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GENERALIZED CHILLINGWORTH CLASSES ON SUBSURFACE TORELLI GROUPS

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Abstract

The contraction of the image of the Johnson homomorphism is called the Chillingworth class. In this paper, we derive a combinatorial description of the Chillingworth class for Putman's subsurface Torelli groups. We also prove the naturality and uniqueness properties of the map whose image is the dual of the Chillingworth classes of the subsurface Torelli groups. Moreover, we relate the Chillingworth class of the subsurface Torelli group to the partitioned Johnson homomorphism.

1. Introduction

The Torelli group of an oriented surface with genus g and n boundary components $\Sigma_{g,n}$, $\mathcal{I}(\Sigma_{g,n})$, is the normal subgroup of the mapping class group $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ that acts trivially on $H_1(\Sigma_{g,n};\mathbb{Z})$. In the study of the Torelli group, the Johnson homomorphism has an important role. The Johnson homomorphism determines the abelianization of $\mathcal{I}(\Sigma_{g,1})$ mod torsion [6]. As no finite presentation for the Torelli group is known, the finiteness information inherent in the abelianization of the Torelli group is a fundamental tool.

The tensor contraction of the image of the Johnson homomorphism is the Chillingworth class. The Chillingworth class was first defined by Earle [4] by using complex analytic methods. Johnson in [5] defined the Chillingworth class by considering Chillingworth's conjecture in [2]. In [5], Johnson called the homomorphism $t : \mathcal{I}(\Sigma_{g,1}) \to H_1(\Sigma_{g,1};\mathbb{Z})$ sending each $f \in \mathcal{I}(\Sigma_{g,1})$ to the Chillingworth class of f the Chillingworth homomorphism.

In [7], Putman defined the subsurface Torelli groups in order to use inductive arguments in the Torelli group. An embedding of a subsurface $\Sigma_{g,n}$ into a larger surface $\Sigma_{g'}$ gives a partition $\mathcal{P}$ of the boundary components of $\Sigma_{g,n}$ and this partition records which of the boundary components of $\Sigma_{g,n}$ become homologous in $\Sigma_{g'}$ [3]. Putman [7] defined the subsurface Torelli group $\mathcal{I}(\Sigma_{g,n},\mathcal{P})$ by restricting $\mathcal{I}(\Sigma_{g'})$ to $\Sigma_{g,n}$. The subsurface Torelli groups $\mathcal{I}(\Sigma_{g,n},\mathcal{P})$ restore functoriality and are therefore of central importance to the study of the Torelli group.

In this paper, we construct a combinatorial description of the Chillingworth class of the subsurface Torelli groups via winding numbers in the projective tangent bundle of $\Sigma_{g,n}$. Given the definition of Putman's subsurface Torelli groups, the difficulty in finding a combinatorial description via winding numbers is to make sense of the winding number of an arc with end points on the boundary of the subsurface. By defining a difference cocycle on the projective tangent bundle of the surface we are able to make sense of the winding number

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of the difference of two arcs.

The rest of this paper is structured as follows:

In Section 2, we review basic definitions related to the Torelli group and the subsurface Torelli groups. We define a symplectic basis for the homology group $H_1^{\mathcal{P}}(\Sigma_{g,n};\mathbb{Z})$ defined by Putman [7].

In Section 3, we construct a well-defined map $\widetilde{e}_X : \mathcal{I}(\Sigma_{g,n},\mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma_{g,n};\mathbb{Z}),\mathbb{Z})$ using the projective tangent bundle of $\Sigma_{g,n}$. Here, X is a nonvanishing vector field on $\Sigma_{g,n}$. We show that $\widetilde{e}_X$ is a homomorphism. We call the dual of $\widetilde{e}_X(f)$ *the Chillingworth class of* f. One reason for calling this dual the Chillingworth class, is that it is shown to factor through the partitioned Johnson homomorphism. Therefore, we obtain a combinatorial description of the Chillingworth class of the subsurface Torelli groups using the projective tangent bundle of $\Sigma_{g,n}$.

We use the Torelli category $\mathcal{T}$Surf defined by Church [3], which is the refinement of the category TSur defined by Putman [7]. The Torelli group is a functor from $\mathcal{T}$Surf to the category of groups and homomorphisms [7]. For a morphism $i : (\Sigma_{g,n},\mathcal{P}) \to (\Sigma_{g',n'},\mathcal{P}')$ of $\mathcal{T}$Surf and a nonvanishing vector field X on $\Sigma_{g',n'}$, we prove the following:

Main Theorem. *There exists a homomorphism i'_* such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{I}(\Sigma_{g,n},\mathcal{P}) \arrow[r, "i_*"] \arrow[d, "\widetilde{e}_Y"'] & \mathcal{I}(\Sigma_{g',n'},\mathcal{P}') \arrow[d, "\widetilde{e}_X"] \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma_{g,n};\mathbb{Z}),\mathbb{Z}) \arrow[r, "i'_*"'] & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma_{g',n'};\mathbb{Z}),\mathbb{Z}) \end{tikzcd} \tag{1.1}$$

Here Y is the restriction of X to $\Sigma_{g,n}$.

We also prove that $\widetilde{e}_Y$ is unique in the sense that it is the only nontrivial homomorphism such that diagram (1.1) commutes. A commutative diagram for the Chillingworth homomorphism $t_{(\Sigma_{g,n},\mathcal{P})} : \mathcal{I}(\Sigma_{g,n},\mathcal{P}) \to H_1^{\mathcal{P}}(\Sigma_{g,n};\mathbb{Z})$ is also obtained.

2. Preliminaries

In this section, we review some background knowledge and give some preliminary definitions that will be used throughout the paper.

The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ onto itself which fix the boundary components of $\Sigma_{g,n}$ pointwise.

Throughout this paper, we will be working with representatives of mapping classes that fix a neighborhood of the boundary pointwise. We will use the notation $f \circ h$ or fh to denote the composition of maps, where h is assumed to be applied first.

The subgroup of $\mathcal{M}(\Sigma_{g,1})$ acting trivially on $H_1(\Sigma_{g,1};\mathbb{Z})$ is a normal subgroup of $\mathcal{M}(\Sigma_{g,1})$ and is called the *Torelli group*. In other words, the Torelli group is the kernel of the symplectic representation $\rho : \mathcal{M}(\Sigma_{g,1}) \to \mathrm{Sp}(2g,\mathbb{Z})$. It will be denoted by the symbol $\mathcal{I}(\Sigma_{g,1})$.

Winding Number: If a surface $\Sigma_{g,n}$ has nonempty boundary, a nonvanishing vector field X on $\Sigma_{g,n}$ exists. By choosing an appropriate parametrisation for a smooth closed curve, it can be assumed without loss of generality that the curve has a nonvanishing tangent vector at each point of the curve.

Let us choose a Riemannian metric on $\Sigma_{g,n}$ with which we define a norm on $T_x\Sigma_{g,n}$, the tangent space to $\Sigma_{g,n}$ at $x \in \Sigma_{g,n}$, for each $x \in \Sigma_{g,n}$.

Informally, given a nonvanishing vector field X, the *winding number $w_X(\gamma)$ of a smooth closed oriented curve* γ on a surface is defined as the number of rotations its tangent vector makes with respect to X as γ is traversed once in the positive direction [1].

The Chillingworth Class: In [5], Johnson defined the following homomorphism

$$e : \mathcal{I}(\Sigma_{g,1}) \to H^1(\Sigma_{g,1}; \mathbb{Z})$$

such that $e(f)([\gamma]) = w_X(f\gamma) - w_X(\gamma)$.

