JGSP **48** (2018) 47–77

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

SYMMETRY GROUPS OF SYSTEMS OF ENTANGLED PARTICLES

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Communicated by Jan J. Slawianowski

Abstract. A Lorentz transformation of signature (m, n), $m, n \in \mathbb{N}$, is a pseudo-rotation in a pseudo-Euclidean space of signature (m, n). Accordingly, the Lorentz transformation of signature $(1, 3)$ is the common Lorentz transformation of special relativity theory. It is known that entangled particles involve Lorentz symmetry violation. Hence, the aim of this article is to expose and illustrate the symmetry groups of systems of entangled particles uncovered in [44]. It turns out that the Lorentz transformations of signature (m, n) form the symmetry group by which systems of m n-dimensional entangled particles can be understood, just as the common Lorentz group of signature $(1, 3)$ forms the symmetry group by which Einstein's special theory of relativity can be understood. Consequently, it is useful to extend special relativity theory by incorporating Lorentz transformation groups of signature $(m, 3)$ for all $m \geq 2$. The resulting extended special relativity theory, then, provides not only the symmetry group of the $(1 + 3)$-dimensional spacetime of particles, but also the symmetry group of the $(m + 3)$-dimensional spacetime of systems of m entangled three-dimensional particles, for each $m \geq 2$.

MSC: 81M05, 83A05, 51M10
Keywords: Lorentz (Galilei) transformations of signature (m, n), pseudo-Euclidean spaces, quantum entanglement, special relativity

Contents

1. Introduction

A plethora of peculiar phenomena emerges in quantum mechanics, the quintessential being *entanglement*. Quantum entanglement is a physical phenomenon that occurs when groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others, *even when the particles are separated by a large distance*. Instead, a quantum state must be described for a system of particles as a whole.

Nature organizes itself using the language of symmetries. Thus, in particular, the underlying symmetry by which Einstein's special relativity theory can be understood is the Lorentz group $\mathrm{SO}_c(1,3)$. A physical system has Lorentz symmetry if the relevant laws of physics are invariant under Lorentz transformations. Lorentz symmetry is one of the cornerstones of modern physics. However, it is known that entangled particles involve Lorentz symmetry violation. Indeed, several explorers exploit entangled particles to observe Lorentz symmetry violation, see for instance [7], [9], [10], [21] and [22].

The aim of this article is, therefore, to expose and illustrate extended Lorentz transformations that provide the missing symmetry groups of systems of entangled particles. The underlying geometry of our exploration is an extension of analytic hyperbolic geometry, called analytic *bi-hyperbolic geometry* of signature (m,n),

$m, n \in \mathbb{N}$, $\mathbb{N}$ being the set of all positive integers. Analytic hyperbolic geometry [38], in turn, is the hyperbolic geometry of Bolyai and Lobachevsky studied analytically. An introduction to analytic hyperbolic geometry is presented in [32], and its application in Einstein's special relativity theory is presented in [34]. Analytic bi-hyperbolic geometry of signature (m, n) is presented in [44], where symmetry groups of the spacetime of entangled particles emerge.

The extended Lorentz group $\mathrm{SO}(m, n)$ of signature (m, n), $m, n \in \mathbb{N}$, also known as the group of pseudo-rotations in a pseudo-Euclidean space of signature (m, n), is well-known in algebra [14]. It descends to the common homogeneous, proper, orthochronous Lorentz transformation group SO(1,3) of Einstein's special relativity theory in the special case when $(m, n) = (1, 3)$. Following studies in [25, 40, 41, 43, 44], we realize the extended Lorentz group $\mathrm{SO}_c(m, n)$ parametrically, calling the elements of $\mathrm{SO}_c(m, n)$ Lorentz transformations of signature (m, n) or, in short, (m, n)-Lorentz transformations, in m time dimensions and n space dimensions. Then, the Lorentz transformation group $\mathrm{SO}_c(1, 3)$ is the special relativistic Lorentz transformation group $\mathrm{SO}(1, 3)$ with $c > 0$ being an arbitrarily fixed constant that, in physical applications, represents the vacuum speed of light. Accordingly, $\mathrm{SO}_{c=1}(m, n) = \mathrm{SO}(m, n)$.

The parametric realization (23) of each extended Lorentz group $\mathrm{SO}_c(m, n)$, $m, n \in \mathbb{N}$, allows us to associate extended Lorentz groups to corresponding extended Galilei groups. Indeed, by means of the additive decomposition of the Lorentz bi-boost (28), the set of all Lorentz bi-boosts of signature (m, n) is uniquely associated to the set of all Galilei bi-boosts of same signature (m, n) and, conversely, the set of all Galilei bi-boosts of signature (m, n) is uniquely associated to the set of all Lorentz bi-boosts of same signature (m, n). A Lorentz (Galilei) bi-boost of signature (m, n) is an extended Lorentz (Galilei) transformation of signature (m, n) without rotations.

Unlike extended Lorentz groups, extended Galilei groups are intuitively clear, as shown in Sections 6 – 7. The resulting clear interpretation of a Galilei transformation of signature (m, n), in turn, induces interpretation for its associated Lorentz transformation of signature (m, n). Having the induced interpretation for any $m, n \in \mathbb{N}$, the Lorentz transformation group $\mathrm{SO}_c(m, n)$ turns out to be the symmetry group of the $(m + n)$-dimensional spacetime of particle systems that consist of m n-dimensional entangled particles. The incorporation of the extended Lorentz groups into Einstein's special relativity theory gives rise to our extended special relativity theory. In geometry $m, n \geq 1$, while in physical applications $m \geq 1$ and $n = 3$ (see [30] for elegant remarks about the unique features of $n = 3$ space dimensions). It is indicated that a Lorentz group of signature $(m, 3)$, $m > 1$, describes physical reality for m entangled moving particles in extended special relativity theory, just as the Lorentz group of signature $(1, 3)$ describes physical reality in Einstein's special relativity theory for moving particles which are not entangled.

An (m, n)-particle is a system on m n-dimensional entangled subparticles. It is an element of the $(m + n)$-dimensional pseudo-Euclidean spacetime having m time dimensions and n space dimensions ($n = 3$ in physical applications, but $n \in \mathbb{N}$ in geometry). Each subparticle of an (m, n)-particle possesses its own clock that measures its time, so that the time of an (m, n)-particle is measured by m clocks, implying that an (m, n)-particle has m time dimensions. The m subparticles of an (m, n)-particle inhabit the same n-dimensional space, implying that an (m, n)-particle has n space dimensions. Accordingly, an (m, n)-particle is an $(m + n)$-dimensional spacetime event, having m time and n space dimensions. It is indicated that a Lorentz transformation group $\mathrm{SO}_c(m, n)$ of signature (m, n) is the symmetry group of $(m + n)$-dimensional spacetime of any system of m n-dimensional entangled subparticles, just as the common Lorentz group $\mathrm{SO}_c(1, 3)$ is the symmetry group of $(1 + 3)$-dimensional spacetime in special relativity.

The main result of this article, identifying the symmetry group of a system of m n-dimensional entangled particles, $m, n \in \mathbb{N}$, stems from several theorems established in [44]. Suggestively, the extended Lorentz group of signature $(m, 3)$, $m \geq 2$, is the symmetry group by which the quantum mechanical phenomenon of entanglement of m particles can be understood as entanglement in the context of our extended special relativity theory into which extended Lorentz groups of signature $(m, 3)$, $m \geq 2$, are incorporated.

In this paper and, in more details, in [44], our model of m entangled particles, $m \geq 2$, is established mathematically in terms of patterns and analogies that the model shares with Einstein's special relativity theory. Therefore, it remains to explore whether the model is supported experimentally for $m \geq 2$, just as it is supported experimentally for $m = 1$ in special relativity theory.

2. Pseudo-Euclidean Spaces and Pseudo-Rotations

The study of pseudo-Euclidean spaces and pseudo-rotations is well-known in linear algebra [14]. A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n), $m, n \in \mathbb{N}$, is an $(m + n)$-dimensional space with an orthogonal basis e_i, $i = 1, \ldots, m + n$

$$e_i \cdot e_j = \epsilon_i \delta_{ij} \tag{1}$$

where

$$\epsilon_i = \begin{cases} 1\,, & i = 1, \ldots, m \\ -\dfrac{1}{c^2}, & i = m+1, \ldots, m+n \end{cases} \tag{2}$$

and where c is an arbitrarily fixed positive constant.

Without loss of generality one may select $c = 1$. However, we prefer to view c as a free positive parameter, $0 < c < \infty$, in order to allow limits as $c \to \infty$ to be available, where the modern and new descends to the classical and familiar. An illustrative point in case is the Lorentz transformation of special relativity theory, which descends to the Galilei transformation of classical mechanics when the vacuum speed of light c tends to infinity, as shown in (25) – (28) with any signature (m, n).

