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DEFORMATIONS OF THE VIRASORO ALGEBRA OF KRICHEVER – NOVIKOV TYPE

MARTIN SCHLICHENMAIER

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Abstract. It is the general impression that deformation problems are always governed by cohomology spaces. In this contribution we consider the deformation of Lie algebras. There this close connection is true for finite-dimensional algebras, but fails for infinite dimensional ones. We construct geometric families of infinite dimensional Lie algebras over the moduli space of complex one-dimensional tori with marked points. These algebras are algebras of Krichever-Novikov type which consist of meromorphic vector fields of certain type over the tori. The families are non-trivial deformations of the (infinite dimensional) Witt algebra, and the Virasoro algebra respectively, despite the fact that the cohomology space associated to the deformation problem of the Witt algebra vanishes, and hence the algebra is formally rigid. A similar construction works for current algebras. The presented results are jointly obtained with Alice Fialowski.

1. Introduction

In this write-up of a talk, presented at the Białowieża meeting on "Geometric Methods in Physics" in 2005, I will give families of Lie algebras, which are non-trivial deformations of "formally rigid" infinite dimensional Lie algebras. The Lie algebras deformed are the Witt algebra, its universal central extension (i.e., the Virasoro algebra), the current algebras, and their central extensions (i.e., the affine Lie algebras). These algebras play an important role in Conformal Field Theory (CFT). The deformed algebras are of Krichever-Novikov type [7] and appear in particular in the context of a global operator approach to CFT [15], [16].

The algebras to be deformed are formally rigid, i.e., they only admit trivial deformations over the formal power series. Nevertheless, the constructed families are such that the deformations are locally non-trivial, where "locally" means that they are considered over small Zariski open or analytically open subsets of the deformation space containing the special point, corresponding to the algebra to be deformed. This phenomena is peculiar to, and in fact only possible, for infinite dimensional algebras. If the Lie algebra is finite-dimensional, then formal rigidity implies rigidity in the geometric or analytic sense (see below).

Formal rigidity has to do with the Lie algebra cohomology space $\mathrm{H}^2(\mathcal{L},\mathcal{L})$. In particular, $\mathcal{L}$ will be formally rigid if this space is trivial. This is true e.g. for the Witt algebra. Nevertheless, our families are non-trivial deformations of the Witt algebra.

The results presented here were obtained jointly with Alice Fialowski [5], [6]. Some explicit constructions are based on some older work of mine [12]. For the proofs, the explicit calculations, and further references I refer to these articles.

2. What are Deformations of Lie Algebras

Let $\mathcal{L}$ be a Lie algebra over $\mathbb{C}$ [1] with Lie bracket $\mu_0 : \mathcal{L}\times\mathcal{L}\to\mathcal{L}$; $\mu_0(x,y) = [x,y]$. Consider on the same vector space $\mathcal{L}$ is modeled on, a family of Lie structures

$$\mu_t = \mu_0 + t\cdot\phi_1 + t^2\cdot\phi_2 + \cdots \tag{1}$$

with bilinear maps $\phi_i : \mathcal{L}\times\mathcal{L}\to\mathcal{L}$, such that $\mathcal{L}_t := (\mathcal{L},\mu_t)$ is a Lie algebra and $\mathcal{L}_0$ is the Lie algebra we started with. The family $\{\mathcal{L}_t\}$ is a **deformation** of $\mathcal{L}_0$.

Up to now I did not mention what the "parameter" t should be. Indeed we have different possibilities:

1. The parameter t is a variable which allows to plug in numbers $\alpha\in\mathbb{C}$. In this case $\mathcal{L}_\alpha$ is a Lie algebra for every α for which the expression (1) is defined. The family can be considered as deformation over the affine line $\mathbb{C}[t]$ or over the convergent power series $\mathbb{C}\{\{t\}\}$. The deformation is called a **geometric** or an **analytic deformation** respectively.

2. We consider t as a formal variable, and we allow infinitely many terms in (1). It might be the case that μ_t does not exist if we plug in for t any other value different from 0. In this way we obtain deformations over the ring of formal power series $\mathbb{C}[[t]]$. The corresponding deformation is a **formal deformation**.