For $f \in \mathcal{I}(\Sigma_{g,1})$, in Section 5 of [5], Johnson dualized the class $e(f)$ to a homology class $t(f)$ defined as follows: $[\gamma] \cdot t(f) = e(f)[\gamma]$. The homology class $t(f)$ is called the *Chillingworth class* of f. In [5], Johnson proved that $C(\tau(f)) = t(f)$ holds for any $f \in \mathcal{I}(\Sigma_{g,1})$, where τ is the Johnson homomorphism and C is the tensor contraction map. The Johnson homomorphism $\tau : \mathcal{I}(\Sigma_{g,1}) \to \bigwedge^3 H_1(\Sigma_{g,1}; \mathbb{Z})$ is a surjective homomorphism. The tensor contraction map $C : \bigwedge^3 H_1(\Sigma_{g,1}; \mathbb{Z}) \to H_1(\Sigma_{g,1}; \mathbb{Z})$ is defined as follows:

$$C(x \wedge y \wedge z) = 2[(x \cdot y)z + (y \cdot z)x + (z \cdot x)y],$$

where $\cdot$ denotes the intersection pairing of homology classes.

2.1. Subsurface Torelli Groups. A partitioned surface is a pair $(\Sigma, \mathcal{P})$ consisting of a surface Σ and a partition $\mathcal{P}$ of the boundary components of Σ. Note that when the genus and the number of boundary components are not important, we use Σ to denote the surface. Each element P_k of $\mathcal{P}$ is called a block. If each element of the partition contains only one boundary component, it is called a *totally separated surface* [3].

For a given embedding $i : \Sigma \hookrightarrow \Sigma_g$, let the connected components of $\Sigma_g \backslash \Sigma^\circ$ be $\{S_0, S_1, \ldots, S_m\}$ and let P_k denote the set of boundary components of S_k for each $k \in \{0, \ldots, m\}$. Here, Σ° denotes the interior of Σ. Consider the partition

$$\mathcal{P} = \{P_0, P_1, \ldots, P_m\}.$$

Then $i : \Sigma \hookrightarrow \Sigma_g$ is called a capping of $(\Sigma, \mathcal{P})$ (c.f. [7]).

For a partitioned surface $(\Sigma, \mathcal{P})$, in [7] Putman defined the subsurface Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ to be the subgroup $i_*^{-1}(\mathcal{I}(\Sigma_g))$ of $\mathcal{M}(\Sigma)$ for any capping $i : \Sigma \hookrightarrow \Sigma_g$.

In [7], Section 3, a special homology group $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ is defined on a partitioned surface $(\Sigma, \mathcal{P})$ such that $\mathcal{I}(\Sigma, \mathcal{P})$ is the kernel of $\mathcal{M}(\Sigma) \to \mathrm{Aut}(\mathrm{H}_1^{\mathcal{P}}(\Sigma; \mathbb{Z}))$.

Consider a partition

$$\mathcal{P} = \{\{\partial_1^1, \ldots, \partial_{k_1}^1\}, \ldots, \{\partial_1^m, \ldots, \partial_{k_m}^m\}\}.$$

Suppose the boundary components ∂_i^j are oriented so that $\sum_{i,j}[\partial_i^j] = 0$ in $H_1(\Sigma; \mathbb{Z})$. Define the homology group

$$\overline{H}_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) := H_1(\Sigma; \mathbb{Z})/\partial H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}),$$

where

$$\partial H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) = \left\langle ([\partial_1^1] + \ldots + [\partial_{k_1}^1]), \ldots, ([\partial_1^m] + \ldots + [\partial_{k_m}^m]) \right\rangle \subset H_1(\Sigma; \mathbb{Z}).$$

DEFINITION 2.1 ([7], SECTION 3.1). Let $(\Sigma, \mathcal{P})$ be a partitioned surface, and let Q denote a set containing one point from each boundary component of Σ. The homology group $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ is defined to be the image of the following subgroup of $H_1(\Sigma, Q; \mathbb{Z})$ in $H_1(\Sigma, Q; \mathbb{Z})/\partial H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$:

$$\langle\{[h] \in H_1(\Sigma, Q; \mathbb{Z}) | \; h \text{ is either a simple closed curve or a properly embedded arc } a \text{ connecting two boundary curves in the same block of } \mathcal{P} \text{ and with } \partial a \in Q\}\rangle$$

One can easily see that $\mathcal{M}(\Sigma)$ acts on $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. In Theorem 3.3 of [7], Putman proves that the subsurface Torelli group $\mathcal{I}(\Sigma, \mathcal{P})$ is the subgroup of $\mathcal{M}(\Sigma)$ that acts trivially on $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$.

A $\mathcal{P}$-separating curve on a partitioned surface $(\Sigma, \mathcal{P})$ is a simple closed curve γ with $[\gamma] = 0$ in $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. A twist about $\mathcal{P}$-bounding pair is defined to be $T_{\gamma_1} T_{\gamma_2}^{-1}$, where γ_1 and γ_2 are disjoint, nonisotopic simple closed curves and $[\gamma_1] = [\gamma_2]$ in $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. For $g \geq 1$, $\mathcal{I}(\Sigma_{g,n}, \mathcal{P})$ is generated by twists about $\mathcal{P}$-separating curves and twists about $\mathcal{P}$-bounding pairs [7].

A category TSur was defined in [7] such that $\mathcal{I}(\Sigma_{g,n}, \mathcal{P})$ is a functor from TSur to the category of groups and homomorphisms. The objects of TSur are partitioned surfaces $(\Sigma, \mathcal{P})$ and the morphisms from $(\Sigma_{g_1,n_1}, \mathcal{P}_1)$ to $(\Sigma_{g_2,n_2}, \mathcal{P}_2)$ are exactly those embeddings $i : \Sigma_{g_1,n_1} \hookrightarrow \Sigma_{g_2,n_2}$ which induce morphisms $i_* : \mathcal{I}(\Sigma_{g_1,n_1}, \mathcal{P}_1) \to \mathcal{I}(\Sigma_{g_2,n_2}, \mathcal{P}_2)$. The embeddings satisfy the following condition: for any $\mathcal{P}_1$-separating curve γ, the curve $i(\gamma)$ must be a $\mathcal{P}_2$-separating curve. In this paper, we will use the refinement of this category defined by Church in [3].

Before giving the definition of the category defined by Church, we need to describe a construction of a minimal totally separated surface $\widehat{\Sigma}$ containing Σ.

REMARK 2.2 ([3]). Given a partitioned surface $(\Sigma, \mathcal{P})$, a minimal totally separated surface containing Σ can be constructed as follows: For each $P \in \mathcal{P}$ with $|P| = n$, we attach a sphere with $n + 1$ boundary components to the n boundary components in P of Σ to obtain $\widehat{\Sigma}$ with a partition $\widehat{\mathcal{P}}$. Each element of the partition $\widehat{\mathcal{P}}$ contains only one boundary component.

NOTATION. Given a partitioned surface $(\Sigma, \mathcal{P})$, the partitioned surface $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ will denote a minimal totally separated surface containing Σ.

Note that $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z})$ is isomorphic to $H_1^{P}(\Sigma; \mathbb{Z})$.

DEFINITION 2.3 ([3], SECTION 2.3). Objects of the Torelli category $\mathcal{T}$Surf are partitioned surfaces $(\Sigma, \mathcal{P})$. A morphism from $(\Sigma_1, \mathcal{P}_1)$ to $(\Sigma_2, \mathcal{P}_2)$ is an embedding $i : \Sigma_1 \hookrightarrow \Sigma_2$ satisfying the following conditions:

- i takes $\mathcal{P}_1$-separating curves to $\mathcal{P}_2$-separating curves.
- i extends to an embedding $\widehat{i} : \widehat{\Sigma}_1 \hookrightarrow \widehat{\Sigma}_2$.

