



BRANES ON G -MANIFOLDS

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Abstract. Let X be Calabi-Yau manifold acted by a group G . We give a definition of G -equivariance for branes on X , and assign to each equivariant brane an element of the equivariant cohomology of X that can be considered as a charge of the brane. We prove that the spaces of strings stretching between equivariant branes support representations of G . This fact allows us to give formulas for the dimension of some of such spaces, when X is a flag manifold of G .

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1. Introduction

Let X be a compact Kähler n -manifold analytically acted by a Lie group G . Some objects related with X admit an “equivariant” version, when they are equipped with a G -action compatible with its structure, for example, the equivariant vector bundles on X .

In the same way, it seems natural to consider “equivariant” B -branes on X , i.e., B -branes endowed with a G -action which lifts the G -action on X . In this article we will concern with such branes. We will associate them “equivariant charges”, consider its equivariant cohomology, study the spaces of strings stretching between “equivariant” branes and the correlation functions for vertex operators for these strings, etc.

Besides the Introduction, the article consists of three sections: Equivariant branes, Cohomology of equivariant branes and an Appendix. In the Appendix, we present detailed proofs of two propositions which are stated in Section 2. The following is a brief explanation of the key points considered in the first two sections.

Equivariant branes. A D -brane of type B on X can be considered as an object of the derived category of coherent sheaves on X (see monograph [2], which includes a large list of specific references). Particular B -branes are the coherent sheaves. There exists a definition of G -equivariance for sheaves, which generalizes the one for equivariant vector bundles and, obviously, applicable to branes which are coherent sheaves. (In Subsection 2.1, we recall this concept).

To explain the extension of that definition to a general brane, we introduce some notations. In the definition of B -branes is involved the category \mathcal{Coh} of coherent sheaves on X , which is a subcategory of \mathcal{Mod} , the category of \mathcal{O}_X -modules [21]. We put \mathcal{Mod}^G for the subcategory of \mathcal{Mod} whose objects are G -equivariant sheaves and we denote by \mathcal{Coh}^G the full subcategory of \mathcal{Mod}^G whose objects belong to \mathcal{Coh} . Hence, we have the following subcategories of \mathcal{Mod}

$$\begin{array}{ccc} \mathcal{Coh}^G & \subset & \mathcal{Mod}^G \\ \cap & & \cap \\ \mathcal{Coh} & \subset & \mathcal{Mod} \end{array}$$

As we said, a brane is an object of $D(\mathcal{Coh})$, the bounded derived category of \mathcal{Coh} . One possible translation of the concept of equivariance to more general branes, is to define the G -equivariant branes as the objects of $D(\mathcal{Coh}^G)$, the bounded derived category of the abelian category \mathcal{Coh}^G . In this article we will adopt this point of view.

In the definition of the spaces of strings between branes are involved the \mathcal{Ext} groups. More precisely, let \mathcal{F} and \mathcal{G} be two general B -branes, then an open string between \mathcal{F} and \mathcal{G} with ghost number k is an element of the Ext group $\mathcal{Ext}^k(\mathcal{F}, \mathcal{G})$ [1, Section 5.2]. On the other hand, the space of local operators for strings with ghost number k stretching from \mathcal{F} to \mathcal{G} is (see [29])

$$\bigoplus_q H^q(X, \mathcal{Ext}^k(\mathcal{F}, \mathcal{G})). \quad (1)$$

Although \mathcal{Coh}^G has not enough injectives, the category \mathcal{Mod}^G is abelian and it has sufficient injectives [16]. The existence of "sufficient injectives" allows us to construct in \mathcal{Mod}^G the $\mathcal{E}xt$ groups, and the $\mathcal{E}xt$ functors. That is, the space of strings between two objects of $D(\mathcal{Coh}^G)$ can be defined by considering them as objects of $D(\mathcal{Mod}^G)$, the derived category of \mathcal{Mod}^G . In this way, the equivariance of branes gives rise to representations of G on the corresponding spaces of strings and on the vertex operators. These results are stated in Propositions 5 and 6.