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n) is equipped with an inner product of signature (m, n). The inner product $\mathbf{x}{\cdot}\mathbf{y}$ of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$

$$\mathbf{x} = \sum_{i=1}^{m+n} x_i e_i \qquad \text{and} \qquad \mathbf{y} = \sum_{i=1}^{m+n} y_i e_i \tag{3}$$

is

$$\mathbf{x}{\cdot}\mathbf{y} = \sum_{i=1}^{m+n} \epsilon_i x_i y_i = \sum_{i=1}^{m} x_i y_i - \frac{1}{c^2} \sum_{i=m+1}^{m+n} x_i y_i \,. \tag{4}$$

Accordingly, the squared norm of $\mathbf{x} \in \mathbb{R}^{m,n}$ is

$$\|\mathbf{x}\|^2 = \mathbf{x}^2 = \mathbf{x}{\cdot}\mathbf{x} = \sum_{i=1}^{m+n} \epsilon_i x_i^2 = \sum_{i=1}^{m} x_i^2 - \frac{1}{c^2} \sum_{i=m+1}^{m+n} x_i^2 \,. \tag{5}$$

Let I_m be the $m \times m$ identity matrix, and let η be the $(m+n) \times (m+n)$ diagonal matrix

$$\eta = \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -\frac{1}{c^2} I_n \end{pmatrix} \tag{6}$$

where $0_{m,n}$ is the $m \times n$ zero matrix. Then, the matrix representation of the inner product (4) is

$$\mathbf{x}{\cdot}\mathbf{y} = \mathbf{x}^t \eta \mathbf{y} \tag{7}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m,n}$ are the column vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m+n} \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+n} \end{pmatrix} \tag{8}$$

and exponent t denotes transposition.

Let Λ be an $(m+n)\times(m+n)$ matrix that leaves the inner product (7) invariant. Then, for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^{m,n}$

$$(\Lambda\mathbf{x})^t\eta\Lambda\mathbf{y}=\mathbf{x}^t\eta\mathbf{y} \tag{9}$$

implying $\mathbf{x}^t\Lambda^t\eta\Lambda\mathbf{y}=\mathbf{x}^t\eta\mathbf{y}$, so that [20, p. 193]

$$\Lambda^t\eta\Lambda=\eta\,. \tag{10}$$

The determinant of the matrix equation (10) yields

$$(\mathrm{det}\Lambda)^2=1 \tag{11}$$

noting that $\det(\Lambda^t\eta\Lambda)=(\mathrm{det}\Lambda^t)(\mathrm{det}\eta)(\mathrm{det}\Lambda)$ and $\mathrm{det}\Lambda^t=\mathrm{det}\Lambda$. Hence, $\mathrm{det}\Lambda=\pm1$.

The special transformations Λ that can be reached continuously from the identity transformation of $\mathbb{R}^{m,n}$ constitute the special pseudo-orthogonal group $\mathrm{SO}_c(m,n)$, called the group of pseudo-rotations of signature (m,n). A pseudo-rotation of signature (m,n) is also called a Lorentz transformation of signature (m,n) or, in short, an (m,n)-Lorentz transformation. The Lorentz transformation of signature $(1,3)$ turns out to be the common homogeneous, proper, orthochronous Lorentz transformation of Einstein's special theory of relativity [34]. The formal definition of $\mathrm{SO}_c(m,n)$ follows.

Definition 1 ($\mathrm{SO}_c(m,n)$)**.** *Let $m,n\in\mathbb{N}$ be two positive integers. A linear transformation Λ of the pseudo-Euclidean space $\mathbb{R}^{m,n}$ is called a pseudo-rotation of signature (m,n), or a Lorentz transformation of signature (m,n), if it leaves the inner product* (4) *invariant, and if it can be reached continuously from the identity transformation of $\mathbb{R}^{m,n}$. The group of all Lorentz transformations of signature (m,n) is denoted by $\mathrm{SO}_c(m,n)$.*

If $\Lambda\in\mathrm{SO}_c(m,n)$ then its determinant is equal to one, and the determinant of its first m rows and columns is positive [15, p. 478].

3. Matrix Balls of Radius c

Definition 2 (Spectrum, Spectral Radius [17, p. 35]**).** *The set of all complex numbers that are eigenvalues of a square matrix $A\in\mathbb{R}^{n\times n}$, $n\in\mathbb{N}$, is called the spectrum of A and is denoted by $\sigma(A)$. The spectral radius of A is the nonnegative number*

$$\rho(A)=\max\{|\lambda|;\ \lambda\in\sigma(A)\}. \tag{12}$$

The spectral radius $\rho(A)$ is thus the radius of the smallest disc centered at the origin of the complex plane that includes all the eigenvalues of A.

For Definition 3 below, we note that for any $V \in \mathbb{R}^{n\times m}$ the set of nonzero eigenvalues of $VV^t \in \mathbb{R}^{n\times n}$ equals the set of nonzero eigenvalues of $V^tV \in \mathbb{R}^{m\times m}$.

Definition 3 (Matrix Ball, Matrix Spectral Norm). *For any $m, n \in \mathbb{N}$ and $c > 0$, the c-ball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices is given by*

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m};\ \lambda \in \sigma(VV^t),\ \sqrt{\lambda} < c\} \\ &= \{V \in \mathbb{R}^{n\times m};\ \lambda \in \sigma(V^tV),\ \sqrt{\lambda} < c\}\,.\end{aligned} \tag{13}$$

The matrix spectral norm $\|V\|$ of $V \in \mathbb{R}^{n\times m}$, or norm in short, is defined by

$$\|V\| = \max\{\sqrt{\lambda};\ \lambda \in \sigma(VV^t)\} = \max\{\sqrt{\lambda};\ \lambda \in \sigma(V^tV)\}\,. \tag{14}$$

It is clear from Defenition 3 that

$$\mathbb{R}_c^{n\times m} = \{V \in \mathbb{R}^{n\times m};\ \|V\| < c\} \tag{15}$$

and that

$$\|-V\| = \|V\| \tag{16}$$

for all $V \in \mathbb{R}_c^{n\times m}$.

Well known properties of the matrix spectral norm are [17, p. 290]

$$\begin{array}{ll} \|A\| \geq 0 & \text{Nonnegative} \\ \|A\| = 0 \text{ if and only if } A = 0 & \text{Positive} \\ \|rA\| = |r|\|A\| & \text{Homogeneity Property} \\ \|A+B\| \leq \|A\| + \|B\| & \text{Triangle Inequality} \\ \|AB\| \leq \|A\|\|B\| & \text{Submultiplicity} \end{array} \tag{17}$$

for any $r \in \mathbb{R}$ and square matrices $A, B \in \mathbb{R}^{k\times k}$, $k \in \mathbb{N}$.

Theorem 4. ([44, Section 5.3]). *For any $m, n \in \mathbb{N}$ and $c > 0$, let $V \in \mathbb{R}^{n\times m}$. Then, $V \in \mathbb{R}_c^{n\times m}$ if and only if the real matrix*

$$\Gamma_V^L = \Gamma_{n,V,c}^L := \sqrt{I_n - c^{-2}VV^t}^{\,-1} \in \mathbb{R}^{n\times n} \tag{18}$$

exists, and similarly, $V \in \mathbb{R}_c^{n\times m}$ if and only if the real matrix

$$\Gamma_V^R = \Gamma_{m,V,c}^R := \sqrt{I_m - c^{-2}V^tV}^{\,-1} \in \mathbb{R}^{m\times m} \tag{19}$$

exists.

The obvious limits

$$\lim_{c\to\infty}\Gamma_V^L = I_n, \qquad \lim_{c\to\infty}\Gamma_V^R = I_m \tag{20}$$

will prove useful in (25).

We use the abbreviated notation Γ_V^L and Γ_V^R in (18) – (19), rather than the full notation $\Gamma_{n,V,c}^L$ and $\Gamma_{m,V,c}^R$, since the values of m, n and c are always known from the context. We call Γ_V^L and Γ_V^R, respectively, the left and right gamma factors of signature (m, n).

In view of (20), it is interesting to note that for any fixed $m, n \in \mathbb{N}$ and $c > 0$, the left and right gamma factors of signature (m, n) satisfy the equations [44, Lemma 5.82]

$$\Gamma_V^L = I_n + \frac{1}{c^2}\frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L}VV^t, \qquad \Gamma_V^R = I_m + \frac{1}{c^2}\frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R}V^tV. \tag{21}$$

In (21) we use the convenient *matrix division notation* A/B to denote either AB^{-1} or $B^{-1}A$ when no confusion may arise, that is, when the matrices A and B satisfy $AB^{-1} = B^{-1}A$. The identities in (21) prove useful in establishing the important additive decomposition (28).

In the special case when $m = 1$, the right gamma factor, Γ_V^R in (19), descends to the Lorentz gamma factor, γ_V, of special relativity theory [34]

$$\Gamma_V^R = \frac{1}{\sqrt{1-c^{-2}\|V\|^2}} =: \gamma_V, \qquad m = 1 \tag{22}$$

$V \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$, where $\|V\|^2 = V^tV$.

4. V-Parametric Realization of the Lorentz Transformations of Signature (*m,n*)

The following theorem realizes the group $\mathrm{SO}_c(m, n)$ parametrically. Each element $\Lambda \in \mathrm{SO}_c(m, n)$ is a Lorentz transformation of signature (m, n). It is parametrized by a matrix parameter in the ball, $V \in \mathbb{R}_c^{n\times m}$, and two orientation parameters $O_m \in \mathrm{SO}(m)$ and $O_n \in \mathrm{SO}(n)$, according to the following theorem.