In [3], given a surface $(\Sigma, \mathcal{P})$ with $\mathcal{P} = \{P_0, P_1, \ldots, P_k\}$, Church defined the partitioned Johnson homomorphism $\tau_{(\Sigma,\mathcal{P})}$ with image $W_{(\Sigma,\mathcal{P})}$ given in Definition 5.8 of [3]. The definition of the partitioned Johnson homomorphism is similar to the definition of the Johnson homomorphism. Church stated in [3], Definition 5.12, that $W_{(\Sigma,\mathcal{P})}$ can be considered to be a subspace of $\bigwedge^3 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}) \oplus (\mathbb{Z}^k \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma}; \mathbb{Z}))$. Basis elements of $W_{(\Sigma,\mathcal{P})}$ is shown to be $a \wedge b \wedge c$

for the $\bigwedge^3 H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$ component and as $z_i \wedge x$ for $\mathbb{Z}^k \otimes H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$, where $a, b, c, x \in H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$ and z_i is the boundary component of $\widehat{\Sigma}$ corresponding to $P_i \in \mathcal{P}$ for each $1 \le i \le k$.

2.2. Symplectic Basis for $\mathbf{H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})}$. In this section, we introduce a symplectic basis for $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$.

Let $(\Sigma, \mathcal{P})$ be a partitioned surface of genus g with the partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, $|P_l| = n_l$. For an $l = 1, 2, \ldots, n$, suppose that $P_l = \{\partial_1^l, \partial_2^l, \ldots, \partial_{n_l}^l\}$. Let $\mathcal{Q}$ be a subset of the boundary $\partial\Sigma$ containing exactly one point from each boundary component.

Let us choose a set of simple closed curves $\{x_1, y_1, x_2, y_2, \ldots, x_g, y_g\}$ on Σ satisfying

- $x_i \cap x_j = \emptyset, \quad x_i \cap y_j = \emptyset, \quad y_i \cap y_j = \emptyset$ for $i \neq j$,
- x_i intersects y_i transversely at one point, and
- under the filling map

$$H_1(\Sigma;\mathbb{Z}) \to H_1(\overline{\Sigma};\mathbb{Z})$$

  $\{[x_i], [y_i] \mid i = 1, \ldots, g\}$ maps to a symplectic basis of $H_1(\overline{\Sigma};\mathbb{Z})$. Here, $\overline{\Sigma}$ denotes the closed surface obtained by gluing a disc along each boundary component and the filling map is induced by inclusion.

For each $l = 1, 2, \ldots, n$, choose oriented arcs h_j^l connecting $\partial_j^l \cap \mathcal{Q}$ to $\partial_{j+1}^l \cap \mathcal{Q}$ for $j = 1, 2, \ldots, n_l - 1$ such that

- h_j^l are disjoint from x_i, y_i,
- h_j^l are pairwise disjoint except perhaps at endpoints,
- each h_j^l is oriented so that the algebraic intersection number of the homology classes $[h_j^l]$ and $[\partial_1^l + \cdots + \partial_j^l]$ is 1, where the orientations of the boundary components are induced from the orientation of the surface.

The union of the sets

- $\{[x_1], [y_1], \ldots, [x_g], [y_g]\}$,
- $\{[h_1^1], [h_2^1], \ldots, [h_{n_1-1}^1], [h_1^2], \ldots, [h_{n_2-1}^2], \ldots, [h_1^n], \ldots, [h_{n_n-1}^n]\}$,
- $\{[\partial_1^1], [\partial_1^1 + \partial_2^1], \ldots, [\partial_1^1 + \cdots + \partial_{n_1-1}^1], [\partial_1^2], \ldots, [\partial_1^2 + \cdots + \partial_{n_2-1}^2], \ldots, [\partial_1^n], \ldots, [\partial_1^n + \cdots + \partial_{n_n-1}^n]\}$

is a basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$.

In this basis, $\{x_i, y_i\}$ are closed curves, the $\{h_j^l\}$s are arcs, and $\{\partial_j^l\}$s are boundary curves as shown in Figure 1.

This basis $\mathscr{B}$ has the following properties:

- $\widehat{i}([x_i], [x_j]) = \widehat{i}([y_i], [y_j]) = 0, \qquad \widehat{i}([x_i], [y_j]) = \delta_{ij}, \quad$ for all $1 \le i, j \le g$,
- $\widehat{i}([h_i^l], [\partial_1^l + \cdots + \partial_j^l]) = \delta_{ij}, \quad$ for all $1 \le i, j \le n_l - 1, 1 \le l \le n$,
- $\widehat{i}([h_j^l], [x_i]) = \widehat{i}([h_j^l], [y_i]) = 0,$ for all $1 \le i \le g, 1 \le j \le n_l - 1, 1 \le l \le n$,
- $\widehat{i}([\partial_1^l + \cdots + \partial_j^l], [x_i]) = \widehat{i}([\partial_1^l + \cdots + \partial_j^l], [y_i]) = 0,$ for all $1 \le i \le g, 1 \le j \le n_l - 1, 1 \le l \le n$,
- $\widehat{i}([h_i^l], [h_j^n]) = 0,$ for any i, j, l, n.

Here, δ_{ij} denotes the Kronecker delta and $\widehat{i}(\cdot,\cdot)$ denotes the algebraic intersection number.

Remark 2.4. The pairing $\widehat{i}(\cdot,\cdot)$ is induced from the pairing on $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. We can define an isomorphism from $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ to $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$ as follows: Suppose that $\widehat{\Sigma}$ is the totally separated

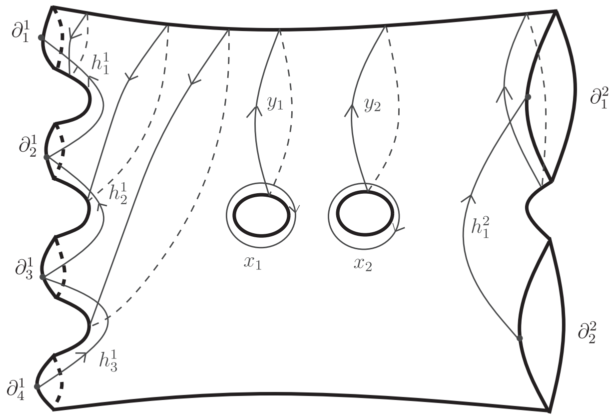

Fig. 1. An example illustrating homology basis elements of $H_1^{\mathcal{P}}(\Sigma_{2,6};\mathbb{Z})$, where $\mathcal{P} = \{\{\partial_1^1, \partial_2^1, \partial_3^1, \partial_4^1\}, \{\partial_1^2, \partial_2^2\}\}$.

surface, with the partition $\widehat{\mathcal{P}}$, obtained by gluing a sphere S_l with $n_l + 1$ holes along the boundary components in P_l, i.e. the minimal totally separated surface containing Σ (c.f. Remark 2.2). For each $j = 1, 2, \ldots, n_l - 1$, choose smooth arcs k_j^l on the complement $\widehat{\Sigma} \setminus \Sigma^\circ$ connecting $Q \cap \partial_j^l$ to $Q \cap \partial_{j+1}^l$. Here, k_j^l are pairwise disjoint except perhaps at endpoints. Let us orient each k_j^l so that concatenation $h_j^l * k_j^l$ is a smooth closed oriented curve in $\widehat{\Sigma}$, where $[h_j^l]$ is an element of the basis $\mathscr{B}$. Let $\mathcal{P}_l = \{P_l, \{z_l\}\}$ be the partition of the boundary of S_l, where z_l is the boundary component of $\widehat{\Sigma}$. Then $K_l = \{[k_j^l]\}$ is a set of basis elements with arc representatives of $H_1^{\mathcal{P}_l}(S_l;\mathbb{Z})$. Let $\mathscr{K}$ denote the union $K_1 \cup K_2 \cup \cdots \cup K_n$. We then have an isomorphism

$$\psi_{\mathscr{K}} : H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \to H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$$

by mapping basis elements with closed curve representatives to themselves and $[h_j^l]$ to $[h_j^l * k_j^l]$. One can observe that $\psi_{\mathscr{K}}$ preserves the algebraic intersection form, i.e. for any $a, b \in H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ we have $\widehat{i}(a, b) = \widehat{i}(\psi_{\mathscr{K}}(a), \psi_{\mathscr{K}}(b))$.