When \mathcal{F} and \mathcal{G} are locally free sheaves, we will prove that the correlation functions of vertex operators for strings between these branes are G -invariant (Proposition 10).

The Borel-Weil-Bott theorem permits to determine the dimension of the space of strings between branes which are G -equivariant locally free sheaves on a flag manifold of G and the result is stated in Proposition 8. When the branes are rank one locally free sheaves, we determine the highest weight of the representation of G supported by the corresponding spaces of vertex operators (Proposition 9). In this way, the computation of the dimension of these spaces reduces to the application of the Weyl's dimension formula.

Cohomology of equivariant branes. By \bar{X} we denote the homotopy quotient $\bar{X} := EG \times_G X$, where EG is the universal bundle of the group G . Given an equivariant brane \mathcal{F} on X , it admits a lift to an object $\bar{\mathcal{F}}$ of the derived category of sheaves on \bar{X} . In fact, the pair $(\bar{\mathcal{F}}, \mathcal{F})$ determines an object in the equivariant derived category $D_G(X)$, introduced by Bernstein and Lunts in [4]. Thus, one can define the equivariant cohomology of the brane \mathcal{F} , as the cohomology $H(\bar{X}, \bar{\mathcal{F}})$.

In the case that the group G is a compact torus T , the localization theorems for the equivariant cohomology are applicable to the groups $H^p(\bar{X}, \bar{\mathcal{F}})$. In this way we give, in Corollary 15, a necessary condition for two T -equivariant branes of \mathcal{Coh}^T be equivalent.

When X is an algebraic variety and under certain hypotheses on the G -action, each object of \mathcal{Coh}^G admits a resolution consisting of G -equivariant locally free sheaves [28]. Thus, it is possible to define G -equivariant charges for the branes of $D(\mathcal{Coh}^G)$, i.e., elements of the equivariant cohomology $H_G(X)$, which coincide with the usual charges when $G = \{1\}$. For a brane $\mathcal{F} \in D(\mathcal{Coh}^G)$, we define the equivariant charge $Q^G(\mathcal{F})$ as the product of the equivariant Chern character of \mathcal{F} and the equivariant Todd class of X .

Some equivariant charges admit interpretations in terms of the index of elliptic operators. For example, when the equivariant brane is the sheaf $\mathcal{O}(V)$, of sections of the holomorphic vector bundle V , and G acts on X as a group of isometries, then the Dirac operator D defined on $(\bigwedge T^*X) \otimes V$ is G -equivariant, and

$\ker(D) - \text{coker}(D)$ supports a virtual representation of G . The character of this representation is equal to the evaluation of $Q^G(V)$ on X (see Proposition 17).

Let X be a toric manifold and T be the torus whose action on X defines the toric structure. Given a T -equivariant brane on X which is a locally free sheaf $\mathcal{O}(V)$, applying the localization formulas of equivariant cohomology, we will evaluate the equivariant charge $Q^T(\mathcal{O}(V))$ in terms of data associated to the fixed points of the T -action on X . The result is stated in Proposition 18.

2. Equivariant Branes

In this section, we introduce the category \mathfrak{Mod}^G of G -equivariant \mathcal{O} -modules, define a representation on the cohomology groups of the branes which are objects of $D(\mathfrak{Coh}^G)$, and characterize the space of strings stretching between some equivariant branes on a flag manifold of G .

2.1. Equivariant sheaves

By \mathcal{O}_X , or simply by \mathcal{O} , we denote the sheaf of regular functions on X . Let $\mu : G \times X \rightarrow X$ be an *analytic* action of a reductive Lie group G on X . Essentially, a G -equivariant structure on the \mathcal{O} -module \mathcal{H} is given by a family $\{\lambda_{g,x}\}$ of isomorphisms between the stalks

$$\lambda_{g,x} : \mathcal{H}_x \rightarrow \mathcal{H}_{\mu(g,x)}, \quad \text{for all } g \in G, x \in X \quad (2)$$

compatible with the multiplication in G (i.e., satisfying the cocycle condition).

