F(G')$.

Proof. 1) and 2) may be clear. To prove 3), suppose that $\Sigma \in F(G')$ and $g \in N(G' : G)$. From (3.1) and 1) of the proposition, we have

$$G(g^*\Sigma) = g^{-1}G(\Sigma)g = g^{-1}G'g = G'$$

and hence $g^*\Sigma \in F(G')$.

4) From (3.1) and 1) of proposition, we have

$$g^{-1}G'g = g^{-1}G(\Sigma_2)g = G(g^*\Sigma_2) = G(\Sigma_1) = G'$$

and hence $g \in N(G' : G)$.

By virtue of this proposition, the equivalence probleme and the classification are reduced to the following problems:

(i) determine all conjugate classes of the subgroups of G.

(ii) For a representative G' of each conjugate class, determine the set $F(G')$.

(iii) Observe the action of $N(G' : G)$ on $F(G')$.

Since there are, in general, many subgroups G' of G such that $F(G')$ is empty set, this reduction of the problems is not always the best one. Moreover, the Pfaffian systems to be considered are not always defined globally on M . Therefore, instead of ordinary Lie groups, we have to consider Lie pseudogroups [4, 13, 14, 16]. Then the subject of the study is *invariants* of a Pfaffian system with respect to a given Lie pseudogroup. At this level, we can recognize that the subgroup $G(\Sigma)$ plays an important role in the problems.

4 Equivalence with respect to the A(2,$\mathbb{R}$)

Let G be a finite dimensional Lie group and let Σ be a left-invariant completely integrable Pfaffian system on G. We denote by $I_g(\Sigma)$ the maximal integral manifold through $g \in G$ and we set

$$G_g(\Sigma) = \{h \in G;\ L_h(I_g(\Sigma)) = I_g(\Sigma)\}\ ,$$

Since Σ is left-invariant, $G_g(\Sigma)$, $g \in G$, are mutually conjugate in G.

4.1 Invariant forms of A(2,$\mathbb{R}$)

Let $A(2,\mathbb{R})$ be the affine transformation group on $\mathbb{R}^2$. By making use of the matrix representation

$$A(2,\mathbb{R}) = \left\{ \begin{bmatrix} x_3 & x_4 & x_1 \\ x_5 & x_6 & x_2 \\ 0 & 0 & 1 \end{bmatrix};\ x_3x_6 - x_4x_5 \neq 0,\ x_i \in \mathbb{R},\ i = 1,2,\ldots,6 \right\},$$

we have a basis of invariant forms of $A(2,\mathbb{R})$

$$\begin{aligned}
w^1 &= \frac{1}{D}\left(x_6dx_1 - x_4dx_2\right), \\
w^2 &= \frac{1}{D}\left(x_3dx_2 - x_5dx_1\right), \\
w^3 &= \frac{1}{D}\left(x_6dx_3 - x_4dx_5\right), \\
w^4 &= \frac{1}{D}\left(x_6dx_4 - x_4dx_6\right), \\
w^5 &= \frac{1}{D}\left(x_3dx_5 - x_5dx_3\right), \\
w^6 &= \frac{1}{D}\left(x_3dx_6 - x_5dx_4\right),
\end{aligned}$$

where we put $D = x_3x_6 - x_4x_5$. We have then the structure equation

$$\begin{aligned}
dw^1 &= w^1 \wedge w^3 + w^2 \wedge w^4, \\
dw^2 &= w^1 \wedge w^5 + w^2 \wedge w^6, \\
dw^3 &= -w^4 \wedge w^5, \\
dw^4 &= -w^3 \wedge w^4 - w^4 \wedge w^6, \\
dw^5 &= w^3 \wedge w^5 + w^5 \wedge w^6, \\
dw^6 &= w^4 \wedge w^5.
\end{aligned} \tag{4.1}$$

We remark that changing the basis of invariant forms by the formula

$$\begin{aligned}
\overline{w}^1 &= a'w^1 + b'w^2, \ \overline{w}^2 = c'w^2, \ \overline{w}^3 = w^3 + \frac{b'}{a'}w^5, \\
\overline{w}^4 &= \frac{a'}{c'}w^4 - \frac{b'}{c'}w^3 - \frac{b'^2}{a'c'}w^5 + \frac{b'}{c'}w^6, \ \overline{w}^5 = \frac{c'}{a'}w^5, \\
\overline{w}^6 &= w^6 - \frac{b'}{a'}w^5,
\end{aligned} \tag{4.2}$$

where a', b' and c' denote arbitrary constants with $a'c' \neq 0$, the structure equation (4.1) does not alter. $\Big(A(2,\mathbb{R}), C = (w^1, \ldots, w^6)\Big)$ determines a Cartan system.