We now define the dual of a homology class of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ by using this intersection form. Note that the intersection form $\widehat{i}$ is nondegenerate. Therefore the map

$$D : H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}), \mathbb{Z})$$

sending $[x] \in H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ to $\widehat{i}(\cdot, [x])$ is an isomorphism.

3. Results

In this section, we construct a well-defined map $\widetilde{e}_X$ by means of the projective tangent bundle. We prove that $\widetilde{e}_X$ and the homomorphism obtained by taking the dual of $\widetilde{e}_X(f)$ for any $f \in \mathcal{I}(\Sigma, \mathcal{P})$ satisfy the naturality property. We define the homomorphism from the subsurface Torelli groups to $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ obtained by taking dual of $\widetilde{e}_X(f)$ to be the Chillingworth homomorphism of the subsurface Torelli groups. Moreover, we show that $\widetilde{e}_X$ is the unique nontrivial homomorphism satisfying naturality. Finally, we relate the Chillingworth classes of the subsurface Torelli groups to the partitioned Johnson homomorphism defined by Church.

In this section, if $(\Sigma_1, \mathcal{P}_1)$ and $(\Sigma_2, \mathcal{P}_2)$ are partitioned surfaces, then by an embedding $i : (\Sigma_1, \mathcal{P}_1) \hookrightarrow (\Sigma_2, \mathcal{P}_2)$ of partitioned surfaces, we mean a morphism $i : (\Sigma_1, \mathcal{P}) \to (\Sigma_2, \mathcal{P})$ of $\mathcal{T}$Surf.

3.1. Winding Number In The Projective Tangent Bundle. This section starts with the definition of the projective tangent bundle and we introduce the winding number in the projective tangent bundle.

Let Σ be a smooth compact connected oriented surface with nonempty boundary. Let us choose a Riemannian metric on Σ. Let $UT(\Sigma)$ be the unit tangent bundle of Σ. Since Σ has nonempty boundary, there are nonvanishing vector fields on Σ. Choice of two nonvanishing vector fields which are orthogonal to each other gives a parallelization of Σ. The unit tangent bundle $UT(\Sigma)$ is therefore homeomorphic to $\Sigma \times S^1$.

By using this unit tangent bundle, the projective tangent bundle $PT(\Sigma)$ is constructed as follows: By identifying antipodal points in each fiber S^1, we obtain a fiber bundle whose fiber is $\mathbb{RP}^1$, which is homeomorphic to S^1. The projective tangent bundle $PT(\Sigma)$ is also homeomorphic to the product $\Sigma \times S^1$ since Σ has nonempty boundary.

Let $\{[\alpha_i]\}_{i\in I} \cup \{[x_j], [y_j]\}_{j\in J}$ be a basis for $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. Here, I and J are finite index sets, each α_i is an arc, and each x_j, y_j is a simple closed curve. We assume that all representatives are smooth.

In this paper, we always take representatives of mapping classes that fix points in a regular neighborhood of each boundary component. Therefore, $f(\alpha_i)$ and α_i have the same tangent vectors on a small neighborhood of the boundary components. We denote by $f(\alpha_i) * \alpha_i^{-1}$ the closed curve obtained by first traversing the arc $f(\alpha_i)$ then α_i with the reversed orientation. The resulting closed curve has two nondifferentiable points on the boundary of the subsurface. Since $f(\alpha_i)$ and α_i have the same tangent vectors at the end points, in the projective tangent bundle we can calculate the winding number of closed oriented curves having two such nondifferentiable points on the boundary. When we concatenate arcs to obtain a closed curve, we will assume that the tangent spaces of the arcs at the end points coincide.

The winding number in the projective tangent bundle is defined in analogy to the winding number in the tangent bundle. We define winding number in the projective tangent bundle for smooth closed oriented curves or for closed oriented curves constructed by concatenating a pair of smooth arcs as just described.

Let us denote the winding number in the projective tangent bundle of a closed oriented curve γ with respect to a nonvanishing vector field X by $\widetilde{w}_X(\gamma)$. Since S^1 is a double cover of $\mathbb{RP}^1$, for a smooth closed oriented curve γ we have $w_X(\gamma) = \frac{\widetilde{w}_X(\gamma)}{2}$.

3.2. Construction of $\widetilde{\mathbf{e}}_{\mathbf{X}}$. In this section our aim is to define a well-defined map $\widetilde{e}_X$: $\mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$.

Let X be a nonvanishing vector field on a partitioned surface $(\Sigma, \mathcal{P})$ and f be an element of the subsurface Torelli group of $(\Sigma, \mathcal{P})$. Choose a set of simple closed curves representing a basis of $H_1(\Sigma; \mathbb{Z})$. Assigning an integer to each basis element determines a homomorphism from $H_1(\Sigma; \mathbb{Z})$ to $\mathbb{Z}$. This integer is chosen to be the total number of times that X rotates relative to $f^{-1}X$ as we traverse the basis element. This homomorphism, denoted by $d(X, f^{-1}X)$, is defined in [1]. By Lemma 4.1 in [1], we have

$$d(X, f^{-1}X)[\gamma] = w_X(f\gamma) - w_X(\gamma),$$

for any smooth closed oriented curve γ. In the projective tangent bundle we get

$$d(X, f^{-1}X)[\gamma] = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Since f fixes every boundary component of Σ, $d(X, f^{-1}X)[\partial] = 0$ for any boundary component ∂. Therefore, $d(X, f^{-1}X)$ induces a homomorphism $\overline{d}(X, f^{-1}X) : \overline{H}_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \to \mathbb{Z}$ defined by

$$\overline{d}(X, f^{-1}X)[\gamma] = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Now our aim is to get a well-defined map

$$\widetilde{d}(X, f^{-1}X) : H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \to \mathbb{Z}$$

mapping an element $[\alpha]$ of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ to the half of the number of times that X rotates relative to $f^{-1}X$ in the projective tangent bundle as we traverse α.

For a closed oriented curve γ, we define

$$\widetilde{d}(X, f^{-1}X)[\gamma] = \overline{d}(X, f^{-1}X)[\gamma].$$

Now, let h be a smooth oriented arc whose endpoints are on the boundary components of Σ contained in the same element of $\mathcal{P}$ and let $f \in \mathcal{I}(\Sigma, \mathcal{P})$. Since f fixes all points of a regular neighborhood of the boundary components, h and $f(h)$ have the same tangent spaces at the end points and $f(h) * h^{-1}$ is a closed oriented curve with two cusps. We define

$$\widetilde{d}(X, f^{-1}X)[h] := \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}.$$

For each $P \in \mathcal{P}$ with $|P| = n$, let us attach a sphere with $n + 1$ boundary components to the n boundary components in P of Σ to obtain $\widehat{\Sigma}$ with a partition $\widehat{\mathcal{P}}$ as in Remark 2.2. Thus, $(\widehat{\Sigma}, \widehat{\mathcal{P}})$ is totally separated. Extend X to the obtained larger surface $\widehat{\Sigma}$ so that it is again a nonvanishing vector field on $\widehat{\Sigma}$. For simplicity, the extension will also be denoted by X. Let h_1 be a smooth oriented arc in the complement of Σ whose end points are ∂h. Let $\gamma := h * h_1$ denote the smooth closed oriented curve obtained by concatenating h and h_1. Notice that we choose a consistent orientation for h_1 to get a closed oriented curve γ. We parametrize γ such that its initial and terminal points are on one of the boundary components of the subsurface Σ. Then $f\gamma$ is isotopic to $f(h) * h_1$.