To formulate the cocycle condition, we introduce the following notations

$$\begin{aligned} m : G \times G \rightarrow G, \quad m(g_1, g_2) = g_1 g_2, \quad b : G \times X \mapsto x \in X, \quad b(g, x) = x \\ p : G \times G \times X \rightarrow G \times X, \quad p(g_1, g_2, x) = (g_2, x). \end{aligned}$$

Thus, together with the maps b and μ one has the maps $p, 1_G \times \mu$ and $m \times 1_X$ from $G \times G \times X$ to $G \times X$ and the corresponding functors

$$b^*, \mu^* : \mathfrak{Mod}(\mathcal{O}_X) \rightarrow \mathfrak{Mod}(\mathcal{O}_{G \times X}) \quad (3)$$

and

$$p^*, (1_G \times \mu)^*, (m \times 1_X)^* : \mathfrak{Mod}(\mathcal{O}_{G \times X}) \rightarrow \mathfrak{Mod}(\mathcal{O}_{G \times G \times X}) \quad (4)$$

where an asterisk as superscript is used for denoting the inverse image functor between the corresponding categories, and $\mathfrak{Mod}(\mathcal{O}_Z)$ stands for the category of \mathcal{O}_Z -modules. The equalities

$$b \circ (m \times 1_X) = b \circ p, \quad b \circ (1_G \times \mu) = \mu \circ p, \quad \mu \circ (1_G \times \mu) = \mu \circ (m \times 1_X)$$

give rise to equalities between the respective compositions of the functors of (3) with the functors of (4).

Let \mathcal{H} be an \mathcal{O} -module and λ an isomorphism $\lambda : b^*\mathcal{H} \rightarrow \mu^*\mathcal{H}$, which satisfies the cocycle condition

$$(m \times 1_X)^*(\lambda) = (1_G \times \mu)^*(\lambda) \circ p^*(\lambda). \quad (5)$$

We say that the pair (\mathcal{H}, λ) is a G -equivariant sheaf of \mathcal{O} -modules, or simply a G -equivariant \mathcal{O} -module.

One defines the category \mathfrak{Mod}^G , whose objects are the G -equivariant \mathcal{O} -modules. If (\mathcal{H}', λ') and (\mathcal{H}, λ) are objects in this category, a morphism in \mathfrak{Mod}^G from (\mathcal{H}', λ') to (\mathcal{H}, λ) is a morphism of \mathcal{O} -modules $f : \mathcal{H}' \rightarrow \mathcal{H}$, such that $\lambda b^*(f) = \mu^*(f)\lambda'$.

Given $g \in G$, we put L_g for the map defined by $x \in X \mapsto \mu(g, x) = gx \in X$. Since the G -action on X is analytic, the composition with μ defines a morphism of sheaves of rings $\mathcal{O}_X \rightarrow \mathcal{O}_{G \times X}$. In particular, given an open subset $U \subset X$ and $g \in G$, the map

$$h \in \mathcal{O}(U) \mapsto h \circ L_{g^{-1}} \in \mathcal{O}(gU)$$

determines a ring isomorphism $\mathcal{O}(U) \rightarrow \mathcal{O}(gU)$.

In other terms, we have an isomorphism of sheaves of rings

$$\mathcal{O} \rightarrow (L_{g^{-1}})_*\mathcal{O}. \quad (6)$$

We have the following proposition.

Proposition 1. *Let (\mathcal{H}, λ) be a G -equivariant \mathcal{O} -module and g an element of G , then λ determines an isomorphism of \mathcal{O} -modules*

$$\lambda_g : \mathcal{H} \rightarrow (L_{g^{-1}})_*\mathcal{H}$$

where the \mathcal{O} -structure of $(L_{g^{-1}})_*\mathcal{H}$ is defined through the isomorphism (6).

The image of $\sigma_U \in \mathcal{H}(U)$ by the isomorphism $\mathcal{H}(U) \xrightarrow{\sim} \mathcal{H}(gU)$ will be denoted $g \cdot \sigma_U$.

A consequence of the cocycle condition is the following proposition.