4.2 Classification under action of A(2,$\mathbb{R}$)

The systems to be considered are given by $\Sigma = \langle dx_2 - f\ (x_1,x_2)dx_1 \rangle$ where f denotes a differentiable function. In this paper I given some ideas for the classification. Since

$$dx_2 - f(x_1,x_2)dx_1 = (x_5 - x_3 f(x_1,x_2))w^1 + (x_6 - x_4 f(x_1,x_2))w^2,$$

and

$$(x_6 - x_4 f(x_1,x_2))^{-1}(dx_2 - f(x_1,x_2)dx_1 = (x_6 - x_4 f(x_1,x_2))^{-1}(x_5 - x_3 f(x_1,x_2))w^1 + w^2,$$

then $(x_6 - x_4 f)^{-1}(x_5 - x_3 f)$ forms a characteristic invariant system. By using this invariant, we reduce the Cartan system to be submanifold M_0 defined by the equation $x_5 - x_3 f(x_1,x_2) = 0$. The equation to be integrated is now given by $w^2 = 0$. On the submanilfold M_0 , we have

$$\begin{aligned}
w^5 &= aw^1 + bw^2, \\
da &= 2aw^3 - a\overline{w}^6 + u_1 w^1 + u_2 w^2, \\
db &= bw^3 + aw^4 + (u_2 - b^2)w^1 + u_3 w^2.
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
dw^1 &= w^1 \wedge w^3 + w^2 \wedge w^4, \\
dw^2 &= w^2 \wedge \overline{w}^6, \\
dw^3 &= aw^1 \wedge w^4 + bw^2 \wedge w^4, \\
dw^4 &= -w^3 \wedge w^4 - w^4 \wedge \overline{w}^6 + bw^1 \wedge w^4, \\
d\overline{w}^6 &= -2bw^2 \wedge w^4 + u_3 w^1 \wedge w^2,
\end{aligned} \tag{4.4}$$

where we put $\overline{w}^6 = w^6 - bw^1$ and a, b, u_1, u_2, u_3 denote definite functions on M_0. These functions are all invariants of induced Cartan system.

4.2.1 From now on, we shall determine all the equations which admit at least 2-dimensional Lie subgroup of $A(2,\mathbb{R})$ as an invariant group. Therefore we suppose always that the forms w^1, w^2 are linearly independent.

I. The case $a=0$. From (4.3) , we have $u_1=u_2=0$. Moreover

$$du_3 = u_3\overline{w}^6 + u_3w^3 - 2bu_3w^1 + cw^2 \tag{4.5}$$

where c denotes a definite function on M.

1. $b=0$. The manifold M_0 is given by the maximal integral manifold of $w^5=0$. Hence we obtain the first type:

$$\begin{aligned}
\Sigma &: w^5 = 0,\\
dw^1 &= w^1\wedge w^3 + w^2\wedge w^4,\\
dw^2 &= w^2\wedge w^6,\\
dw^3 &= 0,\\
dw^4 &= -w^3\wedge w^4 - w^4\wedge w^6,\\
dw^6 &= 0.
\end{aligned}$$

Integrating the system Σ, we have the result:

Theorem 4.1. *The equation $y'=c$ (constant) admits a 5-dimensional Lie subgroup in $A(2,\mathbb{R})$ and can be transformed to the equation $y'=0$ by an element of $A(2,\mathbb{R})$.*