Theorem 5. (Lorentz Transformation Bi-Gyration Decomposition and the V-Parametric Realization [44, Theorem 5.20]). *A matrix* $\Lambda \in \mathbb{R}^{(m+n)\times(m+n)}$, $m, n \in \mathbb{N}$, *is the matrix representation of a Lorentz transformation* Λ *of signature*

(m,n), $\Lambda \in \mathrm{SO}_c(m,n)$, *if and only if it possesses the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{23}$$

$$\Lambda \in \mathrm{SO}_c(m,n) = \mathrm{SO}(m) \times \mathbb{R}_c^{n\times m} \times \mathrm{SO}(n)$$

parametrized by the main parameter $V \in \mathbb{R}_c^{n\times m}$*, and the two orientation parameters* $O_m \in \mathrm{SO}(m)$ *and* $O_n \in \mathrm{SO}(n)$*.*

It follows from (23) that a Lorentz transformation Λ of signature (m,n) without rotations is what we call a Lorentz bi-boost, $B_c(V)$, of signature (m,n)

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in \mathrm{SO}_c(m,n)\,. \tag{24}$$

The matrix $B_c(V)$ is a bi-boost (as opposed to a boost) in the sense that it admits in (23) the bi-rotation $(O_m, O_n) \in \mathrm{SO}(m) \times \mathrm{SO}(n)$.

An elegant, straightforward demonstration that $B_c(V) \in \mathrm{SO}_c(m,n)$, as stated in (24), is presented in [44, Section 5.8].

In the special case when the signature is $(m,n) = (1,n)$, the Lorentz bi-boost $B_c(V)$ in (24) descends to the common Lorentz transformation of Einstein's special theory of relativity, as shown in [44, Equation (5.173)], where the parameter $V \in \mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ represents relativistically admissible velocities.

The gyration decomposition (23) presents the generic Lorentz transformation of signature (m,n) as a bi-boost along with a left rotation $O_n \in \mathrm{SO}(n)$ and a right rotation $O_m \in \mathrm{SO}(m)$ acting on $V \in \mathbb{R}_c^{n\times m}$. Collectively, the pair (O_n, O_m) of a left and a right rotation, taking V into $O_n V O_m$, is called a *bi-rotation*.

In the limit of large c, the Lorentz bi-boost $B_c(V)$ of signature (m,n) tends to its Galilean counterpart $B_\infty(V)$, called the Galilei bi-boost of signature (m,n). Indeed, by means of (20) we have

$$\lim_{c\to\infty} B_c(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} =: B_\infty(V) \in \mathrm{SO}_\infty(m,n) \tag{25}$$

in which $V \in \mathbb{R}_c^{n\times m} \xrightarrow[c\to\infty]{} \mathbb{R}^{n\times m}$ and where $\mathrm{SO}_\infty(m,n)$ is the group of all Galilei transformations of signature (m,n). The Galilei bi-boost of signature $(1,3)$ is the common Galilei transformation of classical mechanics. The physical interpretation of the Galilei bi-boost of signature $(1,3)$ is intuitively clear. We will see in Section 7 that the physical interpretation of the Galilei bi-boost of signature $(m,3)$ for any $m > 1$ is intuitively clear as well. The physical interpretation of the Galilei bi-boost of signature $(m, 3)$, in turn, leads to a physical interpretation of the Lorentz bi-boost of signature $(m, 3)$, by means of the additive decomposition of the Lorentz bi-boost that we present in Section 5.

5. Additive V-Decomposition of the Lorentz Bi-Boost

Theorem 6. (Additive V-Decomposition of the Lorentz Bi-boost [44, Theorem 5.83]) . *Let*

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix} \in \mathrm{SO}_c(m,n), \qquad V \in \mathbb{R}_c^{n\times m} \subset \mathbb{R}^{n\times m} \tag{26}$$

be the Lorentz bi-boost of signature (m, n)*,* $m, n \in \mathbb{N}$*, and let*

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}, \qquad V \in \mathbb{R}^{n\times m} \tag{27}$$

be the Galilei bi-boost of signature (m, n)*.*

Then, $B_c(V)$ *and* $B_\infty(V)$ *are related to each other by the additive decomposition of the Lorentz bi-boost*

$$B_c(V) = B_\infty(V) + \frac{1}{c^2}\begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R} V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L} V V^t V & \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L} V V^t \end{pmatrix}, \qquad V \in \mathbb{R}_c^{n\times m}. \tag{28}$$

The additive decomposition (28) in Theorem 6 provides a correspondence between Lorentz and Galilei bi-boosts. Thus, the set of all Lorentz bi-boosts $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, of signature (m, n) is associated to the set of all Galilei bi-boosts $B_\infty(V)$, $V \in \mathbb{R}^{n\times m}$, of same signature (m, n) and, conversely, the set of all Galilei bi-boosts of signature (m, n) is associated to the set of all Lorentz bi-boosts of same signature (m, n), $m, n \in \mathbb{N}$. This association is important, enabling us to interpret extended Lorentz transformations in terms of the intuitively clear interpretation of corresponding extended Galilei transformations of any signature (m, n). It suggests the following formal definition.

Definition 7 (Additive V-Decomposition of the Lorentz Bi-Boost). *Let*

$$B_c(V) = \begin{pmatrix} \Gamma_V^R & \frac{1}{c^2}\Gamma_V^R V^t \\ \Gamma_V^L V & \Gamma_V^L \end{pmatrix}, \qquad V \in \mathbb{R}_c^{n\times m} \tag{29}$$

and

$$B_\infty(V) = \begin{pmatrix} I_m & 0_{m,n} \\ V & I_n \end{pmatrix}, \qquad V \in \mathbb{R}^{n\times m} \tag{30}$$

be the Lorentz bi-boost of signature (m, n) *and its corresponding Galilei bi-boost of same signature* (m, n)*,* $m, n \in \mathbb{N}$*. Furthermore, let*

$$E(V) = \begin{pmatrix} \frac{(\Gamma_V^R)^2}{I_m+\Gamma_V^R} V^t V & \Gamma_V^R V^t \\ \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L} V V^t V & \frac{(\Gamma_V^L)^2}{I_n+\Gamma_V^L} V V^t \end{pmatrix}, \qquad V \in \mathbb{R}_c^{n\times m} \tag{31}$$

so that, by Theorem 6*, we have the Lorentz bi-boost additive* V*-decomposition*

$$B_c(V) = B_\infty(V) + \frac{1}{c^2} E(V), \qquad V \in \mathbb{R}_c^{n\times m}. \tag{32}$$

Following (32)*, we say that* $B_\infty(V)$ *is the Galilean part and* $c^{-2}E(V)$ *is the entanglement part of the Lorentz bi-boost* $B_c(V)$ *of signature* (m, n)*.*

Owing to the presence of the factor c^{-2} in the entanglement part $c^{-2}E(V)$ of the additive V-decomposition (32), non-Galilean relativistic effects are directly noticeable only at very high speeds.

The physical interpretation of the Lorentz bi-boost of signature $(m, 3)$ is well-known when $m = 1$, being the Lorentz boost of Einstein's special theory of relativity. The entanglement part of the Lorentz bi-boost $B_c(V)$ of signature $(1, 3)$ is responsible for the relativistic intertwining of space and time, as well as for other relativistic effects like time dilation, length contraction, Thomas precession and quantum mechanical energy levels.

In order to set the road to extending the physical interpretation of the Lorentz bi-boost of signature $(m, 3)$ from $m = 1$ to $m \geq 1$, we present in Sections 6 and 7 the intuitively clear physical interpretation of the Galilei bi-boost of signature $(m, 3)$, $m \geq 1$.

6. Application of the Galilei Bi-Boost of Signature (1,3)

In this section about the Galilei bi-boost of signature $(1, 3)$, we set the stage for the presentation of the Galilei bi-boost of signature $(2, 3)$ and, hence, of signature $(m, 3)$, $m > 1$, in Section 7.

A Galilei bi-boost of signature $(1, 3)$ is the common Galilei boost of classical mechanics. Let $B_\infty(V) = B_\infty(\mathbf{v})$ be the Galilei bi-boost of signature $(m, n) =$

$(1,3)$, parametrized by the velocity parameter $V = \mathbf{v}$

$$V = \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 = \mathbb{R}^{3\times 1}\,. \tag{33}$$

In order to conform to the formalism of bi-boosts of signature (m,n), $m,n > 1$, in Section 7, we view V as a 3×1 matrix in $\mathbb{R}^{3\times 1}$, and $\mathbf{v}$ as a vector in $\mathbb{R}^3$, noting that $\mathbb{R}^{3\times 1} = \mathbb{R}^3$. Accordingly, V is a matrix the single column of which is $\mathbf{v}$.

Let

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^{4\times 1} \tag{34}$$

be a 4×1 matrix that represents the time-space coordinates of a particle with position $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. The point $(t, \mathbf{x})$ is said to be a particle of signature (m,n)=$(1,3)$ with position $\mathbf{x} \in \mathbb{R}^3$ at time $t \in \mathbb{R}$. A particle of signature (m,n) is also called an (m,n)-particle, in short.

The application of the Galilei bi-boost $B_\infty(V)$ of signature $(m,n) = (1,3)$ to a $(1,3)$-particle $(t,\mathbf{x})$ in $m+n = 1+3$ time-space dimensions yields

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} := B_\infty(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ v_1 t + x_1 \\ v_2 t + x_2 \\ v_3 t + x_3 \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix} \tag{35}$$

where $B_\infty(V)$ is given by (30) with $m = 1$ and $n = 3$.

Accordingly, the Galilei bi-boost $B_\infty(V)$ of signature $(1,3)$ is a Galilei boost that keeps the time invariant, $t' = t$, and boosts the position $\mathbf{x} \in \mathbb{R}^3$ of the particle $(t,\mathbf{x})^t$ into the position $\mathbf{x}' = \mathbf{x} + \mathbf{v}t \in \mathbb{R}^3$, $\mathbf{v} \in \mathbb{R}^3$, of the boosted particle $(t, \mathbf{x} + \mathbf{v}t)^t$, at time t.