Proposition 2. *For each $g \in G$*

1. $\lambda_h \circ \lambda_g = \lambda_{hg}$.
2. The map $g \mapsto \hat{\lambda}_g := \lambda_g(X)$ is a group homomorphism from G to the group of automorphisms of the complex vector space $\mathcal{H}(X)$.

Although the results stated in Proposition 1 and Proposition 2 are easy to understand, we give detailed proofs of these propositions in Appendix.

In summary, the cocycle condition for (\mathcal{H}, λ) gives rise to a representation of G in the space $H^0(X, \mathcal{H})$.

The following step will be to define a representation in the cohomologies $H^i(X, \mathcal{G})$, when \mathcal{G} is an object of $D(\mathfrak{Coh}^G)$.

Proposition 3. *If \mathcal{G} is a brane of the category $D(\mathfrak{Coh}^G)$, then for each i the cohomology group $H^i(X, \mathcal{G})$ supports a representation of G induced by the G -structure of \mathcal{G} . When \mathcal{G} is an object of \mathfrak{Coh}^G the representation on $H^0(X, \mathcal{G})$ is the one described in the second item of the Proposition 2.*

Proof: As the category \mathfrak{Mod}^G has enough injectives [16], following the well-known procedure, it is possible to construct an equivariant Cartan-Eilenberg resolution $\mathcal{J}^{\bullet\bullet}$ of \mathcal{G}^\bullet in \mathfrak{Mod}^G [25, Thm. 10.45]. Then \mathcal{G}^\bullet is quasi-isomorphic to the total complex $\mathcal{I}^\bullet = (\text{Tot}(\mathcal{J}^\bullet), \partial^\bullet)$, a complex of in \mathfrak{Mod}^G consisting of injective objects.

By the second item in Proposition 2, the space the $\mathcal{I}^i(X)$ carries the representation ρ^i of G . Since the diagrams

$$\begin{array}{ccc} \mathcal{I}^i(X) & \xrightarrow{\partial^i(X)} & \mathcal{I}^{i+1}(X) \\ \downarrow \rho_g^i & & \downarrow \rho_g^{i+1} \\ \mathcal{I}^i(X) & \xrightarrow{\partial^i(X)} & \mathcal{I}^{i+1}(X) \end{array}$$

are commutative, one has a representation of G on each cohomology group of the complex $\mathcal{I}^\bullet(X)$. That is, a representation on the cohomology $H^i(X, \mathcal{G})$. Thus, we have the proposition. \blacksquare

If in the statement of Proposition 3 \mathcal{G} is a locally free \mathcal{O} -module, the representation on $H^i(X, \mathcal{G})$ can be constructed by means of the Dolbeault resolution. Let $\mathcal{G} = \mathcal{O}(V)$ be the sheaf of germs of sections of the holomorphic vector bundle V . We put $\mathcal{A}^{0,q}$ for the sheaf of germs of holomorphic differential forms on X of type $(0, q)$. Since $\mathcal{O}(V)$ is a flat \mathcal{O} -module, the tensor product of this module by the Dolbeault resolution of \mathcal{O} gives rise to the following fine resolution of $\mathcal{O}(V)$

$$0 \rightarrow \mathcal{O}(V) \rightarrow \mathcal{A}^{0,0}(V) \xrightarrow{1 \otimes \bar{\partial}} \mathcal{A}^{0,1}(V) \xrightarrow{1 \otimes \bar{\partial}} \mathcal{A}^{0,2}(V) \rightarrow \dots \quad (7)$$

where $\mathcal{A}^{0,q}(V) := \mathcal{O}(V) \otimes_{\mathcal{O}} \mathcal{A}^{0,q}$. Thus

$$H^q(X, \mathcal{O}(V)) = h^q(\mathcal{A}^{0,\bullet}(V)) \quad (8)$$

where $A^{0,q} = \Gamma(X, \mathcal{A}^{0,q}(V))$.