2. $b\neq 0$. We can reduce M_0 to the submanifold M_1 defined by the equation $b=const.(\neq 0)$. Taking $a'=b$, $b'=0$, $c'=b$ in (4.2) we can assume that the constant is equal to 1: $M_1=\{g\in M_0;\ b(g)=1\}$. From (4.3), (4.4) and (4.5) we have on M_1

$$\begin{aligned}
w^3 &= w^1 - u_3w^2,\ \ w^5 = w^2,\\
du_3 &= u_3\overline{w}^6 - u_3w^1 + (c-u_3^2)w^2,\\
dw^1 &= -u_3w^1\wedge w^2 + w^2\wedge w^4,\\
dw^2 &= w^2\wedge \overline{w}^6,\\
dw^4 &= u_3w^2\wedge w^4 - w^4\wedge\overline{w}^6,\\
d\overline{w}^6 &= -2w^2\wedge w^4 + u_3w^1\wedge w^2.
\end{aligned} \tag{4.6}$$

2.1. $\mathbf{u}_3=0$. From the second equation of (4.6) we have $c=0$. We have thus obtained the second type:

$$\begin{aligned}
\Sigma &: w^3 = w^1,\ \ w^5 = w^2.\\
dw^1 &= w^2\wedge w^4,\\
dw^2 &= w^1\wedge w^2 + w^2\wedge w^6,\\
dw^4 &= -w^1\wedge w^4 - w^4\wedge w^6,\\
dw^6 &= -w^2\wedge w^4.
\end{aligned}$$

Integrating the system Σ, we obtain the result:

Theorem 4.2. *The equation $y' = (x+a)^{-1}(y+b)$, a, b constants, admits a 4-dimensional Lie subgroup in $A(2,\mathbb{R})$ and can be transformed to $y' = x^{-1}y$ by an element of $A(2,\mathbb{R})$.*

2.2. $\mathbf{u}_3 \neq 0$. We can reduce M_1 to be submanifold M_2 defined by the equation $u_3 =$ constant $(\neq 0)$. Taking $a' = 1$, $b' = 0$, $c' = u_3$ in (4.2), we can assume that the constant is equal to 1. From (4.6) , we have on M_2

$$\begin{aligned} w^3 &= w^1 - w^2,\ w^5 = w^2,\ \overline{w}^6 = w^1 - ew^2 \ \ (e = c-1), \\ de &= -3w^4 + (e-2)w^1 + rw^2, \\ dw^1 &= -w^1 \wedge w^2 + w^2 \wedge w^4, \\ dw^2 &= -w^1 \wedge w^2, \\ dw^4 &= (1-e)w^2 \wedge w^4 + w^1 \wedge w^4, \end{aligned} \tag{4.7}$$

where r denotes a definite function on M_2. The equations (4.7) does not determines a 3-dimensional Lie group. By using the invariant e, we can reduce M_2. Taking $a' = 1$, $b' = 1$, $c' = \frac{(2-e)}{3}$ in (4.2), we can reduce M_2 to the manifold $M_3 = \{g \in M_2\,; e(g) = 2\}$. From (4.7) we obtain on M_3

$$\begin{aligned} w^5 &= w^2,\ w^3 = w^1 - w^2,\ \overline{w}^6 = w^1 - 2w^2,\ w^4 = \frac{1}{3}rw^2, \\ dr &= 2rw^1 + r_0 w^2 \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} dw^1 &= -w^1 \wedge w^2, \\ dw^2 &= -w^1 \wedge w^2. \end{aligned} \tag{4.9}$$

Although the equations (4.9) do not contain any functions, the function r is an invariant of the group. Therefore (4.8) and (4.9) determine a 2-dimensional Lie group if and only if r is a constant on M_3. In the case, we have $r = 0$ and

$$\begin{aligned} \Sigma \quad &: \quad w^3 = w^1 - w^2,\ w^4 = 0,\ w^5 = w^2,\ w^6 = 2w^1 - 2w^2. \\ dw^1 &= -w^1 \wedge w^2,\ \ dw^2 = -w^1 \wedge w^2. \end{aligned}$$

Integrating the system Σ, we obtain the result:

Theorem 4.3. *Let f be a function satisfying the Clairaut equation*

$$y = xf + \frac{af^2 + bf + c}{\alpha f + \beta} \quad \text{with } a,\ b,\ c,\ \alpha,\ \beta \text{ constants.}$$

Then $y' = f(x,y)$ admits a 2-dimensional Lie group in $A(2,\mathbb{R})$ and can be transformed to $y' = x - (x^2 - 2y)^{\frac{1}{2}}$ by an element $A(2,\mathbb{R})$.