7. Application of the Galilei Bi-Boost of Signature (*m,n*)

We are now in the position to explore the Galilei bi-boost of signature (m,n) for all $m,n \in \mathbb{N}$, paying special attention to the case when $(m,n) = (2,3)$ as an illustrative example.

Let $B_\infty(V) = B_\infty(\mathbf{v}_1, \mathbf{v}_2)$ be the Galilei bi-boost of signature $(2,3)$, parametrized by the velocity matrix $V = (\mathbf{v}_1\ \mathbf{v}_2)$

$$V = (\mathbf{v}_1\ \mathbf{v}_2) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{pmatrix} \in \mathbb{R}^{3\times 2} \tag{36}$$

of two velocities $\mathbf{v}_k = (v_{1k},\ v_{2k},\ v_{3k})^t \in \mathbb{R}^3$, $k = 1,2$. The two velocities $\mathbf{v}_k$ form the two columns of the velocity matrix V.

Furthermore, let

$$\begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \in \mathbb{R}^{5\times 2} \tag{37}$$

be a 5×2 matrix that represents a $(2,3)$-particle consisting of the time-space coordinates of two subparticles, $(t_k, \mathbf{x}_k)$, $k = 1,2$, with positions $\mathbf{x}_k = (x_{1k}, x_{2k}, x_{3k})^t \in \mathbb{R}^3$, at time $t_k \in \mathbb{R}$, respectively. Here

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \tag{38}$$

$t_1, t_2 > 0$, is a 2×2 diagonal, positive definite matrix that represents the times t_1 and t_2 when two subparticles are observed at positions $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$, respectively, and

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} = (\mathbf{x}_1\ \mathbf{x}_2) \in \mathbb{R}^{3\times 2} \tag{39}$$

is a 3×2 matrix the columns of which represent the positions $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ of two subparticles at times $t_1, t_2 \in \mathbb{R}$, respectively.

Accordingly, the point $\begin{pmatrix} T \\ X \end{pmatrix} = (T, X) \in \mathbb{R}^{5\times 2}$ represents a $(2,3)$-particle, which is a particle system consisting of two subparticles $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$ with positions $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathbb{R}^3$ at times t_1 and t_2, respectively, $t_1, t_2 \in \mathbb{R}$. Here we use the *displayed notation*, $\begin{pmatrix} T \\ X \end{pmatrix}$, and the *inline notation*, (T, X), interchangeably.

The collective application of the Galilei bi-boost $B_\infty(V)$ of signature $(2,3)$ to the pair of subparticles (T, X) in $m + n = 2 + 3$ time-space dimensions yields

$$\begin{pmatrix} T' \\ X' \end{pmatrix} := B_\infty(V) \begin{pmatrix} T \\ X \end{pmatrix} \tag{40}$$

that is

$$\begin{pmatrix} t'_1 & 0 \\ 0 & t'_2 \\ \mathbf{x}'_1 & \mathbf{x}'_2 \end{pmatrix} := B_\infty(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ v_{11} & v_{12} & 1 & 0 & 0 \\ v_{21} & v_{22} & 0 & 1 & 0 \\ v_{31} & v_{32} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}$$
$$= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ v_{11}t_1 + x_{11} & v_{12}t_2 + x_{12} \\ v_{21}t_1 + x_{21} & v_{22}t_2 + x_{22} \\ v_{31}t_1 + x_{31} & v_{32}t_2 + x_{32} \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix} \tag{41}$$

where $B_\infty(V)$ is given by (30) with $m = 2$ and $n = 3$.

The chain of equations (41) describes the application of a Galilei bi-boost $B_\infty(V)$ of signature $(2, 3)$ to collectively bi-boost two subparticles, $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$, into the two bi-boosted subparticles, $(t_1, \mathbf{x}_1 + \mathbf{v}_1 t_1)$ and $(t_2, \mathbf{x}_2 + \mathbf{v}_2 t_2)$, by 3-dimensional velocities $\mathbf{v}_1 = (v_{11}, v_{21}, v_{31})^t$ and $\mathbf{v}_2 = (v_{12}, v_{22}, v_{32})^t$ in $\mathbb{R}^3$. It is important to note that the two collectively bi-boosted subparticles are not *entangled* in the sense that the boost of each boosted subparticle is independent of the boost of the other boosted subparticle. Interestingly, this observation fails when we replace Galilei bi-boosts of signature $(m, 3)$, $m \geq 2$, by corresponding Lorentz bi-boosts of signature $(m, 3)$, as we will see in the sequel.

Each of the two particles $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$ possesses a one-dimensional time, $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$, respectively. Accordingly, the system consisting of the two particles possesses the two-dimensional time, $(t_1, t_2) \in \mathbb{R}^2$. Each of the two particles possesses its own clock, so that the two-dimensional time of the system is measured by two clocks. In general, a system consisting of m particles possesses an m-dimensional time, measured by m clocks, $m \in \mathbb{N}$.

The extension of (36) – (41) from signature $(2, 3)$ to signature (m, n), $m, n \in \mathbb{N}$, is now obvious. The Galilei bi-boost $B_\infty(V)$ of signature (m, n) is parametrized by a velocity matrix $V \in \mathbb{R}^{n\times m}$ of order $n \times m$ that consists of m columns, $V = (\mathbf{v}_1\ \mathbf{v}_2\ \ \ldots\ \ \mathbf{v}_m)$, which represent the m velocities $\mathbf{v}_1, \mathbf{v}_2,\ \ldots,\ \mathbf{v}_m \in \mathbb{R}^n$ relative to some inertial frame. Furthermore, when $B_\infty(V)$ is applied to collectively bi-boost m particles in $\mathbb{R}^n$ (i) it keeps invariant each of the times t_k, $k = 1, \ldots, m$ of the m particles $(t_k, \mathbf{x}_k)$, that is, $t'_k = t_k$, and (ii) it bi-boosts their positions $\mathbf{x}_k \in \mathbb{R}^n$ into the bi-boosted positions $\mathbf{x}_k + \mathbf{v}_k t_k \in \mathbb{R}^n$ at times t_k, respectively. The m collectively bi-boosted particles are not entangled in the sense that i) the boost of each boosted particle is independent of the boosts and times of the other boosted particles and ii) the time of each boosted particle is independent of the times and boosts of the other boosted particles.

A Galilei bi-boost of signature $(m, 3)$, applied collectively to the m subparticles of an $(m, 3)$-particle is thus equivalent to m Galilei boosts applied individually to each subparticle, yielding no entanglement. In contrast, we will see that the Lorentz bi-boost of signature $(m, 3)$, applied collectively to the m subparticles of an $(m, 3)$-particle, yields subparticle entanglement, where the times and the positions of the subparticles are instantaneously entangled, no matter how far apart the subparticles are.

The chain of equations (41) for the action of Galilei bi-boosts of signature $(2, 3)$ and its obvious extension to the action of Galilei bi-boosts of any signature (m, n), $m, n \in \mathbb{N}$, demonstrate that the extension of the common Galilei boost of signature $(1, 3)$ to Galilei bi-boosts of any signature (m, n) is quite natural. The correspondence (28) between Galilei bi-boosts, $B_\infty(V)$, $V \in \mathbb{R}^{n\times m}$, of signature (m, n) and Lorentz bi-boosts, $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, of same signature (m, n) indicates that the extension of the common Lorentz boost of signature $(1, 3)$ to Lorentz bi-boosts of any signature (m, n) is quite natural as well. Yet, unlike Lorentz bi-boosts, Galilei bi-boosts of signature (m, n) are intuitively clear. We, therefore, explore the Lorentz bi-boosts of any signature (m, n) in Section 8.

8. Application of the Lorentz Bi-Boost of Signature (*m,n*)

We are now in the position to explore the interpretation of the Lorentz bi-boost of signature (m, n) for all $m, n \in \mathbb{N}$, paying special attention to the case when $(m, n) = (2, 3)$ as an illustrative example.

The collective application of the Lorentz bi-boost $B_c(V)$ of signature $(2, 3)$ to the pair of particles (T, X) in $2 + 3$ time-space dimensions is obtained by replacing $B_\infty(V)$ by $B_c(V)$ in (41) and employing the Lorentz bi-boost additive decomposition (32). Accordingly, we obtain the chain of equations

$$\begin{aligned}\begin{pmatrix} T' \\ X' \end{pmatrix} &= \begin{pmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \\ \mathbf{x}'_1 & \mathbf{x}'_2 \end{pmatrix} := B_c(V) \begin{pmatrix} T \\ X \end{pmatrix} = B_c(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \left\{ B_\infty(V) + \frac{1}{c^2} E(V) \right\} \\ &\times \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 + \mathbf{v}_1 t_1 & \mathbf{x}_2 + \mathbf{v}_2 t_2 \end{pmatrix} + \frac{1}{c^2} E(V) \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \\ \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}\end{aligned} \tag{42}$$

where T and X are given by (38)–(39), $V \in \mathbb{R}_c^{3\times 2} \subset \mathbb{R}^{3\times 2}$ is given by (36), and $E(V)$ is given by (31) with $(m, n) = (2, 3)$. Owing to the presence of the factor $1/c^2$ in (42), the entangled time components t'_{21} and t'_{12} in (42) are directly noticeable only at very high speeds.