REMARK 3.1. The winding number in the projective tangent bundle of the concatenation of smooth closed oriented curves is equal to the sum of the winding numbers of each smooth closed oriented curve if the tangent spaces of the curves at the end points are the same. Therefore, we obtain the following equalities:

$$\begin{aligned}\frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2} &= \frac{\widetilde{w}_X(f\gamma * \gamma^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(h) * h_1 * (h * h_1)^{-1})}{2}\end{aligned}$$

$$= \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}.$$

One can easily observe that the obtained equality

$$\frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2} = \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}$$

does not depend on the choice of the arc representative h_1 on $\widehat{\Sigma} \setminus \Sigma^\circ$.

Lemma 3.2. *Let h be a smooth oriented arc representing a homology class $[h]$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Then the number $\frac{\widetilde{w}_X(f(h)*h^{-1})}{2}$ is independent of the choice of the representative of $[h]$.*

Proof. Let $[h] = [h']$ be in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Then we have $[h * h'^{-1}] = 0$ in $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$. Since the embedding $(\Sigma, \mathcal{P}) \hookrightarrow (\widehat{\Sigma}, \widehat{\mathcal{P}})$ of partitioned surfaces takes $\mathcal{P}$-separating curves to $\widehat{\mathcal{P}}$-separating curves by the first condition of Definition 2.3, we get $[h * h'^{-1}] = 0$ in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. We have $[\gamma] = [\gamma']$ by using the following equalities:

$$[h * h'^{-1}] = [h * h_1 * h_1^{-1} * h'^{-1}] = [h * h_1] - [h' * h_1] = 0,$$

where $[\gamma] = [h * h_1]$ and $[\gamma'] = [h' * h_1]$.

Since we have

$$\frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2} = \frac{\widetilde{w}_X(f\gamma') - \widetilde{w}_X(\gamma')}{2},$$

for any smooth homologous simple closed curves γ and γ' in $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$, we get

$$\frac{\widetilde{w}_X(f(h) * h^{-1})}{2} = \frac{\widetilde{w}_X(f(h') * h'^{-1})}{2}.$$

□

Lemma 3.3. *The map $\widetilde{d}(X, f^{-1}X) : H_1^{\mathcal{P}}(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ is a homomorphism.*

Proof. For smooth closed oriented curves γ_1 and γ_2 by the definition of $d(X, f^{-1}X)$, we have

$$\widetilde{d}(X, f^{-1}X)[\gamma_1 * \gamma_2] = \widetilde{d}(X, f^{-1}X)[\gamma_1] + \widetilde{d}(X, f^{-1}X)[\gamma_2].$$

Let h_1 and h_2 be smooth oriented arcs whose endpoints are on the boundary components of Σ contained in the same element of $\mathcal{P}$ and let us assume that the initial point of h_2 is the same as the terminal point of h_1. Let $[h]$ denote the sum of two homology classes $[h_1]$ and $[h_2]$. We obtain the following equalities:

$$\begin{aligned}
\widetilde{d}(X, f^{-1}X)[h_1] + \widetilde{d}(X, f^{-1}X)[h_2] &= \frac{\widetilde{w}_X(f(h_1) * h_1^{-1})}{2} + \frac{\widetilde{w}_X(f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(h_1^{-1} * f(h_1))}{2} + \frac{\widetilde{w}_X(f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(h_1^{-1} * f(h_1) * f(h_2) * h_2^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(h_1) * f(h_2) * h_2^{-1} * h_1^{-1})}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\widetilde{w}_X(f(h_1 * h_2) * (h_1 * h_2)^{-1})}{2} \\
&= \widetilde{d}(X, f^{-1}X)[h_1 * h_2] \\
&= \widetilde{d}(X, f^{-1}X)[h].
\end{aligned}
$$

Now let $[h']$ denote a homology class whose representatives are arcs. Let γ be a smooth oriented arc whose homology class $[\gamma]$ is the sum of $[h']$ and a homology class $[\alpha]$ with closed curve representatives. As in the previous paragraph of Remark 3.1, we can obtain a smooth closed oriented curve α' by concatenating h' with a smooth oriented arc in the complement of Σ. Hence, we have

$$
\begin{aligned}
\widetilde{d}(X, f^{-1}X)[h'] + \widetilde{d}(X, f^{-1}X)[\alpha] &= \frac{\widetilde{w}_X(f(h') * h'^{-1})}{2} + \frac{\widetilde{w}_X(f\alpha) - \widetilde{w}_X(\alpha)}{2} \\
&= \frac{\widetilde{w}_X(f\alpha') - \widetilde{w}_X(\alpha')}{2} + \frac{\widetilde{w}_X(f\alpha) - \widetilde{w}_X(\alpha)}{2} \\
&= \widetilde{d}(X, f^{-1}X)[\alpha' * \alpha] \\
&= \widetilde{d}(X, f^{-1}X)[h' * \alpha] \quad \text{(by Remark 3.1)} \\
&= \widetilde{d}(X, f^{-1}X)[\gamma].
\end{aligned}
$$

□

DEFINITION 3.4. The map $\widetilde{e}_X : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ is defined to be $\widetilde{e}_X(f) := \widetilde{d}(X, f^{-1}X)$. More explicitly, it is defined as follows:

If $[\gamma]$ has a smooth closed curve representative γ,

$$
\widetilde{e}_X(f)[\gamma] := \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.
$$

If h is a smooth oriented arc representing a homology class $[h]$ in $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$,

$$
\widetilde{e}_X(f)[h] := \frac{\widetilde{w}_X(f(h) * h^{-1})}{2}.
$$

Lemma 3.5. *The map* $\widetilde{e}_X : \mathcal{I}(\Sigma, \mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$ *is a homomorphism.*

Proof. By Lemma 5B of [5], it is easy to see that $\widetilde{e}_X(fg)[\gamma] = \widetilde{e}_X(f)[\gamma] + \widetilde{e}_X(g)[\gamma]$ for a smooth closed oriented curve γ.

For a smooth oriented arc α_i,

$$
\begin{aligned}
\widetilde{e}_X(fg)[\alpha_i] &= \frac{\widetilde{w}_X(fg(\alpha_i) * \alpha_i^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(g\alpha_i) * g(\alpha_i^{-1}) * g(\alpha_i) * \alpha_i^{-1})}{2} \\
&= \frac{\widetilde{w}_X(f(g\alpha_i) * g(\alpha_i^{-1}))}{2} + \frac{\widetilde{w}_X(g(\alpha_i) * \alpha_i^{-1})}{2} \\
&= \widetilde{e}_X(f)[g(\alpha_i)] + \widetilde{e}_X(g)[\alpha_i].
\end{aligned}
$$

Since $g \in \mathcal{I}(\Sigma, \mathcal{P})$, $g(\alpha_i)$ and α_i represent the same element of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. Hence we get

$$
\widetilde{e}_X(fg) = \widetilde{e}_X(f) + \widetilde{e}_X(g).
$$

□

Notice that $\widetilde{e}_X$ depends on the choice of the nonvanishing vector field X.