3. The parameter t is considered as an infinitesimal variable, i.e., we take $t^2 = 0$. We obtain **infinitesimal deformations** defined over the quotient $\mathbb{C}[X]/(X^2) = \mathbb{C}[[X]]/(X^2)$.

[1] In fact, the general considerations are true also for arbitrary fields (at least if the characteristic is 0); But our examples are Lie algebras over $\mathbb{C}$.

Of course, deformations over more general base (e.g. multi-dimensional, etc.) might be considered. See Appendix A for a mathematically precise definition of a deformation.

There is always the trivially deformed family given by $\mu_t = \mu_0$ for all values of t. Two families μ_t and μ'_t deforming the same μ_0 are **equivalent** if there exists a linear automorphism (with the same vagueness about the meaning of t)

$$\psi_t = \mathrm{id} + t \cdot \alpha_1 + t^2 \cdot \alpha_2 + \cdots \tag{2}$$

with $\alpha_i : \mathcal{L} \to \mathcal{L}$ linear maps such that

$$\mu'_t(x, y) = \psi_t^{-1}(\mu_t(\psi_t(x), \psi_t(y))). \tag{3}$$

Definition 1. *A Lie algebra $(\mathcal{L}, \mu_0)$ is called* **rigid** *if every deformation μ_t of μ_0 is locally equivalent to the trivial family.*

Here "locally" means that we only consider the situation for t "near 0". Of course, this depends on the category we consider. As on the formal and the infinitesimal level there exists only one closed point, i.e., the point 0 itself, every deformation over $\mathbb{C}[[t]]$ or $\mathbb{C}[X]/(X^2)$ is already local. This is different on the geometric and analytic level. Here it means that there exists an (Zariski or analytically) open neighbourhood U of 0, such that the family restricted to it is equivalent to the trivial one. In particular, this implies $\mathcal{L}_\alpha \cong \mathcal{L}_0$ for all $\alpha \in U$.

3. Deformations and Cohomology

There is the wide-spread conviction that deformations can always be described in terms of cohomology objects. This is only true to a certain extent. For Lie algebra deformations the relevant cohomology space is $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$, the space of Lie algebra two-cohomology classes with values in the adjoint module $\mathcal{L}$. Recall that the cohomology classes are classes of two-cocycles modulo coboundaries. An antisymmetric map $\phi : \mathcal{L} \to \mathcal{L}$ is a Lie algebra **two-cocycle** if

$$\phi([x, y], z) + \phi([y, z], x) + \phi([z, x], y) - [x, \phi(y, z)] + [y, \phi(z, x)] - [z, \phi(x, y)] = 0. \tag{4}$$

It is a **coboundary** if there exists a linear map $\psi : \mathcal{L} \to \mathcal{L}$ with

$$\phi(x, y) = (d_1\psi)(x, y) = \psi([x, y]) - [x, \psi(y)] + [y, \psi(x)]. \tag{5}$$

From (1) we get that the Jacobi identity for μ_t implies that the first non-vanishing ϕ_i is a two-cocycle. Furthermore, if μ_t and μ'_t are equivalent then the corresponding ϕ_i and ϕ'_i are cohomologous.

The following results are well-known:

1. $\mathrm{H}^2(\mathcal{L},\mathcal{L})$ classifies infinitesimal deformations [2]
2. If $\dim \mathrm{H}^2(\mathcal{L},\mathcal{L}) < \infty$, then all formal deformations up to equivalence can be realized in this vector space [4]
3. If $\mathrm{H}^2(\mathcal{L},\mathcal{L}) = 0$, then $\mathcal{L}$ is infinitesimally and formally rigid (follows from 1 and 2)
4. If $\dim \mathcal{L} < \infty$, then $\mathrm{H}^2(\mathcal{L},\mathcal{L}) = 0$ implies that $\mathcal{L}$ is also rigid in the geometric and analytic sense [2], [9], [10].

As our examples show, Point 4 is not true anymore if one drops the condition $\dim \mathcal{L} < \infty$.

For the Witt algebra $\mathcal{W}$ one has $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$ ([3], see also [5]). Hence it is formally rigid. For the classical current algebras $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$ with $\mathfrak{g}$ a finite-dimensional simple Lie algebra, Lecomte and Roger [8] (partly based on results of Garland [1]) showed that $\overline{\mathfrak{g}}$ is formally rigid. Nevertheless, for both types of algebras including their central extensions we have deformations which are both locally geometrically and analytically non-trival [5], [6].