II. The case $a \neq 0$. We go back to the manifold M_0. Suppose that $b \neq 0$ on M_0 . Consider the submanifold N_0 defined by the equation $b = const$ $(\neq 0)$. By setting

$$\begin{aligned} \overline{w}^1 &= bw^1 + \frac{b^2}{a} w^2, \ \overline{w}^2 = \frac{b}{a} w^2, \ \overline{w}^3 = w^3 + \frac{b}{a} w^5, \\ \overline{w}^4 &= aw^4 - bw^3 - \frac{b^2}{a} w^5 + bw^6, \ \overline{w}^5 = \frac{1}{a} w^5, \ \overline{w}^6 = w^6 - \frac{b}{a} w^5, \end{aligned}$$

we can assume that $a = 1, b = 0$ on M_0. Therefore we have only to examine the case $a \neq 0$, $b = 0$ on M_0. We can reduce M_0 to the submanifold

$$N_0 = \{g \in M_0;\ a(g) = 1, b(g) = 0\}$$

on which we have

$$\begin{aligned} w^4 &= -u_2 w^1 - u_3 w^2, \ w^5 = w^1, \ w^6 = 2w^3 + u_1 w^1 + u_2 w^2, \\ dw^1 &= w^1 \wedge w^3 + u_2 w^1 \wedge w^2, \\ dw^2 &= 2w^2 \wedge w^3 - u_1 w^1 \wedge w^2, \\ dw^3 &= -u_3 w^1 \wedge w^2. \end{aligned} \tag{4.10}$$

By differentiating (4.9) we obtain

$$\begin{aligned} du_1 &= u_1 w^3 + v_1 w^1 + v_2 w^2, \\ du_2 &= 2u_2 w^3 + v_3 w^1 + v_4 w^2, \\ du_3 &= 3u_3 w^3 + v_5 w^1 + v_6 w^2, \\ 0 &= -v_5 + v_4 + 2(u_1 u_3 - u_2^2), \\ 0 &= v_2 - v_3 + 3u_3. \end{aligned} \tag{4.11}$$

1. $u_1 = u_2 = u_3 = 0$. From (4.10), we have

$$\begin{aligned} \Sigma \ &: \ \ w^4 = 0, \ w^5 = w^1, \ w^6 = 2w^3. \\ dw^1 &= w^1 \wedge w^3, \\ dw^2 &= 2w^2 \wedge w^3, \\ dw^3 &= 0. \end{aligned}$$

Integrating the system Σ, we obtain the result:

Theorem 4.4. *The equations of this type can be transformed to* $y' = x$ *by an element of* $A(2,\mathbb{R})$ *and admit a 3-dimensional Lie group in* $A(2,\mathbb{R})$.

2. $u_1 \neq 0$, $u_2 = 0$, $u_3 = 0$. We reduce N_0 to the submanifold defined by the equation $u_1 = const.(\neq 0)$. From (4.11) we have $v_2 = v_3 = v_4 = v_5 = v_6 = 0$ and hence

$$\begin{aligned}
\Sigma \quad &: \quad w^3 = -\frac{v_1}{u_1}w^1,\ w^4 = 0,\ w^5 = w^1,\ w^6 = (u_1 - \frac{2v_1}{u_1})w^1. \\
dw^1 &= 0, \\
dw^2 &= -(u_1 - \frac{2v_1}{u_1})w^1 \wedge w^2, \\
dv_1 &= 0 (\operatorname{mod} w^1).
\end{aligned}$$

If these equations determine a 2-dimensional Lie group, v_1 must be a constant. In this case, integrating the system Σ, we obtain the result:

Theorem 4.5. *All the equations in this case are transformed by an element of* $A(2,\mathbb{R})$ *to one the following three types*

i) $y' = \log x$, *ii)* $y' = e^x$, *iii)* $y' = x^a$ $(a\ const. \neq 0,1)$, *which admit 2-dimensional Lie group in* $A(2,\mathbb{R})$.