3.3. Naturality and Uniqueness of $\widetilde{\mathbf{e}}_{\mathbf{X}}$. In this section, we show that $\widetilde{e}_X$ is natural and that it is the unique nontrivial natural homomorphism from $\mathcal{I}(\Sigma, \mathcal{P})$ to $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z})$.

REMARK 3.6. Suppose that $(\Sigma, \mathcal{P})$ is a totally separated surface with boundary components $z_1, z_2, \ldots, z_n$, so that $\mathcal{P} = \{\{z_1\}, \ldots, \{z_n\}\}$. Suppose also that Σ' is a partitioned surface with a partition $\mathcal{P}'$ such that there is an embedding $(\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ of partitioned surfaces. For $1 \leq j \leq n$, let V_j be a connected component of $\Sigma' \setminus \Sigma^\circ$ containing z_j as a boundary component and let $\mathcal{P}_j$ be the partition of the boundary of V_j consisting of $\{z_j\}$ and a subset of $\mathcal{P}'$. By identifying $H_1^{\mathcal{P}_j}(V_j; \mathbb{Z})$ and $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$ with their images in $H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z})$, we can write

$$H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}) = H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}) \oplus H_1^{\mathcal{P}_1}(V_1; \mathbb{Z}) \oplus \cdots \oplus H_1^{\mathcal{P}_n}(V_n; \mathbb{Z}).$$

If Σ is totally separated with the partition $\mathcal{P}$ and if $i : (\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ is an embedding of partitioned surfaces, then there is a natural projection

$$r_* : H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}) \to H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$$

which gives a natural homomorphism

$$r^* : \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}), \mathbb{Z}).$$

Proposition 3.7. *Let Σ be a totally separated surface with the partition $\mathcal{P}$ and let i : $(\Sigma, \mathcal{P}) \hookrightarrow (\Sigma', \mathcal{P}')$ be an embedding of partitioned surfaces. Let X be a nonvanishing vector field on Σ' and let Y denote the restriction of X to Σ. Then the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

$$\begin{tikzcd} \mathcal{I}(\Sigma, \mathcal{P}) \arrow[r, "i_*"] \arrow[d, "\widetilde{e}_Y"'] & \mathcal{I}(\Sigma', \mathcal{P}') \arrow[d, "\widetilde{e}_X"] \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma; \mathbb{Z}), \mathbb{Z}) \arrow[r, "r^*"'] & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma'; \mathbb{Z}), \mathbb{Z}) \end{tikzcd} \tag{3.1}$$

commutes.

Proof. Let $f \in \mathcal{I}(\Sigma, \mathcal{P})$, and let $i_*(f) = \widetilde{f}$. Thus (the class of) the diffeomorphism $\widetilde{f}$ is equal to f on Σ and is the identity on the complement $\Sigma' \setminus \Sigma$. We show that $r^*(\widetilde{e}_Y(f)) = \widetilde{e}_X(\widetilde{f})$.

Let γ be a smooth oriented simple closed curve in Σ representing a basis element of $H_1^{\mathcal{P}}(\Sigma; \mathbb{Z})$. Then, we have

$$r^*(\widetilde{e}_Y(f))[\gamma] = \widetilde{e}_Y(f)(r_*[\gamma]) = \widetilde{e}_Y(f)[\gamma] = \frac{\widetilde{w}_Y(f\gamma) - \widetilde{w}_Y(\gamma)}{2}$$

and

$$\widetilde{e}_X(\widetilde{f})[\gamma] = \frac{\widetilde{w}_X(\widetilde{f}\gamma) - \widetilde{w}_X(\gamma)}{2} = \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.$$

Since Y is the restriction of X to Σ, we have $r^*(\widetilde{e}_Y(f))[\gamma] = \widetilde{e}_X(\widetilde{f})[\gamma]$.

Now let γ' be a smooth closed oriented curve or smooth oriented arc in some V_j representing a homology basis element in $H_1^{\mathcal{P}_j}(V_j; \mathbb{Z})$. In this case, $r^*(\widetilde{e}_Y(f))[\gamma'] = \widetilde{e}_Y(f)(r_*([\gamma'])) = 0$ because $r_*([\gamma']) = 0$. Since $f(\gamma') = \gamma'$, we have

$$\widetilde{e}_X(\widetilde{f})[\gamma'] = \frac{\widetilde{w}_X(\widetilde{f}\gamma') - \widetilde{w}_X(\gamma')}{2} = \frac{\widetilde{w}_X(\gamma') - \widetilde{w}_X(\gamma')}{2} = 0.$$

Since $H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ is the direct sum of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ and $H_1^{\mathcal{P}_j}(V_j;\mathbb{Z})$, it follows that $r^*(\widetilde{e}_Y(f)) = \widetilde{e}_X(\widetilde{f})$ for every f in $\mathcal{I}(\Sigma,\mathcal{P})$, and hence $r^*\widetilde{e}_Y = \widetilde{e}_X i_*$. □

Suppose now that Σ is any surface with a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, $|P_l| = n_l$. Let us fix the symplectic basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ defined as in Subsection 2.2.

By using $\psi_{\mathscr{K}}$ in Remark 2.4, we get the isomorphism

$$\psi^*_{\mathscr{K}} : \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$$

defined to be $\psi^*_{\mathscr{K}}(\chi) = \chi \circ \psi_{\mathscr{K}}$ for any $\chi \in \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z})$.

Proposition 3.8. *Let $(\Sigma,\mathcal{P})$ be a partitioned surface and let $(\widehat{\Sigma},\widehat{\mathcal{P}})$ be a minimal totally separated surface containing Σ. Let $i : (\Sigma,\mathcal{P}) \hookrightarrow (\widehat{\Sigma},\widehat{\mathcal{P}})$ be an inclusion coming from an embedding of partitioned surfaces. Let X be a nonvanishing vector field on $\widehat{\Sigma}$ and let Y denote the restriction of X to Σ. Then the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

$$\begin{tikzcd}
\mathcal{I}(\Sigma,\mathcal{P}) \arrow[r, "i_*"] \arrow[d, "\widetilde{e}_Y"'] & \mathcal{I}(\widehat{\Sigma},\widehat{\mathcal{P}}) \arrow[d, "\widetilde{e}_X"] \\
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) & \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z}) \arrow[l, "\psi^*_{\mathscr{K}}"']
\end{tikzcd} \tag{3.2}$$

commutes.

Proof. Let $f \in \mathcal{I}(\Sigma,\mathcal{P})$, and let $i_*(f) = \widetilde{f}$. Thus $\widetilde{f}$ is equal to f on Σ and is the identity on the complement $\widehat{\Sigma} \setminus \Sigma$. We show that $\widetilde{e}_Y(f) = \psi^*_{\mathscr{K}}\widetilde{e}_X(\widetilde{f})$.

For any homology basis element $[\gamma] \in H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ with a smooth closed oriented curve representative γ, we have

$$\widetilde{e}_Y(f)[\gamma] = \frac{\widetilde{w}_Y(f\gamma) - \widetilde{w}_Y(\gamma)}{2}$$

and

$$\begin{aligned}
\psi^*_{\mathscr{K}}\widetilde{e}_X(\widetilde{f})([\gamma]) &= \widetilde{e}_X(\widetilde{f})(\psi_{\mathscr{K}}[\gamma]) \\
&= \widetilde{e}_X(\widetilde{f})[\gamma] \\
&= \frac{\widetilde{w}_X(\widetilde{f}\gamma) - \widetilde{w}_X(\gamma)}{2} \\
&= \frac{\widetilde{w}_X(f\gamma) - \widetilde{w}_X(\gamma)}{2}.
\end{aligned}$$

Since $X = Y$ on Σ, we get the desired equality.