4. The Geometric Families

The **Witt algebra** is the Lie algebra consisting of those meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = S^2$, which are holomorphic outside $\{0,\infty\}$. It has the following basis and the corresponding structure equations

$$l_n = z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z},\ n \in \mathbb{Z}, \quad \text{and} \quad [l_n, l_m] = (m-n)\, l_{n+m}. \tag{6}$$

By defining $\deg(l_n) := n$, it becomes a graded Lie algebra. The **Virasoro algebra** is its universal central extension.

The **Krichever-Novikov vector field algebras** [7], [11] are generalization of the Witt algebra to arbitrary higher genus compact Riemann surfaces X. They consist of meromorphic vector fields on X, which are holomorphic outside a certain finite set of points. In general the algebras are not graded anymore, but only almost-graded (see [11], [14] for the details and further references). Here, I will only deal with examples and will also ignore central extensions.

Let $T = \mathbb{C}/L$ be a complex one-dimensional torus, i.e., a Riemann surface of genus 1. Here L denotes the lattice $L = \langle 1, \tau \rangle_{\mathbb{Z}}$, $\tau \in \mathbb{C}$, with $\operatorname{im} \tau > 0$. The field of meromorphic functions on T is generated by the doubly-periodic Weierstraß $\wp$ function and its derivative $\wp'$ fulfilling the differential equation

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) = 4\wp^3 - g_2\wp - g_3 \tag{7}$$

with the e_i pairwise distinct and given by

$$\wp(\frac{1}{2}) = e_1, \qquad \wp(\frac{\tau}{2}) = e_2, \qquad \wp(\frac{\tau+1}{2}) = e_3, \quad e_1 + e_2 + e_3 = 0. \tag{8}$$

The function $\wp$ is an even meromorphic function with poles of order two at the points of the lattice and holomorphic elsewhere. The function $\wp'$ is an odd meromorphic function with poles of order three at the points of the lattice and holomorphic elsewhere. It has zeros of order one at the points $1/2, \tau/2$ and $(1+\tau)/2$ and all its translates under the lattice.

We have to pass here to the algebraic-geometric picture. The map

$$T \to \mathbb{P}^2(\mathbb{C}), \qquad z \bmod L \mapsto \begin{cases} (\wp(z) : \wp'(z) : 1), & z \notin L \\ (0 : 1 : 0), & z \in L. \end{cases} \tag{9}$$

realizes T as a complex-algebraic smooth curve in the projective plane. The affine coordinates are $X = \wp(z,\tau)$ and $Y = \wp'(z,\tau)$. From (7) it follows that its affine part can be given by the cubic curve defined by

$$Y^2 = 4(X - e_1)(X - e_2)(X - e_3) = 4X^3 - g_2X - g_3 =: f(X). \tag{10}$$

The point at infinity on the curve is the point $\infty = (0 : 1 : 0)$. We consider the subalgebra of those vector fields, which are holomorphic outside of $\bar{z} = \bar{0}$ and $\bar{z} = \overline{1/2}$ (respectively in the algebraic-geometric picture, outside the points ∞ and $(e_1, 0)$). A basis of such vector fields is given (with $k \in \mathbb{Z}$)

$$V_{2k+1} := (X - e_1)^k Y \frac{\mathrm{d}}{\mathrm{d}X}, \quad V_{2k} := \frac{1}{2} f(X)(X - e_1)^{k-2} \frac{\mathrm{d}}{\mathrm{d}X}. \tag{11}$$

If we vary the points e_1 and e_2 ($e_3 = -(e_1 + e_2)$) we obtain families of curves and associated families of vector field algebras as long as the curves are non-singular. Let us describe them in detail. Let

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2; e_2 = s \cdot e_1\},\ s \in \mathbb{C}, \qquad D_\infty := \{(0, e_2) \in \mathbb{C}^2\} \tag{12}$$

and

$$B := \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}). \tag{13}$$

Then the curves are non-singular exactly over the points of B. Over the exceptional D_s at least two of the e_i are the same. For the algebra we obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & n, m \text{ odd} \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + (e_1 - e_2)(e_1 - e_3)V_{n+m-4}\big) & n, m \text{ even} \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \quad + (m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{14}$$

In fact these relations define Lie algebras for every pair (e_1, e_2). We denote by $\mathcal{L}^{(e_1,e_2)}$ the Lie algebra corresponding to (e_1, e_2). Obviously, $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.