Now, we use the fact that V is a G -equivariant vector bundle. If σ a holomorphic section of V and ω a $(0, q)$ -form on X , we put

$$g \cdot (\sigma \otimes \omega) := (g \cdot \sigma) \otimes L_{g^{-1}}^* \omega \quad (9)$$

$L_{g^{-1}}^* \omega$ being the pullback of ω by the diffeomorphism $L_{g^{-1}}$ and $g \cdot \sigma$ the section defined as just after Proposition 1.

If f is a holomorphic function on X , then

$$g \cdot (f\sigma \otimes \omega) = g \cdot (\sigma \otimes f\omega).$$

On the other hand, since the G acts analytically on X , $1 \otimes \bar{\partial}$ commutes with the G -action (9). So, we have a representation of G on $H^q(X, \mathcal{O}(V))$, which is equivalent to the one of Proposition 3.

2.2. Vertex Operators

In this subsection, we consider the spaces of vertex operators for strings stretching between equivariant branes.

We put $\mathcal{H}om(\cdot, \cdot)$ for the sheaf functor $\mathcal{H}om$ of the category \mathfrak{Mod} (see [21, page 87])

$$\mathcal{H}om(\cdot, \cdot) : \mathfrak{Mod}^{\text{op}} \times \mathfrak{Mod} \rightarrow \mathfrak{Sh}$$

where \mathfrak{Sh} is the category of sheaves of abelian groups over X .

Let $(\mathcal{F}, \gamma), (\mathcal{G}, \beta)$ be G -equivariant \mathcal{O} -modules. We set $\mathcal{K} := \mathcal{H}om(\mathcal{F}, \mathcal{G})$ for the sheaf of homomorphisms from \mathcal{F} to \mathcal{G} . Given an open subset $U \subset X$, a section $\Phi \in \mathcal{K}(U) = \mathcal{H}om_{\mathcal{O}(U)}(\mathcal{F}(U), \mathcal{G}(U))$ and $g \in G$, we define $g \cdot \Phi \in \mathcal{K}(gU)$ as follows

Let $\tau \in \mathcal{F}(gU)$ be a section of \mathcal{F} on gU , then $g^{-1} \cdot \tau \in \mathcal{F}(U)$, with the notation introduced below Proposition 1. Thus, $\Phi(g^{-1} \cdot \tau) \in \mathcal{G}(U)$ and $g \cdot \Phi(g^{-1} \cdot \tau) \in \mathcal{G}(gU)$. We put

$$(g \cdot \Phi)(\tau) := g \cdot (\Phi(g^{-1} \cdot \tau)). \quad (10)$$

So, we have constructed an isomorphism

$$\eta_g(U) : \mathcal{K}(U) \longrightarrow \mathcal{K}(gU), \quad \Phi \mapsto g \cdot \Phi. \quad (11)$$

Moreover

$$\eta_h(gU) \circ \eta_g(U) = \eta_{hg}(U). \quad (12)$$

Hence, the isomorphisms $\{\eta_g(X)\}_g$ define a representation of G on the space $\mathcal{K}(X)$. Thus, one has the following proposition.

Proposition 4. *Let (\mathcal{F}, γ) , (\mathcal{G}, β) be G -equivariant \mathcal{O} -modules, then*

1. *The isomorphisms (11) convert the \mathcal{O} -module*

$$\mathcal{K} = \mathcal{H}om(\mathcal{F}, \mathcal{G})$$

in a G -equivariant \mathcal{O} -module.

2. *For $g \in G$ and $\Psi \in \mathcal{K}(X) = \mathcal{H}om_{\mathfrak{M}od}(\mathcal{F}, \mathcal{G})$, we put $g \cdot \Psi$ for the element of $\mathcal{K}(X)$ defined by*

$$(g \cdot \Psi)_U(\sigma) := g \cdot (\Psi_{g^{-1}U}(g^{-1} \cdot \sigma))$$

where U is an open subset of X and $\sigma \in \mathcal{F}(U)$. The correspondence $\Psi \rightarrow g \cdot \Psi$ defines a representation of G in vector space $\mathcal{K}(X)$.