3. $u_1 = 0$, $u_2 \neq 0$, $u_3 = 0$. We reduce N_0 to the submanifold defined by the equation $u_2 = const.(\neq 0)$. From (4.11), we have $v_4 = 2u_2^2$ and otherwise $v_i = 0$ and hence

$$\begin{aligned}
\Sigma \quad &: \quad w^3 = -u_2 w^2,\ w^4 = -u_2 w^1,\ w^5 = w^1,\ w^6 = -u_2 w^2. \\
dw^1 &= 0,\ dw^2 = 0.
\end{aligned}$$

3.1. The case $u_2 \gneqq 0$. We can assume always $u_2 = 1$. Integrating the system Σ, we obtain the result:

Proposition 4.6. *All the equations in this type are transformed by an element of* $A(2,\mathbb{R})$ *to the equation* $y' = -xy^{-1}$, *which admits a 2-dimensional Lie group in* $A(2,\mathbb{R})$.

3.2. The case $u_2 \lneqq 0$. We can assume $u_2 = -1$. Integrating the system Σ we obtain the result:

Proposition 4.7. *All equations in this case are transformed by an element of* $A(2,\mathbb{R})$ *to the equation* $y' = -x^{-1}y$, *which admits a 2-dimensional Lie group in* $A(2,\mathbb{R})$.

4. $u_1 = u_2 = 0$, $u_3 \neq 0$. From (4.11) we have $v_i = 0$, $i = 1,2,3,4$. Since $v_2 - v_3 + 3u_3 = 0$, this contradicts the assumption $u_3 \neq 0$.

5. $u_1 = 0$, $u_2 \neq 0$, $u_3 \neq 0$. We reduce N_0 to the submanifold defined by the equations $u_2 = const.(\neq 0)$, $u_3 = const.(\neq 0)$. We can assume $u_3 = 4$. From (4.11), we have $v_1 = v_2 = 0$, $v_3 = 12$, $v_4 - v_5 = 2u_2^2$ and

$$\begin{aligned}
0 &= 2u_2 w^3 + 12w^1 + v_4 w^2, \\
0 &= 12w^3 + v_5 w^1 + v_6 w^2.
\end{aligned} \tag{4.12}$$

By this equations, we obtain

$$v_4 = \frac{72}{u_2} + 2u_2^2,\quad v_5 = \frac{72}{u_2},\quad v_6 = \frac{6}{u_2}(\frac{72}{u_2} + 2u_2^2).$$

Substituting these values to (4.12) we have

$$w^3 = -\frac{6}{u_2}w^1 - (\frac{36}{u_2^2} + u_2)w^2.$$

Substituting this equation to the last equation of (4.10) we have $u_2 = -3$. Hence we have

$$\begin{aligned} \Sigma &: \quad w^3 = 2w^1 - w^2, \;\; w^4 = 3w^1 - 4w^2, \;\; w^5 = w^1, \;\; w^6 = 2w^3 - 3w^2. \\ dw^1 &= -4w^1 \wedge w^2, \;\; dw^2 = -4w^1 \wedge w^2. \end{aligned}$$

Integrating the system Σ, we obtain the result:

Theorem 4.8. *The equation of this type can be transformed by an element of* $A(2,\mathbb{R})$ *to* $y' = x^{-1}y + x^4$, *which admits a 2-dimensional Lie group in* $A(2,\mathbb{R})$.

6. $u_1 \neq 0$, $u_2 = 0$, $u_3 \neq 0$. We reduce N_0 to the submanifold defined by the equations $u_1 = const.(\neq 0)$, $u_3 = const.(\neq 0)$. We can assume $u_1 = -3$. By the same argument as in the case **5**, we have

$$\begin{aligned} \Sigma &: \quad w^3 = 2w^1 - u_3w^2, \; w^4 = -u_3w^2, \; w^5 = w^1, \; w^6 = w^1 - 2u_3w^2. \\ dw^1 &= -u_3w^1 \wedge w^2, \;\; dw^2 = -w^1 \wedge w^2. \end{aligned}$$