Proposition 2. ([5, Proposition 5.1]) *For $(e_1, e_2) \neq (0, 0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra $\mathcal{W}$, but $\mathcal{L}^{(0,0)} \cong \mathcal{W}$.*

If we restrict our two-dimensional family to the lines D_s ($s \neq \infty$) then we obtain a one-dimension family

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m} & n, m \text{ odd} \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + e_1^2(1-s)(2+s)V_{n+m-4}\big) & n, m \text{ even} \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \quad + (m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{15}$$

Here s has a fixed value and e_1 is the deformation parameter. (A similar family exists for $s = \infty$.) It can be shown that as long as $e_1 \neq 0$ the algebras over D_s are pairwise isomorphic, but not isomorphic to the algebra over 0, which is the Witt algebra. Using the result of Fialowski [3] on $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$, we get

Theorem 3. *Despite its infinitesimal and formal rigidity, the Witt algebra $\mathcal{W}$ admits deformations $\mathcal{L}_t$ over the affine line with $\mathcal{L}_0 \cong \mathcal{W}$, which restricted to every (Zariski or analytic) neighbourhood of $t = 0$ are non-trivial.*

5. The Geometric Reason Behind

If we take $e_1 = e_2 = e_3$ in the definition of the cubic curve (10) we obtain the cuspidal cubic E_C, with affine part given by the polynomial $Y^2 = 4X^3$. It has a singularity at $(0, 0)$ and the desingularization is given by the projective line

$\mathbb{P}^1(\mathbb{C})$. This says there exists a surjective (algebraic) map $\pi_C : \mathbb{P}^1(\mathbb{C}) \to E_C$, which outside the singular point is $1:1$. Over the cusp lies exactly one point. The vector fields can be degenerated to E_C and pull-backed to vector fields on $\mathbb{P}^1(\mathbb{C})$. The point $(e_1, 0)$, where a pole is allowed, moves to the cusp. The other point stays at infinity. In particular, by pulling back the algebra we obtain the algebra of vector fields with two possible poles, which is the Witt algebra.

The exceptional lines D_s for $s = 1, -1/2, -2$ are related to interesting geometric situations. Above $D_s \setminus \{(0,0)\}$ with these values of s, two of the e_i are the same, the third one remains distinct. The curve will be a nodal cubic E_N, defined by $Y^2 = 4(X-e)^2(X-e)$. The singularity will be a node with the coordinates $(e, 0)$. Again, the desingularization will be the projective line $\pi_N : \mathbb{P}^1(\mathbb{C}) \to E_N$. But now above the node there will be two points in $\mathbb{P}^1(\mathbb{C})$. For the pull-back of the vector fields we have the following two situations:

1) If $s = 1$ or $s = -2$ then $e = e_1$ and the node is a possible point for a pole. We obtain vector fields on $\mathbb{P}^1(\mathbb{C})$ which might have, beside the pole at ∞, poles at two other places. Hence, we obtain a three-point Krichever-Novikov vector field algebra of genus 0.
2) If $s = -1/2$ then at the node there is no pole. The number of possible poles for the pull-back remains two. But the vector fields obtained by pull-back acquire zeros at the points lying above the node. Hence, we get a certain subalgebra of the Witt algebra. These algebras have been identified and studied in detail in [5] and [12].

These deformed families are of importance in the context of going to the boundary of the moduli space of curves with marked points.

A. The Precise Definition of a Deformation of a Lie Algebra

In the following we will assume that A is a commutative algebra over $\mathbb{K}$ (where $\mathbb{K}$ is a field of characteristic zero) which admits an augmentation $\epsilon : A \to \mathbb{K}$. This says that ϵ is a $\mathbb{K}$-algebra homomorphism, e.g., $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \operatorname{Ker} \epsilon$ is a maximal ideal of A. Vice versa, given a maximal ideal m of A with $A/m \cong \mathbb{K}$, the natural quotient map defines an augmentation.