Given \mathcal{G}^\bullet a complex in the category $\mathfrak{C}oh^G$, we consider \mathcal{I}^\bullet the complex consisting of injective objects in $\mathfrak{M}od^G$ described in the proof of Proposition 3. If \mathcal{F}^\bullet is another complex in $\mathfrak{M}od^G$, we put

$$\mathcal{C}^n := \prod_a \mathcal{H}om(\mathcal{F}^a, \mathcal{I}^{a+n}).$$

By Proposition 4, each $\mathcal{H}om(\mathcal{F}^a, \mathcal{I}^{a+n})$ is an object of $\mathfrak{M}od^G$.

On the other hand, the coboundary operator $\delta^n : \mathcal{C}^n \rightarrow \mathcal{C}^{n+1}$ is defined by ([20, page 17])

$$\delta^n(\Psi_a) = (\partial^{a+n} \circ \Psi_a + (-1)^{n+1} \Psi_{n+1} \circ \partial^a)$$

where

$$\Psi_a \in \mathcal{H}om(\mathcal{F}^a, \mathcal{I}^{a+n}).$$

Since the ∂ 's are G -operators, so is δ^n . Thus, $(\mathcal{C}^\bullet, \delta^\bullet)$ is a complex in the category $\mathfrak{M}od^G$. The complex \mathcal{C}^\bullet is usually denoted by $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$. By definition, $\mathcal{E}xt^p(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is the cohomology object $h^p(\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet))$.

As $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is a complex in the abelian category $\mathfrak{M}od^G$, its cohomologies are also in $\mathfrak{M}od^G$, i.e., they are G -equivariant sheaves. Hence, from Proposition 3, it follows the following proposition.

Proposition 5. *Let \mathcal{F} and \mathcal{G} be branes in the derived category $D(\mathfrak{C}oh^G)$. Then the G -structures of \mathcal{F} and \mathcal{G} induce on the space of vertex operators*

$$H^q(X, \mathcal{E}xt^p(\mathcal{F}, \mathcal{G}))$$

a representation of G .

The spectral sequence of the double complex

$$E^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))$$

converges to the space of strings $\mathcal{E}xt^{p+q}(\mathcal{F}, \mathcal{G})$. Then from Proposition 5, we conclude

Proposition 6. *If \mathcal{F} and \mathcal{G} are branes of the derived category $D(\mathfrak{Coh}^G)$, then the space $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G})$ of strings between \mathcal{F} and \mathcal{G} with ghost number k supports a representation of G induced by the G -structures of \mathcal{F} and \mathcal{G} .*

When \mathcal{F} and \mathcal{G} are locally free \mathcal{O} -modules the representation stated in the previous proposition can be formulated in terms of differential forms.

First of all, one has the following resolution for \mathcal{F} consisting of locally free \mathcal{O} -modules

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathcal{F} \xrightarrow{1} \mathcal{F} \rightarrow 0.$$

By [18, Proposition 6.5, page 234], $\mathcal{E}xt^k(\mathcal{F}, \mathcal{G}) = 0$, for $k \neq 0$. Hence, when \mathcal{F} is a locally free module we will consider only the spaces of vertex operators $H^q(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$. Furthermore, in this case

$$\mathcal{E}xt^k(\mathcal{F}, \mathcal{G}) = H^k(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})). \quad (13)$$

Let V_1 and V_2 be two holomorphic G -equivariant vector bundles on X . We denote by V the holomorphic vector bundle $\text{Hom}(V_1, V_2)$. Given ψ is an element of $\Gamma(X, \mathcal{O}(V))$, putting

$$(g \cdot \psi)(-) = g \cdot (\psi(g^{-1} \cdot (-))) \quad (14)$$

we define a representation of G on $\Gamma(X, \mathcal{O}(V))$; it is the one stated in item 2 of Proposition 4. More explicit, given $x \in X$, the homomorphism $(g \cdot \psi)_x : V_{1x} \rightarrow V_{2x}$ is the composition

$$\lambda_{g, g^{-1}x}^2 \circ \psi_{g^{-1}x} \circ \lambda_{g^{-1}, x}^1 \quad (15)$$

where the λ 's are the isomorphisms (2); that is, $\lambda_{g,y}^i : (V_i)_y \rightarrow (V_i)_{gy}$.