The relativistic squared bi-norm of each subparticle of the $(2,3)$-particle (T,X) remains invariant under the application in (42) of the bi-boost $B_c(V) \in \mathrm{SO}_c(2,3)$, that is

$$\begin{aligned}(t'_{11})^2 + (t'_{21})^2 - c^{-2}(\mathbf{x}'_1)^2 &= t_1^2 - c^{-2}\mathbf{x}_1^2 \\ (t'_{12})^2 + (t'_{22})^2 - c^{-2}(\mathbf{x}'_2)^2 &= t_2^2 - c^{-2}\mathbf{x}_2^2\end{aligned} \tag{43}$$

where we use the notation $\mathbf{x}^2 = \mathbf{x}\cdot\mathbf{x}$ for vectors $\mathbf{x} \in \mathbb{R}^n$.

Moreover, the relativistic bi-inner product of the two subparticles $(\mathbf{t}_k, \mathbf{x}_k)$, $k = 1, 2$, of the $(2,3)$-particle (T,X) remains invariant under a bi-boost application, as well

$$t'_{11}t'_{12} + t'_{21}t'_{22} - c^{-2}\mathbf{x}'_1\cdot\mathbf{x}'_2 = t_1 0 + 0 t_2 - c^{-2}\mathbf{x}_1\cdot\mathbf{x}_2 = -c^{-2}\mathbf{x}_1\cdot\mathbf{x}_2 \tag{44}$$

as implied from [44, Theorem 6.15].

Remark 8. *The invariance in* (44) *results from the theorem in* [44, Theorem 6.15] *for* $(T,X) \in \mathbb{R}^{(m+n)\times m}$*,* $m, n \in \mathbb{N}$*, in the special case when* $(m,n) = (2,3)$*. Hence, it is important to note the following immediate generalization of the theorem. The two theorems in* [44, Theorem 6.14] *and* [44, Theorem 6.15] *are stated for* $(T,X) \in \mathbb{R}^{(m+n)\times m}$ *for all* $m, n \in \mathbb{N}$*, where* $T \in \mathbb{R}^{m\times m}$ *and* $X \in \mathbb{R}^{n\times m}$*. However, these two theorems and their proof can be unified under a generalization of the theorem in* [44, Theorem 6.15] *to the case when* $(T,X) \in \mathbb{R}^{(m+n)\times k}$ *for all* $m, n, k \in \mathbb{N}$*, so that* $T \in \mathbb{R}^{m\times k}$ *and* $X \in \mathbb{R}^{n\times k}$*. Then, each invariance in* (43) *corresponds to* $(m,n,k) = (2,3,1)$*, and the invariance in* (44) *corresponds to* $(m,n,k) = (2,3,2)$*.*

According to (42), the collective application of a Lorentz bi-boost $B_c(V)$ of signature $(m,n) = (2,3)$ to the constituents of a system of two three-dimensional subparticles generates classical effects and bi-relativistic effects.

The classical effects boost the two subparticles of the system individually by velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, as we see from the first term on the extreme right-hand side of (42). The bi-relativistic effects involve collective entanglement of the times and positions of the system constituents, as we see from the second term on the extreme right-hand side of (42).

It is evidenced by (43) that owing to the presence of entanglement, $(2,3)$-particles involve Lorentz symmetry violation. Indeed, the symmetry group of the $(2,3)$-dimensional spacetime of a $(2,3)$-particle is the Lorentz group of signature $(2,3)$, rather than the common, special relativistic Lorentz group of signature $(1,3)$.

The extension of (42) to the application of Lorentz bi-boosts of any signature (m, n), $m, n \in \mathbb{N}$, to systems of m n-dimensional subparticles is now clear. It suggests that Lorentz bi-boosts of signature (m, n) form the symmetry group by which systems of m n-dimensional moving entangled particles can be understood, just as the Lorentz group of signature $(1, 3)$ forms the symmetry group by which Einstein's special theory of relativity can be understood.

9. Geometric Entanglement in Bi-Hyperbolic Geometry

The parametric realization of the special relativistic Lorentz transformation of signature $(1, n)$, $n > 1$, in 1988 [29] gives rise to Einstein gyrogroups and Einstein gyrovector spaces. Einstein gyrovector spaces, in turn, form the algebraic setting for analytic hyperbolic geometry, as demonstrated in [44, Chapters 2-3] and in [31, 32, 34–37, 39].

In full analogy, the parametric realization of Lorentz transformations of signature (m, n), $m, n > 1$, gives rise in [44] to Einstein bi-gyrogroups and Einstein bi-gyrovector spaces of signature (m, n). Einstein bi-gyrovector spaces, in turn, form the algebraic setting for analytic bi-hyperbolic geometry, as demonstrated in [44]. Bi-hyperbolic geometry turns out to form a natural extension of the analytic hyperbolic geometry studied in [31, 32, 34–37, 39]. Remarkably, unlike hyperbolic geometry, bi-hyperbolic geometry of signature (m, n), $m, n > 1$, involves geometric entanglement, a reminiscent of the entanglement observed in physics. The n-dimensional hyperbolic geometry of Lobachevsky and Bolyai is, in fact, a bi-hyperbolic geometry of signature $(1, n)$, $n > 1$. In this sense, bi-hyperbolic geometry is a natural generalization of hyperbolic geometry from signature $(1, n)$ to signature (m, n), $m, n \in \mathbb{N}$.

The entanglement in $(m + n)$-dimensional spacetime can, thus, be considered as a geometric entanglement in bi-hyperbolic geometry of signature (m, n), studied in detail in [44].

10. Quantum Entanglement and Relativistic-Geometric Entanglement

A plethora of peculiar phenomena emerges in quantum mechanics, the quintessential being entanglement [4]. Quantum entanglement is a physical phenomenon (see [6] for a special example) that occurs when groups of particles interact in ways such that the quantum state of each particle cannot be described independently of the others, even when the particles are separated by a large distance. Instead, a quantum state must be described for a system of particles as a whole.

Quantum entanglement is a great mystery in physics. Einstein, Podolsky and Rosen have issued in 1935 a challenge to quantum physics, claiming that the theory is incomplete. They based their argument on the existence of the entanglement phenomenon, which in turn, had been deduced to exist based on mathematical considerations of quantum systems. They, accordingly, claimed that the theory that allows for the "unreal" phenomenon of entanglement has to be incomplete [11].

Contrasting Einstein, Podolsky and Rosen, we see in this article that entanglement is quite real within the frame of a natural extension of special relativity

1. from a theory that stems from the common Lorentz transformation of signature $(1, 3)$ (which is regulated by hyperbolic geometry)
2. to a theory that stems from the Lorentz transformation of signature $(m, 3)$, for all $m \in \mathbb{N}$ (which is regulated by bi-hyperbolic geometry).

The related extension of hyperbolic geometry to bi-hyperbolic geometry, in turn, is associated with the natural emergence of the Lorentz transformation of signature (m, n), $m, n > 1$, into Einstein's special relativity theory. These generalized Lorentz transformations involve m temporal dimensions and n spatial dimensions as well as a geometric entanglement. The resulting geometric entanglement shares with quantum entanglement a characteristic property. They involve particle systems in such a way that the motion of a constituent particle of a system cannot be described independently of the motion of the others, even when the constituent particles of the system are separated by a large distance. Instead, the motion of the constituent particles must be described for the system as a whole. Graphical demonstrations of geometric entanglements are presented in [44].

11. P-Parametric Realization of Lorentz Transformations of Signature (*m*,*n*)

We have seen that bi-boosts $B_c(V)$ of any signature (m, n), parametrized by $V \in \mathbb{R}_c^{n\times m}$, prove useful in modeling entanglement. Hence, in order to understand our model of entanglement it is important to uncover the algebra of the bi-boost $B_c(V)$ in terms of underlying parameters. However, it is found in [44] that in order to uncover the algebra of the bi-boost $B_c(V)$ it is useful to uncover the algebra of the bi-boost $B_c(P)$, parametrized by $P \in \mathbb{R}^{n\times m}$.

In order to uncover the algebra of the bi-boost $B_c(P)$ in terms of underlying parameters, we present Theorem 9 associated with P, which is analogous to Theorem 5 associated with V.

Theorem 9. (Lorentz Transformation Bi-gyration Decomposition and the P-Parametric Realization [44, Theorem 4.12]). *A matrix* $\Lambda \in \mathbb{R}^{(m+n)\times(m+n)}$, $m, n \in \mathbb{N}$, *is the matrix representation of a Lorentz transformation* Λ *of signature* (m, n), $\Lambda \in \mathrm{SO}_c(m, n)$, *if and only if it possesses the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{J_m + c^{-2}P^tP} & \frac{1}{c^2}P^t \\ P & \sqrt{J_n + c^{-2}PP^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{45}$$

$$\Lambda \in \mathrm{SO}_c(m, n) = \mathrm{SO}(m) \times \mathbb{R}_c^{n\times m} \times \mathrm{SO}(n)$$

parametrized by the main parameter $P \in \mathbb{R}^{n\times m}$, *and the two orientation parameters* $O_m \in \mathrm{SO}(m)$ *and* $O_n \in \mathrm{SO}(n)$.

It follows from (45) that a Lorentz transformation Λ of signature (m, n) without rotations is the Lorentz bi-boost, $B_c(P)$, of signature (m, n)

$$B_c(P) = \begin{pmatrix} \sqrt{J_m + c^{-2}P^tP} & \frac{1}{c^2}P^t \\ P & \sqrt{J_n + c^{-2}PP^t} \end{pmatrix} \in \mathrm{SO}_c(m, n)\,. \tag{46}$$

An elegant, straightforward demonstration that $B_c(P) \in \mathrm{SO}_c(m, n)$, as stated in (46), is presented in [44, Section 4.5].