If A is a finitely generated $\mathbb{K}$-algebra over an algebraically closed field $\mathbb{K}$ then $A/m \cong \mathbb{K}$ is true for every maximal ideal m. Hence, in this case every such A admits at least one augmentation and all maximal ideals are coming from augmentations.

Let us consider a Lie algebra $\mathcal{L}$ over the field $\mathbb{K}$, ϵ a fixed augmentation of A, and $m = \operatorname{Ker}\epsilon$ the associated maximal ideal.

Definition 4. ([4]) *A* **deformation** λ *of* $\mathcal{L}$ *with base* (A, m)*, or simply with base* A*, is a Lie* A*-algebra structure on the tensor product* $A \otimes_{\mathbb{K}} \mathcal{L}$*, with bracket* $[.,.]_\lambda$ *such that*

$$\epsilon \otimes \mathrm{id} \ : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L} \tag{16}$$

is a Lie algebra homomorphism.

Specifically, it means that for all $a, b \in A$ *and* $x, y \in \mathcal{L}$*,*

1) $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \mathrm{id})[1 \otimes x, 1 \otimes y]_\lambda$
2) $[.,.]_\lambda$ *is skew-symmetric and satisfies the Jacobi identity*
3) $\epsilon \otimes \mathrm{id}\,([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

By condition 1) to describe a deformation it is enough to give the elements $[1 \otimes x, 1 \otimes y]_\lambda$ for all $x, y \in \mathcal{L}$. If $B = \{z_i\}_{i \in J}$ is a basis of $\mathcal{L}$ it follows from condition (3) that the Lie product has the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \textstyle\sum_i' a_i \otimes z_i \tag{17}$$

with $a_i = a_i(x, y) \in m$, $z_i \in B$. Here $\sum'$ denotes a finite sum. Clearly, condition 2) is an additional condition which has to be satisfied.

If we use $A = \mathbb{C}[t]$ we get exactly the notion of a one parameter geometric deformation discussed above.

A deformation is called **trivial** if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e., (17) reads as $[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y]$.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called **equivalent** if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$, with the two Lie algebra structures compatible with $\epsilon \otimes \mathrm{id}$.

Formal deformations are defined in a similar way. Let A be a complete local algebra over $\mathbb{K}$, so $A = \varprojlim_{n \to \infty} (A/m^n)$, where m is the maximal ideal of A. Furthermore, we assume that $A/m \cong \mathbb{K}$, and $\dim(m^k/m^{k+1}) < \infty$ for all k.

Definition 5. *A* **formal deformation** *of* $\mathcal{L}$ *with base* A *is a Lie algebra structure on the completed tensor product* $A \widehat{\otimes} \mathcal{L} = \varprojlim_{n \to \infty} ((A/m^n) \otimes \mathcal{L})$ *such that*

$$\epsilon \widehat{\otimes} \mathrm{id} \ : A \widehat{\otimes} \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L} \tag{18}$$

is a Lie algebra homomorphism.

If $A = \mathbb{C}[[t]]$, then a formal deformation of $\mathcal{L}$ with base A is the same as a formal one parameter deformation discussed above. There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

B. Families for the Current Algebras

The construction of the families above is not restricted to the vector field case. It also works for the current algebra and affine Lie algebra case. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathcal{A}$ be the algebra of meromorphic functions on a compact Riemann surface with poles only at a given set of points. Then the **higher genus multi-point current algebra** $\overline{\mathfrak{g}}$ of Krichever-Novikov type [7], [13] is defined as

$$\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathcal{A}, \quad \text{with} \quad [x \otimes f, y \otimes g] := [x, y] \otimes fg. \tag{19}$$

For the classical case we take the projective line and two possible points for poles and obtain $\mathfrak{g} \otimes \mathbb{C}[z^{-1}, z]$. Of course, we can now consider $\mathcal{A}$ for the situation discussed in the vector field case. We take as basis for $\mathcal{A}$

$$A_{2k} = (X - e_1)^k, \qquad A_{2k+1} = \frac{1}{2} Y \cdot (X - e_1)^{k-1} \qquad k \in \mathbb{Z} \tag{20}$$

and obtain

$$[x \otimes A_n, y \otimes A_m] = \begin{cases} [x, y] \otimes A_{n+m} & n \text{ or } m \text{ even} \\ [x, y] \otimes A_{n+m} + 3e_1 [x, y] \otimes A_{n+m-2} & \\ \quad + (e_1 - e_2)(2e_1 + e_2)[x, y] \otimes A_{n+m-4} & n \text{ and } m \text{ odd.} \end{cases} \tag{21}$$