As $\mathcal{H}om(\mathcal{O}(V_1), \mathcal{O}(V_2)) = \mathcal{O}(\text{Hom}(V_1, V_2))$, then the equality (8) applied to $V = \text{Hom}(V_1, V_2)$ gives

$$H^q(X, \mathcal{H}om(\mathcal{O}(V_1), \mathcal{O}(V_2))) = h^q(A^{0,\bullet}(\text{Hom}(V_1, V_2))). \quad (16)$$

As in (9), from the G -action (14), we can construct the corresponding representation on the space (16). Thus, given

$$\psi \in \Gamma(X, \mathcal{H}om(\mathcal{O}(V_1), \mathcal{O}(V_2)))$$

and the differential form ω of type $(0, q)$, we have

$$g \cdot (\psi \otimes \omega) = g \cdot \psi \otimes L_{g^{-1}}^* \omega. \quad (17)$$

Thus, Proposition 5 adopts the following form when the branes are locally free sheaves.

Proposition 7. *Let V_1, V_2 be G -equivariant holomorphic vector bundles on X . Then the G -action (17) induces a representation of G on the space of vertex operators*

$$H^q(X, \mathcal{H}om(\mathcal{O}(V_1), \mathcal{O}(V_2))).$$

Vertex operators on flag varieties. When X is a flag manifold, the dimension of the spaces of vertex operators mentioned in Proposition 7 can be determined, in some particular cases, by means of the Borel-Weil-Bott theorem and the Weyl's dimension formula.

Let $G_{\mathbb{C}}$ be a connected complex semi-simple Lie group and $Q \subset G_{\mathbb{C}}$ a parabolic subgroup of $G_{\mathbb{C}}$. Then the flag manifold $X = G_{\mathbb{C}}/Q$ is a compact homogeneous simply connected Kähler algebraic variety (see for example [5, 30]). Moreover, every compact homogeneous simply connected Kähler manifold is isomorphic to such a quotient. In [12], Grantcharov showed several examples of flag manifolds which are also Calabi-Yau.

The holomorphic equivariant vector bundles over X are related with the holomorphic representations of $G_{\mathbb{C}}$. Let $V \rightarrow X$ be a holomorphic vector bundle endowed with a holomorphic $G_{\mathbb{C}}$ -action by bundle maps, that lies over the action on X . Denoting with $x_0 \in X$ the class eQ , then stabilizer of x_0 acts on the fibre V_{x_0} . Hence, there is a holomorphic representation ξ of Q on V , the standard fibre of V . Moreover, one has an isomorphism of holomorphic bundles $V \simeq G_{\mathbb{C}} \times_{\xi} V$. Conversely, equivariant holomorphic bundles on X can be constructed from representations.

We denote by G a compact real form of $G_{\mathbb{C}}$. As it is known, the finite dimensional representations of G are in bijective correspondence with the holomorphic representations of finite dimension of $G_{\mathbb{C}}$. We put $L := G \cap Q$, then one can identify G/L with X . Moreover, each irreducible representation of L on a complex space induces a unique extension to one holomorphic representation of Q . So, an irreducible representation α of L on the complex vector space V gives rise to one homogeneous holomorphic vector bundle

$$V(\alpha) := G_{\mathbb{C}} \times_Q V$$

on X . It is very easy to check that $V(\alpha \oplus \beta) = V(\alpha) \oplus V(\beta)$, and $V(\alpha^*) = (V(\alpha))^*$.

Given α, β irreducible representations of L the representation tensor product $\alpha^* \otimes \beta$ can be written as direct sum of irreducible representations

$$\alpha \otimes \beta = \bigoplus_{\nu} m^{\nu} \nu \quad (18)$$

where ν is an irreducible representation of L and the $m^{\nu} \in \mathbb{Z}_{\geq 0}$ are the corresponding Littlewood-Richardson coefficients.