The gyration decomposition (45) presents the generic Lorentz transformation of signature (m, n) as a bi-boost along with a left rotation $O_n \in \mathrm{SO}(n)$ and a right rotation $O_m \in \mathrm{SO}(m)$ acting on $P \in \mathbb{R}^{n\times m}$. Collectively, the pair (O_n, O_m) of a left and a right rotation, taking P into O_nPO_m, is a *bi-rotation*.

In the limit of large c, the Lorentz bi-boost $B_c(P)$ of signature (m, n) tends to its Galilean counterpart $B_\infty(P)$, which is the Galilei bi-boost of signature (m, n). Indeed, we clearly have

$$\lim_{c\to\infty} B_c(P) = \begin{pmatrix} I_m & 0_{m,n} \\ P & I_n \end{pmatrix} =: B_\infty(P) \in \mathrm{SO}_\infty(m, n), \qquad P \in \mathbb{R}^{n\times m} \tag{47}$$

where $\mathrm{SO}_\infty(m, n)$ is the group of all Galilei transformations of signature (m, n).

12. Additive P-Decomposition of the Lorentz Bi-Boost

Theorem 10 (Additive P-Decomposition of the Lorentz Bi-Boost). *Let*

$$B_c(P) = \begin{pmatrix} B_m^R(P) & \frac{1}{c^2}P^t \\ P & B_n^L(P) \end{pmatrix} \in \mathrm{SO}_c(m, n) \subset \mathbb{R}^{(m+n)\times(m+n)} \tag{48}$$

$P \in \mathbb{R}^{n\times m}$*, be the Lorentz bi-boost of signature* (m,n)*,* $m,n \in \mathbb{N}$*, defined in* (46)*, where*

$$B_m^R(P) = \sqrt{I_m + c^{-2}P^tP} \in \mathbb{R}^{m\times m}, \qquad B_n^L(P) = \sqrt{I_n + c^{-2}PP^t} \in \mathbb{R}^{n\times n} \tag{49}$$

and let

$$B_\infty(P) = \begin{pmatrix} I_m & 0_{m,n} \\ P & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)} \tag{50}$$

be the Galilei bi-boost of signature (m,n)*.*

Then, $B_c(P)$ *and* $B_\infty(P)$ *are related to each other by the additive* P*-decomposition of the Lorentz bi-boost*

$$B_c(P) = B_\infty(P) + \frac{1}{c^2}\begin{pmatrix} P^t(I_n + B_n^L(P))^{-1}P & P^t \\ 0_{n,m} & P(I_m + B_m^R(P))^{-1}P^t \end{pmatrix} \tag{51}$$

for all $P \in \mathbb{R}^{n\times m}$*.*

Proof: By [44, Equations (5.159)-(5.160)], the matrices $B_m^R(P)$ and $B_n^L(P)$ are related to each other by the identities

$$\begin{aligned} B_m^R(P) &= I_m + \frac{1}{c^2}P^t(I_n + B_n^L(P))^{-1}P \\ B_n^L(P) &= I_n + \frac{1}{c^2}P(I_m + B_m^R(P))^{-1}P^t\,. \end{aligned} \tag{52}$$

Substituting (52) into (48) yields (51). ■

Theorem 10 suggests the following formal definition.

Definition 11 (Additive P-Decomposition of the Lorentz Bi-boost). *Let*

$$B_c(P) = \begin{pmatrix} B_m^R(P) & \frac{1}{c^2}P^t \\ P & B_n^L(P) \end{pmatrix} \in \mathrm{SO}_c(m,n) \subset \mathbb{R}^{(m+n)\times(m+n)} \tag{53}$$

and

$$B_\infty(P) = \begin{pmatrix} I_m & 0_{m,n} \\ P & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)} \tag{54}$$

$P \in \mathbb{R}^{n\times m}$*, be a Lorentz bi-boost of signature* (m,n) *and its corresponding Galilei bi-boost of same signature* (m,n)*,* $m,n \in \mathbb{N}$*. Furthermore, let*

$$E(P) = \begin{pmatrix} P^t(I_n + B_n^L(P))^{-1}P & P^t \\ 0_{n,m} & P(I_m + B_m^R(P))^{-1}P^t \end{pmatrix} \tag{55}$$

$P \in \mathbb{R}^{n\times m}$, *so that by* Theorem 10 *we have the Lorentz bi-boost additive* P*-decomposition*

$$B_c(P) = B_\infty(P) + \frac{1}{c^2}E(P) \tag{56}$$

for all $P \in \mathbb{R}^{n\times m}$.

Following (56) *we say that* $B_\infty(P)$ *is the Galilean part and* $c^{-2}E(P)$ *is the entanglement part of the Lorentz bi-boost* $B_c(P)$ *of signature* (m, n).

Owing to the presence of the factor c^{-2} in the entanglement part $c^{-2}E(P)$ of the additive P-decomposition (56), non-Galilean relativistic effects are directly noticeable only at very high speeds.

13. The P-Composition Law

Let

$$B_c(P_k) = \begin{pmatrix} B_m^R(P_k) & \frac{1}{c^2}P_k^t \\ P_k & B_n^L(P_k) \end{pmatrix} \in \mathrm{SO}_c(m,n) \tag{57}$$

$P_k \in \mathbb{R}^{n\times m}$, $k = 1, 2$, be two Lorentz bi-boosts of signature (m, n), $m, n \in \mathbb{N}$. Their product

$$\begin{aligned} \Lambda := B_c(P_1)B_c(P_2) &= \begin{pmatrix} B_m^R(P_1) & \frac{1}{c^2}P_1^t \\ P_1 & B_n^L(P_1) \end{pmatrix} \begin{pmatrix} B_m^R(P_2) & \frac{1}{c^2}P_2^t \\ P_2 & B_n^L(P_2) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{c^2}P_1^tP_2 + B_m^R(P_1)B_m^R(P_2) & \frac{1}{c^2}(B_m^R(P_1)P_2^t + P_1^tB_n^L(P_2)) \\ P_1B_m^R(P_2) + B_n^L(P_1)P_2 & \frac{1}{c^2}P_1P_2^t + B_n^L(P_1)B_n^L(P_2) \end{pmatrix} \end{aligned} \tag{58}$$

need not be a Lorentz bi-boost. However, the product of two Lorentz bi-boosts of signature (m, n) is a Lorentz transformation of signature (m, n) and, hence, possesses the unique bi-gyration decomposition (45), parametrized by a main parameter P and two orientation parameters O_m and O_n, which are uniquely determined by P_1 and P_2. In order to emphasize that these three parameters are determined by P_1 and P_2, they are denoted by

$$P =: P_1 \oplus P_2 \in \mathbb{R}^{n\times m} \tag{59}$$

and

$$O_m =: \mathrm{rgyr}[P_1, P_2] \in \mathrm{SO}(m), \qquad O_n =: \mathrm{lgyr}[P_1, P_2] \in \mathrm{SO}(n)\,. \tag{60}$$

We call $P_1 \oplus P_2$ the composition of P_1 and P_2, viewing $\oplus$ as a binary operation in $\mathbb{R}^{n\times m}$.

We call $\mathrm{rgyr}[P_1, P_2]$ (respectively $\mathrm{lgyr}[P_1, P_2]$) the right (resp. left) gyration generated by P_1 and P_2. Collectively, the pair consisting of a right and a left gyration generated by P_1 and P_2 is called the bi-gyration generated by P_1 and P_2.

Accordingly, the bi-boost product (58) possesses the unique bi-gyration decomposition (45) which, by means of (59) – (60), takes the form

$$\begin{aligned}\Lambda &= B_c(P_1)B_c(P_2)\\ &= \begin{pmatrix}\mathrm{rgyr}[P_1,P_2] & 0_{m,n}\\ 0_{n,m} & I_n\end{pmatrix}\begin{pmatrix}B_m^R(P_1\oplus P_2) & \frac{1}{c^2}(P_1\oplus P_2)^t\\ P_1\oplus P_2 & B_n^L(P_1\oplus P_2)\end{pmatrix}\times\begin{pmatrix}I_m & 0_{m,n}\\ 0_{n,m} & \mathrm{lgyr}[P_1,P_2]\end{pmatrix}\\ &= \begin{pmatrix}\mathrm{rgyr}[P_1,P_2]B_m^R(P_1\oplus P_2) & \frac{1}{c^2}\mathrm{rgyr}[P_1,P_2](P_1\oplus P_2)^t\mathrm{lgyr}[P_1,P_2]\\ P_1\oplus P_2 & B_n^L(P_1\oplus P_2)\mathrm{lgyr}[P_1,P_2]\end{pmatrix}.\end{aligned} \tag{61}$$