Integrating the system Σ, we obtain the result:

Theorem 4.9. *i) If* $u_3 = 1$, *the equations of this type are transformed by an element of* $A(2,\mathbb{R})$ *to* $y' = x^{-1}y + x$.

ii) If $u_3 \neq 1$, *the equations of this type are transformed by an element of* $A(2,\mathbb{R})$ *to the equation*

$$y' = \frac{-u_3x + \sqrt[2]{2(1-u_3)y + u_3x^2}}{1-u_3}.$$

7. $u_1 \neq 0$, $u_2 \neq 0$, $u_3 \neq 0$. By the same argument as in the case **5,** we have

$$\begin{aligned} \Sigma &: \quad w^3 = -\frac{1}{2}u_1w^1 - u_2w^2, \; w^4 = -u_2w^1, \; w^5 = w^1, \; w^6 = 2w^3 + u_1w^1 + u_2w^2. \\ dw^1 &= dw^2 = 0. \end{aligned}$$

Integrating the system Σ, we have the result:

Proposition 4.10. *The equations of this type are transformed by an element of* $A(2,\mathbb{R})$ *to the equation*

$$y' = \frac{-2x + u_1y}{2u_2y} \; .$$

8. $u_1 \neq 0$, $u_2 \neq 0$, $u_3 \neq 0$. By the same argument as in the case **5 ,** we can determine v_i $, 1 \leq i \leq 6$. In particular, we have

$$v_1 = \frac{u_1(6u_2 - 2u_1^2 + u_1u_2^2)}{2u_2^2 - 3u_1}, \quad v_2 = \frac{u_1(18 - 4u_1u_2 + 2u_2^3)}{2u_2^2 - 3u_1}.$$

If $2u_2^2 = 3u_1$, then $u_1 = 6$, $u_2 = 3$, $u_3 = 2$, $v_1 = 2$ and $v_2 = 1$. We have

$$w^3 = -\frac{v_1}{u_1}w^1 - \frac{v_2}{u_1}w^2.$$

Substituting this relation to the last equation of (4.10) we have a certain algebraic equation with respect to the quantities u_1, u_2. Hence we obtain

$$\Sigma \quad : \quad w^3 = -\frac{v_1}{u_1}w^1 - \frac{v_2}{u_1}w^2,\ w^4 = -u_2w^1 - u_3w^2,\ w^5 = w^1,\ w^6 = 2w^3 + u_1w^1 + u_2w^2.$$

$$dw^1 \quad = \quad (u_2 - \frac{v_2}{u_1})w^1 \wedge w^2, \quad dw^2 = -(u_1 - \frac{2v_1}{u_1})w^1 \wedge w^2.$$

Integrating the system Σ, we obtain the result:

Theorem 4.11. *The equations of this type are transformed by an element of* $A(2,\mathbb{R})$ *to* $y' = x^{-1}y + x^a$ (a *const* $\neq 4, 1, 0, -1$), *which admits a 2-dimensional Lie group in* $A(2,\mathbb{R})$.

4.2.2. As for the determination of the equations admitting a 1-dimensional Lie group in $A(2,\mathbb{R})$, we can use the method developed in section 3. Here is the table of the standard forms and the invariant groups. We denote by a the parameter of a 1-dimensional Lie group.

Standard Forms	Invariant Groups
$y' = F(x)$	$X = x, Y = y + a$
$y' = \frac{y}{x}F(\frac{y^r}{x^s})$	$X = a^r x, Y = a^s y$
$y' = yF(ye^{-x})$	$X = x + a, Y = e^a y$
$y' = \frac{y}{x} + F(x)$	$X = x, Y = ax + y$
$y' = \frac{y}{x} + F(xe^{r\frac{y}{x}})$	$X = e^{ar}x, Y = ae^{ar}x + e^{ar}y$
$y' = \frac{y - xF(x^2+y^2)}{x + yF(x^2+y^2)}$	rotation group
$\frac{y - xy'}{x + yy'} = F\left(\frac{y - x\tan(r\log\sqrt[2]{x^2+y^2})}{x + y\tan(r\log\sqrt[2]{x^2+y^2})}\right)$	1-dimensional conformal transformation group

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