For any homology basis element $[h_j^l] \in H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ with a smooth oriented arc representative h_j^l, we have

$$\widetilde{e}_Y(f)[h_j^l] = \frac{w_Y(f(h_j^l) * (h_j^l)^{-1})}{2}$$

and

$$\begin{aligned}\psi_{\mathscr{K}}^{*}\widetilde{e}_X(\widetilde{f})[h_j^l] &= \widetilde{e}_X(\widetilde{f})(\psi_{\mathscr{K}}[h_j^l]) \\ &= \widetilde{e}_X(\widetilde{f})[h_j^l * k_j^l] \\ &= \frac{\widetilde{w}_X(\widetilde{f}(h_j^l * k_j^l)) - \widetilde{w}_X(h_j^l * k_j^l)}{2}.\end{aligned}$$

Since we are working in the projective tangent bundle and we assume that representatives of mapping classes fix a regular neighborhood of the boundary components, we get

$$\begin{aligned}\psi_{\mathscr{K}}^{*}\widetilde{e}_X(\widetilde{f})[h_j^l] &= \frac{\widetilde{w}_X(\widetilde{f}(h_j^l * k_j^l) * (h_j^l * k_j^l)^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(h_j^l) * k_j^l * (k_j^l)^{-1} * (h_j^l)^{-1})}{2} \\ &= \frac{\widetilde{w}_X(f(h_j^l) * (h_j^l)^{-1})}{2} \\ &= \frac{\widetilde{w}_Y(f(h_j^l) * (h_j^l)^{-1})}{2}.\end{aligned}$$

Therefore, we obtain the equality $\widetilde{e}_Y = \psi_{\mathscr{K}}^{*}\widetilde{e}_X i_*$. This concludes the proof. □

Note that commutativity of diagram (3.2) does not depend on the choice of basis $\{[k_j^l]\} \in H_1^{\mathcal{P}_l}(S_l;\mathbb{Z})$.

Proposition 3.7 and Proposition 3.8 imply the following theorem.

Theorem 3.9. *Let $(\Sigma,\mathcal{P})$ and $(\Sigma',\mathcal{P}')$ be partitioned surfaces and $i : (\Sigma,\mathcal{P}) \hookrightarrow (\Sigma',\mathcal{P}')$ be an embedding of partitioned surfaces. Let X be a nonvanishing vector field on Σ' and let Y denote the restriction of X to Σ. Then there exists a homomorphism i'_* such that the homomorphism $\widetilde{e}_Y$ is natural in the sense that the diagram*

(3.3)
$$\begin{array}{ccc}\mathcal{I}(\Sigma,\mathcal{P}) & \xrightarrow{\quad i_* \quad} & \mathcal{I}(\Sigma',\mathcal{P}') \\ \Big\downarrow{\scriptstyle \widetilde{e}_Y} & & \Big\downarrow{\scriptstyle \widetilde{e}_X} \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) & \xrightarrow[i'_*]{} & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z})\end{array}$$

commutes.

Proof. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$, $|P_l| = n_l$ be the partition on Σ. For an $l = 1,2,\ldots,n$, suppose that $P_l = \{\partial_1^l, \partial_2^l, \ldots, \partial_{n_l}^l\}$. For each $j = 1,2,\ldots,n_l - 1$, choose smooth oriented simple arcs k_j^l on the complement $\Sigma' \backslash \Sigma^\circ$ connecting $Q \cap \partial_j^l$ to $Q \cap \partial_{j+1}^l$. Here, k_j^l are pairwise disjoint except perhaps at endpoints. We consider a closed tubular neighbourhood of the union $\partial_1^l \cup \partial_2^l \cup \cdots \cup \partial_{n_l}^l \cup k_1^l \cup \cdots k_{n_l-1}^l$. This tubular neighbourhood is homeomorphic to a sphere S_l with $n_l + 1$ holes. Let us consider now a minimal totally separated surface $(\widehat{\Sigma},\widehat{\mathcal{P}})$ containing Σ and all S_l as a subsurface.

Let us fix bases $\mathscr{B}$ and $\mathscr{K}$ as in Proposition 3.8.

Consider the composition of the embedding $\widehat{j} : (\Sigma,\mathcal{P}) \hookrightarrow (\widehat{\Sigma},\widehat{\mathcal{P}})$ of partitioned surfaces with the embedding $j' : (\widehat{\Sigma},\widehat{\mathcal{P}}) \hookrightarrow (\Sigma',\mathcal{P}')$ of partitioned surfaces. Let $\widehat{Y}$ denote the restriction of X to $\widehat{\Sigma}$. After showing that both diagrams in (3.4) are commutative, our proof will be complete.

$$\begin{tikzcd} \mathcal{I}(\Sigma,\mathcal{P}) \arrow[r, "\widehat{j}_*"] \arrow[d, "\widetilde{e}_Y"'] & \mathcal{I}(\widehat{\Sigma},\widehat{\mathcal{P}}) \arrow[r, "j'_*"] \arrow[d, "\widetilde{e}_{\widehat{Y}}"] & \mathcal{I}(\Sigma',\mathcal{P}') \arrow[d, "\widetilde{e}_X"] \\ \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \arrow[r, "(\psi^*_{\mathscr{K}})^{-1}"'] & \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z}) \arrow[r, "r^*"'] & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \end{tikzcd} \tag{3.4}$$

Proposition 3.7 implies the commutativity of the right-hand side in diagram (3.4). The commutativity of the left-hand side in diagram (3.4) follows from Proposition 3.8. Notice that the composition of $(\psi^*_{\mathscr{K}})^{-1}$ and r^* in diagram (3.4) gives the homomorphism i'_* : $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z})$ in diagram (3.3). For the sake of clarity, for any $\chi \in \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$ and $[\gamma] \in H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ we get

$$i'_*(\chi)[\gamma] = r^* \circ (\psi^*_{\mathscr{K}})^{-1}(\chi)[\gamma] = \chi(\psi^{-1}_{\mathscr{K}} r_*)[\gamma].$$

We conclude that we have $i'_* = r^* \circ (\psi^*_{\mathscr{K}})^{-1}$ such that diagram (3.3) is commutative. □

REMARK 3.10. Theorem 3.9 remains true for any capping $i : (\Sigma,\mathcal{P}) \hookrightarrow \Sigma_g$ under the condition that the chosen vector field X on Σ_g has only one singularity in the complement of $\widehat{\Sigma}$.

Proposition 3.11. *The homomorphism $\widetilde{e}_Y$ is unique in the sense that it is the only nontrivial homomorphism from $\mathcal{I}(\Sigma,\mathcal{P})$ to $\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$ such that diagram* (3.3) *commutes.*

Proof. Let us assume that there is another homomorphism $G : \mathcal{I}(\Sigma,\mathcal{P}) \to \mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z})$ satisfying the naturality condition, $i'_* \circ G = \widetilde{e}_X \circ i_*$. Our aim is to show that $\widetilde{e}_Y = G$, hence proving the proposition. Since both G and $\widetilde{e}_Y$ satisfy the naturality condition, we get $i'* \circ \widetilde{e}_Y = i'_* \circ G$. Observe that $(\psi^*_{\mathscr{K}})^{-1}$ is an isomorphism because $\psi_{\mathscr{K}}$ is an isomorphism. Since r_* is onto, r^* is injective, which implies that the map i'_* is injective. Therefore, we get the equality $\widetilde{e}_Y = G$, as desired. □

3.4. Naturality of the Chillingworth Homomorphism. In this section, we show that the Chillingworth homomorphism is natural. We relate the Chillingworth class of the subsurface Torelli group to the partitioned Johnson homomorphism.