The bi-gyration decomposition (61) of $\mathrm{SO}(m,n)$ and the bi-gyration decomposition (58) of the bi-boost product enable the composition $P_1\oplus P_2$ and the right and left gyrations, $\mathrm{rgyr}[P_1, P_2]$ and $\mathrm{lgyr}[P_1, P_2]$, to be expressed in terms of P_1 and P_2. Indeed, by [44, Theorem 4.18], it follows from (58) and (61) that $P_1\oplus P_2$, $\mathrm{rgyr}[P_1, P_2]$ and $\mathrm{lgyr}[P_1, P_2]$ are given by the equations

$$P_1\oplus P_2 = P_1B_m^R(P_2) + B_n^L(P_1)P_2 \in \mathbb{R}^{n\times m} \tag{62}$$

and

$$\begin{aligned}\mathrm{rgyr}[P_1,P_2] &= \{c^{-2}P_1^tP_2 + B_m^R(P_1)B_m^R(P_2)\}(B_m^R(P_1\oplus P_2))^{-1} \in \mathrm{SO}(m)\\ \mathrm{lgyr}[P_1,P_2] &= (B_n^L(P_1\oplus P_2))^{-1}\{c^{-2}P_1P_2^t + B_n^L(P_1)B_n^L(P_2)\} \in \mathrm{SO}(n)\end{aligned} \tag{63}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

It proves useful to replace the binary operation $\oplus$ in $\mathbb{R}^{n\times m}$ by the binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ given by the *P-composition law*

$$P_1\oplus' P_2 = (P_1\oplus P_2)\mathrm{rgyr}[P_2, P_1]\,. \tag{64}$$

It also proves useful to combine the right and left gyrations, say $\mathrm{lgyr}[P_1, P_2]$ and $\mathrm{rgyr}[P_1, P_2]$ into a composite gyration, $\mathrm{gyr}\,[P_1, P_2]$, according to the equation

$$\mathrm{gyr}\,[P_1,P_2]P = \mathrm{lgyr}[P_1,P_2]P\mathrm{rgyr}[P_2,P_1] \tag{65}$$

for all $P, P_1, P_2 \in \mathbb{R}^{n\times m}$, so that $\mathrm{gyr}\,[P_1, P_2]$ is an automorphism of the gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$.

Remarkably, the construction of gyrations $\mathrm{gyr}\,[P_1, P_2]$ from bi-gyrations

$$(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$$

$P_1, P_2 \in \mathbb{R}^{n\times m}$ in (65), turns bi-gyrogroups into gyrocommutative gyrogroups. Indeed, it is shown in [44] that for any signature (m, n), $m, n \in \mathbb{N}$, the groupoid $(\mathbb{R}^{n\times m}, \oplus')$ forms a gyrocommutative gyrogroup with gyrations given by (65).

In order to indicate the presence of physical significance, we will show in Section 15 that in the special case when $(m, n) = (1, 3)$ the gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ descends to the Einstein gyrogroup $(\mathbb{R}^{3\times 1}, \oplus')$=$(\mathbb{R}^3, \oplus_{\mathrm{E}})$. The Einstein gyrogroup $(\mathbb{R}^3, \oplus_{\mathrm{E}})$, in turn, is the set $\mathbb{R}^3$ of all proper velocities of special relativity theory, equipped with the binary operation $\oplus_{\mathrm{E}} = \oplus'$ in $\mathbb{R}^3$ given by the Einstein proper-velocity addition law.

Einstein gyrogroups are studied, for instance, in [31] and [34]. Einstein gyrogroups and gyrovector spaces of proper velocities are studied in [31, Chapter 5]. The intrinsic beauty, harmony and interdisciplinarity in Einstein gyrogroups and gyrovector spaces is demonstrated in [42]. The universal significance of gyrogroups and gyrovector spaces is illustrated, for instance, in [5] and [1, 2], [8, 18, 19, 46], [3, 12, 13, 23, 28], [24, 26, 27], [45].

14. Gyrogroups

A gyrogroup is a nonassociative group-like object, first arose as an algebraic structure in special relativity theory [29, 31]. The formal definitions of a *gyrogroup* and a *gyrocommutative gyrogroup* are as follows. Let $(G, \oplus)$ be a groupoid, that is, a nonempty set G with a binary operation $\oplus$, and let $\mathrm{Aut}(G, \oplus)$ be the group of automorphisms of the groupoid $(G, \oplus)$.

Definition 12 (Gyrogroups). *A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation obeys the following axioms.*

G1) *There is an element $0 \in G$ such that $0\oplus a = a$ for all $a \in G$.*

G2) *For each $a \in G$, there is an element $b \in G$ such that $b\oplus a = 0$.*

G3) *For all $a, b \in G$ there is an automorphism $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}(G, \oplus)]$ such that*

$$a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}\,[a, b]c \qquad \text{(Left Gyroassociative Law)}$$

for all $c \in G$.

G4) *For all* $a, b \in G$

$$\mathrm{gyr}\,[a, b] = \mathrm{gyr}\,[a \oplus b, b] \qquad \text{(Left Reduction Property)}.$$

Definition 13 (Gyrocommutative Gyrogroups). *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the following axiom.*

G5) *For all* $a, b \in G$

$$a \oplus b = \mathrm{gyr}\,[a, b](b \oplus a) \qquad \text{(Gyrocommutative Law)}.$$

For all $a, b \in G$, gyrations $\mathrm{gyr}\,[a, b]$ are special automorphisms of the gyrogroup $(G, \oplus)$. Hence, they are also called *gyroautomorphisms*. Gyroautomorphisms, in turn, form the mathematical abstraction of the special relativistic effect known as *Thomas precession*. The term *gyration* suggests our *gyrolanguage*, in which we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and in nonassociative algebra. Our gyroterminology thus conveys a world of meaning in an elegant and memorable fashion.

15. Einstein Gyrogroup of Proper Velocities

In order to uncover the physical interpretation of the Lorentz bi-boost $B_c(P)$ parametrized by $P \in \mathbb{R}^{n \times m}$, we consider in this section the special case when the signature is $(m, n) = (1, n)$, where $n = 3$ in physical applications. For $m = 1$ we have $P \in \mathbb{R}^{n \times 1} = \mathbb{R}^n$ and $P^t P = \|P\|^2$, so that the second equation in (52) descends to

$$\sqrt{I_n + c^{-2} P P^t} = I_n + \frac{1}{c^2} \frac{1}{1 + \sqrt{1 + c^{-2}\|P\|^2}} P P^t \,. \tag{66}$$

Hence, by (62) with $m = 1$ and by (66)

$$\begin{aligned} P_1 \oplus P_2 &= P_1 \sqrt{I_m + c^{-2} P_2^t P_2} + \sqrt{I_n + c^{-2} P_1 P_1^t} P_2 \\ &= P_1 \sqrt{1 + c^{-2}\|P_2\|^2} + \Big\{ I_n + \frac{1}{c^2} \frac{1}{1 + \sqrt{1 + c^{-2}\|P_1\|^2}} P_1 P_1^t \Big\} P_2 \\ &= P_1 \sqrt{1 + c^{-2}\|P_2\|^2} + P_2 + \frac{1}{c^2} \frac{1}{1 + \sqrt{1 + c^{-2}\|P_1\|^2}} P_1 P_1^t P_2 \end{aligned} \tag{67}$$

for all $P_1, P_2 \in \mathbb{R}^n$.

Let

$$\beta_P = \frac{1}{\sqrt{1 + c^{-2}\|P\|^2}}. \tag{68}$$

Then, (67) takes the form

$$\begin{aligned} P_1 \oplus P_2 &= P_1 \frac{1}{\beta_{P_2}} + P_2 + \frac{1}{c^2} \frac{1}{1 + \frac{1}{\beta_{P_1}}} (P_1 \cdot P_2) P_1 \\ &= \frac{1}{\beta_{P_2}} P_1 + P_2 + \frac{1}{c^2} \frac{\beta_{P_1}}{1 + \beta_{P_1}} (P_1 \cdot P_2) P_1 \\ &= P_1 + P_2 + \Big\{ \frac{\beta_{P_1}}{1 + \beta_{P_1}} \frac{P_1 \cdot P_2}{c^2} + \frac{1 - \beta_{P_2}}{\beta_{P_2}} \Big\} P_1 \,. \end{aligned} \tag{69}$$

Hence, when the signature is $(m, n) = (1, n)$, the parameter composition $P_1 \oplus P_2$ in (62) descends to the parameter composition law

$$P_1 \oplus P_2 = P_1 + P_2 + \Big\{ \frac{\beta_{P_1}}{1 + \beta_{P_1}} \frac{P_1 \cdot P_2}{c^2} + \frac{1 - \beta_{P_2}}{\beta_{P_2}} \Big\} P_1, \qquad m = 1 \tag{70}$$

for all $P_1, P_2 \in \mathbb{R}^n$.

Finally, (70) is recognized as the common special relativistic proper velocity addition law presented in [31, Equation (5.11)] and [33].

Following (64), we are interested in the binary operation $\oplus'$ rather than $\oplus$ in $\mathbb{R}^{n \times m}$, since unlike the latter, the former is a gyrogroup binary operation. Moreover, when $m = 1$ the two binary operations, $\oplus'$ and $\oplus$, coincide

$$\oplus' = \oplus, \qquad m = 1. \tag{71}$$

Indeed, this result follows immediately from (64) and from (60), asserting that for $m = 1$ we have $\mathrm{rgyr}[P_1, P_2] \in \mathrm{SO}(1) = \{1\}$ implying $\mathrm{rgyr}[P_1, P_2] = 1$.

Hence, like $\oplus$, the binary operation $\oplus'$ descends to the binary operation $\oplus$ in (70), when $m = 1$. This result justifies the notation

$$\oplus_{\mathrm{E}} := \oplus' \tag{72}$$

viewing the binary operation $\oplus_{\mathrm{E}}$ in $\mathbb{R}^{n \times m}$ as the Einstein addition in $\mathbb{R}^{n \times m}$ of proper velocities.