If we let e_1 and e_2 (and hence also e_3) go to zero, we obtain the classical current algebra as degeneration. Again it can be shown that the, on D_s restricted family, is locally non-trivial, see [6]. There also a formula for the central extension is given. Recall that by results of Lecomte and Roger [8] (see also Garland [1]) the current algebra is formally rigid if $\mathfrak{g}$ is simple. But our families show that it is neither geometrically nor analytically rigid.

References

[1] Garland H., *Dedekind's η-Function and the Ccohomology of Infinite Dimensional Lie Algebras*, Proc. Nat. Acad. Sci. USA **72** (1975) 2493–2495.

[2] Gerstenhaber M., *On the Deformation of Rings and Algebras I,II,III* Ann. Math. **79** (1964) 59-10 (1964); **84** (1966) 1-19; **88** (1968) 1-34.

[3] Fialowski A., *Deformations of Some Infinite Dimensional Lie Algebras*, J. Math. Phys. **31** (1990) 1340-1343.

[4] Fialowski A., and Fuchs D., *Construction of Miniversal Deformations of Lie Algebras*, J. Funct. Anal. **161** (1999) 76-110.

[5] Fialowski A., and Schlichenmaier M., *Global Deformations of the Witt algebra of Krichever–Novikov Type*, Comm. Contemp. Math. **5** (2003) 921–945.

[6] Fialowski A. and Schlichenmaier M., *Global Geometric Deformations of Current Algebras as Krichever–Novikov Type Algebras*, Comm. Math. Phys. **260** (2005) 579-612.

[7] Krichever I. and Novikov S., *Algebras of Virasoro Type, Riemann Surfaces and Structures of the Theory of Solitons*, Funktional Anal. i. Prilozhen. **21** (1987) 46–63.

[8] Lecomte P. and Roger C., *Rigidity of Current Lie Algebras of Complex Simple Type*, J. London Math. Soc. **37** (1988) 232–240.

[9] Nijenhuis A. and Richardson R., *Cohomology and Deformations of Algebraic Structures*, Bull. Amer. Math. Soc. **70** (1964) 406-411.

[10] Nijenhuis A. and Richardson R., *Cohomology and Deformations in Graded Lie Algebras*, Bull. Amer. Math. Soc. **72** (1966) 1-29.

[11] Schlichenmaier M., *Central Extensions and Semi-Infinite Wedge Representations of Krichever-Novikov Algebras for More than Two Points*, Lett. Math. Phys. **20** (1991) 33–46.

[12] Schlichenmaier M., *Degenerations of Generalized Krichever-Novikov Algebras on Tori*, J. Math. Phys. **34** (1993) 3809-3824.

[13] Schlichenmaier M., *Higher Genus Affine Lie Algebras of Krichever-Novikov Type*, Moscow Math. J. **3** (2003) 1395–142.

[14] Schlichenmaier M., *Local Cocycles and Central Extensions for Multi-Point Algebras of Krichever-Novikov Type*, J. reine angew. Math. **559** (2003) 53–94.

[15] Schlichenmaier M. and Sheinman O., *Wess-Zumino-Witten-Novikov Theory, Knizhnik-Zamolodchikov Equations, and Krichever-Novikov Algebras,* Russian Math. Surv. **54** (1999) 213-250.

[16] Schlichenmaier M. and Sheinman O., *Knizhnik-Zamolodchikov Equations for Positive Genus and Krichever-Novikov Algebras,* Russian Math. Surv. **59** (2004) 737–770.

Martin Schlichenmaier
University of Luxembourg
Campus Limpertsberg
162A, Avenue de la Faiencerie
L-8706 Luxembourg
GRAND-DUCHY OF LUXEMBOURG
E-mail address: martin.schlichenmaier@uni.lu