For an element $f \in \mathcal{I}(\Sigma,\mathcal{P})$, let us define the dual of $\widetilde{e}_Y(f)$. This will be called the Chillingworth class of f. The algebraic intersection form for $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ gives $t_{(\Sigma,\mathcal{P})}(f)$ defined by:

$$\widehat{i}([\gamma], t_{(\Sigma,\mathcal{P})}(f)) = \widetilde{e}_Y(f)[\gamma].$$

Therefore, we get the Chillingworth homomorphism:

$$t_{(\Sigma,\mathcal{P})} : \mathcal{I}(\Sigma,\mathcal{P}) \to H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}).$$

Let $(\Sigma,\mathcal{P}) \hookrightarrow (\Sigma',\mathcal{P}')$ be an embedding of partitioned surfaces. Fix a symplectic basis $\mathscr{B}$ of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ defined in Section 2.2. Recall that $H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ is isomorphic to $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \oplus H_1^{\mathcal{P}_1}(V_1;\mathbb{Z}) \oplus H_1^{\mathcal{P}_2}(V_2;\mathbb{Z}) \oplus \cdots \oplus H_1^{\mathcal{P}_n}(V_n;\mathbb{Z})$ as in Remark 3.6.

As in the previous section, take a nonvanishing vector field X on Σ'. Restrict X to the subsurface Σ and call the restriction Y.

Lemma 3.12. *Let* $s_* : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \to H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ *be the inclusion map and D be the isomorphism defined in* Section 2.2. *Then the following diagram commutes:*

$$
\begin{tikzcd}
\mathcal{I}(\Sigma,\mathcal{P}) \arrow[r, "i_*"] \arrow[d, "\widetilde{e}_Y"'] & \mathcal{I}(\Sigma',\mathcal{P}') \arrow[d, "\widetilde{e}_X"] \\
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \arrow[r, "i'_*"] \arrow[d, "D^{-1}"] & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \arrow[d, "D^{-1}"] \\
H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \arrow[r, "s_*\circ\psi_{\mathcal{K}}"] & H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})
\end{tikzcd}
\tag{3.5}
$$

Proof. We showed in Theorem 3.9 that the upper square in diagram (3.5) commutes. Hence our aim is to show that the lower square also commutes. Commutativity of the lower square is proven by showing commutativity of diagram (3.6).

$$
\begin{tikzcd}
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \arrow[r, "(\psi^*_{\mathcal{K}})^{-1}"] & \mathrm{Hom}(H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}),\mathbb{Z}) \arrow[r, "r^*"] & \mathrm{Hom}(H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}),\mathbb{Z}) \\
H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) \arrow[u, "D"] \arrow[r, "\psi_{\mathcal{K}}"] & H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \arrow[u, "D"] \arrow[r, "s_*"] & H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}) \arrow[u, "D"]
\end{tikzcd}
\tag{3.6}
$$

First, we show that the square on the left of diagram (3.6) commutes. For any homology class $[x]$ of $H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$, we have

$$(\psi^*_{\mathcal{K}})^{-1}(D([x]))[\gamma] = D([x])(\psi^{-1}_{\mathcal{K}}([\gamma])) = \widehat{i}(\psi^{-1}_{\mathcal{K}}[\gamma],[x])$$

and

$$D(\psi_{\mathcal{K}}([x]))[\gamma] = \widehat{i}([\gamma],\psi_{\mathcal{K}}([x])).$$

By Remark 2.4, $\psi^{-1}_{\mathcal{K}} : H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \to H_1^{\mathcal{P}}(\Sigma;\mathbb{Z})$ preserves the algebraic intersection form. Therefore, we get the desired equality $(\psi^*_{\mathcal{K}})^{-1} \circ D = D \circ \psi_{\mathcal{K}}$.

Now our aim is to show that the square in the right-hand side of diagram (3.6) commutes. Let $[x]$ be an element of $H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z})$. For any homology basis element $[\gamma] \in H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z})$ the lemma follows:

$$r^*(D([x]))[\gamma] = D([x])(r_*([\gamma])) = \widehat{i}(r_*([\gamma]),[x])$$

$$D(s_*([x]))[\gamma] = \widehat{i}([\gamma],s_*([x])).$$

Since the direct sum decomposition

$$H_1^{\mathcal{P}'}(\Sigma';\mathbb{Z}) = H_1^{\widehat{\mathcal{P}}}(\widehat{\Sigma};\mathbb{Z}) \oplus \bigoplus_{i=1}^{n} H_1^{\mathcal{P}_i}(V_i;\mathbb{Z})$$

in Remark 3.6 is orthogonal with respect to the intersection pairing, we obtain $\widehat{i}(r_*([\gamma]),[x]) = \widehat{i}([\gamma],s_*([x]))$ and hence the equality $r^* \circ D = D \circ s_*$ as desired.

Consequently, we have proven that diagram (3.6) commutes. Since the dual maps D are isomorphisms, we obtain that diagram (3.5) is also commutative. We conclude that $s_* \circ \psi_{\mathscr{H}} \circ t_{(\Sigma,\mathcal{P})} = t_{(\Sigma',\mathcal{P}')} \circ i_*$ by diagram (3.5). □

Corollary 3.13. *The following diagram is commutative and hence we get the following equality:* $t_{(\Sigma,\mathcal{P})} = \psi_{\mathscr{H}}^{-1} \circ r_* \circ C \circ p_i \circ \tau_{(\Sigma,\mathcal{P})}$, *where* C *is the contraction map. Here,* $\tau_{(\Sigma,\mathcal{P})}$ *is the partitioned Johnson homomorphism defined in* [3], Definition 5.2, *and* p_i *is the map defined in* [3], Definition 5.13.

(3.7)

$$
\begin{tikzcd}
W_{(\Sigma,\mathcal{P})} \arrow[drr, "p_i"] & & \\
\mathcal{I}(\Sigma,\mathcal{P}) \arrow[u, "\tau_{(\Sigma,\mathcal{P})}"] \arrow[r, "i_*"] \arrow[d, "\widetilde{e}_Y"] \arrow[dd, dotted, bend right=60, "t_{(\Sigma,\mathcal{P})}"'] & \mathcal{I}(\Sigma_{g,1}) \arrow[r, "\tau"] \arrow[d, "\widetilde{e}_X"] \arrow[dr, dotted, "t"] & \bigwedge^3 H_1(\Sigma_{g,1};\mathbb{Z}) \arrow[d, "C"] \\
\mathrm{Hom}(H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}),\mathbb{Z}) \arrow[r, "i'_*"] \arrow[d, "D^{-1}"] & H^1(\Sigma_{g,1};\mathbb{Z}) \arrow[r, "D^{-1}"] & H_1(\Sigma_{g,1};\mathbb{Z}) \arrow[dll, "\psi_{\mathscr{H}}^{-1}\circ r_*"] \\
H_1^{\mathcal{P}}(\Sigma;\mathbb{Z}) & &
\end{tikzcd}
$$

Proof. We need to confirm that each triangle and square is commutative. We showed in Theorem 3.9 that the left square in the middle part is commutative. Here, the homomorphism i_* is defined as in Theorem 3.9. That is, for any $f \in \mathcal{I}(\Sigma,\mathcal{P})$, $i_*(f)$ is equal to f on Σ and is the identity on the complement $\Sigma_{g,1}\backslash\Sigma$. The commutativity of the right square in the middle follows from Theorem 2 of [5] and the definition of the Chillingworth class. The partitioned Johnson homomorphism is natural [3], Theorem 5.14. Hence, the upper triangle is commutative. Finally, the commutativity of the lower triangle follows from Lemma 3.12. □

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