Accordingly, we interpret the parameter $P \in \mathbb{R}^{n \times m}$ as a proper velocity matrix of order $m \times n$. The m columns of P are the m proper velocities of the m constituents of a particle system consisting of m n-dimensional entangled particles.

The Lorentz bi-boost $B_c(P)$ and Einstein proper velocity addition $\oplus_{\rm E}$ in $\mathbb{R}^{n\times m}$ give rise to the P-polar decomposition of bi-boost products

$$B_c(P_1)B_c(P_2) = B_c(P_1 \oplus_{\rm E} P_2)\begin{pmatrix}\mathrm{rgyr}[P_1,P_2] & 0_{m,n}\\ 0_{n,m} & \mathrm{lgyr}[P_1,P_2]\end{pmatrix} \tag{73}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$, where $\oplus_{\rm E}$ is Einstein proper velocity addition in $\mathbb{R}^{n\times m}$, as shown in [44, Equation (4.268)].

Following (72), (64) and (62), Einstein proper velocity addition $\oplus_{\rm E}$ in $\mathbb{R}^{n\times m}$ is given by

$$P_1 \oplus_{\rm E} P_2 = \{P_1 B_m^R(P_2) + B_n^L(P_1)P_2\}\mathrm{rgyr}[P_2,P_1] \in \mathbb{R}^{n\times m} \tag{74}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$, where $\mathrm{rgyr}[P_2,P_1]$ follows from (63).

It is shown in [44] that the bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus_{\rm E})$ of the parameter $P \in \mathbb{R}^{n\times m}$ is a gyrocommutative gyrogroup. As such, it obeys the elegant identities

$$\begin{array}{lll}
P_1 \oplus_{\rm E} P_2 = \mathrm{gyr}\,[P_1,P_2](P_2 \oplus_{\rm E} P_1) & \text{Gyrocommutative Law} \\
P_1 \oplus_{\rm E} (P_2 \oplus_{\rm E} P_3) = (P_1 \oplus_{\rm E} P_2) \oplus_{\rm E} \mathrm{gyr}\,[P_1,P_2]P_3 & \text{Left Gyroassociative Law} \\
(P_1 \oplus_{\rm E} P_2) \oplus_{\rm E} P_3 = P_1 \oplus_{\rm E} (P_2 \oplus_{\rm E} \mathrm{gyr}\,[P_2,P_1]P_3) & \text{Right Gyroassociative Law} \\
\mathrm{gyr}\,[P_1 \oplus_{\rm E} P_2, P_2] = \mathrm{gyr}\,[P_1,P_2] & \text{Left Reduction Property} \\
\mathrm{gyr}\,[P_1, P_2 \oplus_{\rm E} P_1] = \mathrm{gyr}\,[P_1,P_2] & \text{Right Reduction Property} \\
\mathrm{gyr}\,[\ominus_{\rm E} P_1, \ominus_{\rm E} P_2] = \mathrm{gyr}\,[P_1,P_2] & \text{Gyration Even Property} \\
(\mathrm{gyr}\,[P_1,P_2])^{-1} = \mathrm{gyr}\,[P_2,P_1] & \text{Gyration Inversion Law}
\end{array} \tag{75}$$

for all $P_1, P_2, P_3 \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$.

Furthermore, it is shown in [44] that

1. the bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus_{\rm E})$ of the parameter $P \in \mathbb{R}^{n\times m}$ with Einstein proper velocity addition $\oplus_{\rm E}$ of proper velocities in $\mathbb{R}^{n\times m}$ and
2. the bi-gyrogroup $(\mathbb{R}_c^{n\times m}, \oplus_{\rm E})$ of the parameter $V \in \mathbb{R}_c^{n\times m}$ with Einstein addition $\oplus_{\rm E}$ of relativistically admissible velocities in the ball $\mathbb{R}_c^{n\times m}$

are isomorphic gyrogroups. Hence, for instance, (73) has its counterpart, the V-polar decomposition of bi-boost products

$$B_c(V_1)B_c(V_2) = B_c(V_1 \oplus_{\rm E} V_2)\begin{pmatrix}\mathrm{rgyr}[V_1,V_2] & 0_{m,n}\\ 0_{n,m} & \mathrm{lgyr}[V_1,V_2]\end{pmatrix} \tag{76}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$, where $\oplus_{\text{E}}$ is Einstein addition of relativistically admissible velocities in $\mathbb{R}_c^{n\times m}$, presented in [44, Equation (6.54)]. Note that $\oplus_{\text{E}}$ in $\mathbb{R}^{n\times m}$ and $\oplus_{\text{E}}$ in $\mathbb{R}_c^{n\times m}$ are different, but isomorphic, binary operations. Similarly, bi-gyrations in $\mathbb{R}^{n\times m}$ and in $\mathbb{R}_c^{n\times m}$ are different, but isomorphic.

The polar decomposition (76) of bi-boost products determines the Einstein addition $\oplus_{\text{E}}$ in $\mathbb{R}_c^{n\times m}$. Similarly, the identity

$$B_c(r\otimes V) = (B_c(V))^r \tag{77}$$

for all $r \in \mathbb{R}$ and $V \in \mathbb{R}_c^{n\times m}$, determines $r\otimes V \in \mathbb{R}_c^{n\times m}$ in terms of r and V. The resulting Einstein scalar multiplication $r\otimes V$ is presented in [44, Theorem 5.86].

Since $(\mathbb{R}_c^{n\times m}, \oplus_{\text{E}})$ is isomorphic to $(\mathbb{R}^{n\times m}, \oplus_{\text{E}})$, the bi-gyrogroup $(\mathbb{R}_c^{n\times m}, \oplus_{\text{E}})$ is a gyrocommutative gyrogroup, obeying the elegant identities

$$\begin{aligned}
V_1\oplus_{\text{E}}V_2 &= \text{gyr}\,[V_1, V_2](V_2\oplus_{\text{E}}V_1) && \text{Gyrocommutative Law}\\
V_1\oplus_{\text{E}}(V_2\oplus_{\text{E}}V_3) &= (V_1\oplus_{\text{E}}V_2)\oplus_{\text{E}}\text{gyr}\,[V_1, V_2]V_3 && \text{Left Gyroassociative Law}\\
(V_1\oplus_{\text{E}}V_2)\oplus_{\text{E}}V_3 &= V_1\oplus_{\text{E}}(V_2\oplus_{\text{E}}\text{gyr}\,[V_2, V_1]V_3) && \text{Right Gyroassociative Law}\\
\text{gyr}\,[V_1\oplus_{\text{E}}V_2, V_2] &= \text{gyr}\,[V_1, V_2] && \text{Left Reduction Property}\\
\text{gyr}\,[V_1, V_2\oplus_{\text{E}}V_1] &= \text{gyr}\,[V_1, V_2] && \text{Right Reduction Property}\\
\text{gyr}\,[\ominus_{\text{E}}V_1, \ominus_{\text{E}}V_2] &= \text{gyr}\,[V_1, V_2] && \text{Gyration Even Property}\\
(\text{gyr}\,[V_1, V_2])^{-1} &= \text{gyr}\,[V_2, V_1] && \text{Gyration Inversion Law}
\end{aligned} \tag{78}$$

for all $V_1, V_2, V_3 \in \mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$. Furthermore, when $m = 1$, Einstein addition $\oplus_{\text{E}}$ in the ball $\mathbb{R}_c^{n\times m}$ descends to the special relativistic addition of relativistically admissible velocities.

For any $m, n \in \mathbb{N}$, the bi-gyrogroup $(\mathbb{R}_c^{n\times m}, \oplus_{\text{E}})$ admits the Einstein scalar multiplication $\otimes$ in $\mathbb{R}_c^{n\times m}$, turning itself into a bi-gyrovector space $(\mathbb{R}_c^{n\times m}, \oplus_{\text{E}}, \otimes)$, the underlying geometry of which is naturally called *analytic bi-hyperbolic geometry of signature* (m, n) [44]. When $m = 1$, analytic bi-hyperbolic geometry of signature $(m, n) = (1, n)$ descends to the n-dimensional hyperbolic geometry of Lobachevsky and Bolyai, studied analytically in [31,32,34–37,39], which involves no entanglement. When $m \geq 2$, analytic bi-hyperbolic geometry of signature (m, n) involves entanglement, studied in [44], where it is illustrated graphically for signature $(m, n) = (3, 2)$.

16. Conclusion

It is well known that Einstein's special relativity theory stems from the Lorentz transformation group of signature $(1, 3)$ [16]. In this paper and, with more details in [44], we have extended the parametric representation of the Lorentz transformation group from signature $(1,3)$ to signature (m,n) for all $m,n \in \mathbb{N}$, thus obtaining an extended special relativity theory. In physical applications $n=3$, but in geometry, $n \in \mathbb{N}$. Corresponding extended Galilei transformations of signature (m,n), which are intuitively clear, suggest the following physical interpretation of Lorentz transformations of signature $(m,3)$ when $m>1$.

In the same way that in Einstein's special relativity theory the Lorentz group of signature $(1,3)$ is the symmetry group of spacetime of particles, in extended special relativity theory, the Lorentz group of signature $(m,3)$ is the symmetry group of spacetime of particle systems that consist of m three-dimensional entangled particles.

Lorentz symmetry is one of the cornerstones of modern physics. However, it is known that entangled particles involve Lorentz symmetry violation. Hence, following this paper and our book [44], it is appropriate to explore whether our model of $m>1$ entangled particles within the frame of our extended special relativity theory can be tested experimentally.

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