Hence, the bundle of homomorphisms (over the identity) from $V(\alpha)$ to $V(\beta)$ can be written as

$$\mathcal{H}om(V(\alpha), V(\beta)) = V(\alpha)^* \otimes V(\beta) = V(\alpha^* \otimes \beta) = \bigoplus_{\nu} m^{\nu} V(\nu). \quad (19)$$

Since $\mathcal{O}(V(\alpha))$ is a locally free \mathcal{O} -module, it follows from (19) together with (13) that the space of string with ghost number k stretching between $\mathcal{O}(V(\alpha))$ and $\mathcal{O}(V(\beta))$

$$\mathcal{E}xt^k(\mathcal{O}(V(\alpha)), \mathcal{O}(V(\beta))) = \bigoplus_{\nu} m^{\nu} H^k(X, \mathcal{O}(V(\nu))).$$

Notation. We put $\mathfrak{g}_{\mathbb{C}} := \text{Lie}(G_{\mathbb{C}})$ and \mathfrak{g} for $\text{Lie}(G)$ for the Lie algebras of the corresponding groups. Let c be the complex conjugation in $\mathfrak{g}_{\mathbb{C}}$ with respect to \mathfrak{g} . We fix a maximal abelian subalgebra \mathfrak{h} of \mathfrak{g} and denote by $\mathfrak{h}_{\mathbb{C}}$ its complexification with respect to c . Let Δ be the set of roots of the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$.

The system of positive roots Δ^+ for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is chosen so that

$$\text{Lie}(Q) = \mathfrak{q} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\gamma \in \Phi} \mathfrak{g}^{-\gamma}$$

where Φ is a subset of Δ containing Δ^+ and $\mathfrak{g}^{-\gamma}$ being the root space associated to γ . We denote by ρ the half sum of positive roots, write Λ for the weight lattice and put Λ^+ for the set of dominant weights.

$\zeta(\alpha)$ denotes the highest weight of the irreducible representation α of L . By the Borel-Weil-Bott theorem [15], $H^k(X, \mathcal{O}(V(\nu))) = 0$, if $\zeta(\nu) + \rho$ is singular. When $\zeta(\nu) + \rho$ is regular, let w_{ν} the unique element of the Weyl group such that $w_{\nu}(\zeta(\nu) + \rho)$ is dominant and define

$$l(\nu) := \#\{\alpha \in \Delta^+; (\zeta(\nu) + \rho, \alpha) < 0\}$$

where the bilinear product (\cdot, \cdot) is defined by the Killing form. The following proposition is consequence of Borel-Weil-Bott theorem.

Proposition 8. *If α and β are irreducible representations of L , then the dimension of the space of strings $\mathcal{E}xt^k(\mathcal{O}(V(\alpha)), \mathcal{O}(V(\beta)))$ is*

$$\sum_{\tau} m^{\tau} \delta_{k, l(\tau)} \dim \xi_{\tau}$$

where τ runs over the irreducible representations of L , such that $\zeta(\tau) + \rho$ is a regular weight, ξ_{τ} being the irreducible representation of L with highest weight $w_{\tau}(\zeta(\tau) + \rho) - \rho$ and the m^{τ} 's the Littlewood-Richardson coefficients defined in (18).

When the representations α and β are unidimensional the statement of Proposition 8 admits a simpler formulation.

Given $v \in \Lambda$, it determines a holomorphic character $e^v : Q \rightarrow \mathbb{C}^{\times}$. So, one can construct the fiber product $L_v = G_{\mathbb{C}} \times_Q \mathbb{C}$, which is a holomorphic $G_{\mathbb{C}}$ -equivariant line bundle over X . If $\lambda, \mu \in \Lambda$, then

$$L_{-\mu+\lambda} = L_{\mu}^* \otimes L_{\lambda} = \mathcal{H}om(L_{\mu}, L_{\lambda}).$$

Hence, we have isomorphisms among the following sheaves of holomorphic sections

$$\mathcal{H}om(\mathcal{O}(L_{\mu}), \mathcal{O}(L_{\lambda})) = \mathcal{O}(\mathcal{H}om(L_{\mu}, L_{\lambda})) = \mathcal{O}(L_{-\mu+\lambda}).$$

We put $\xi := -\mu + \lambda$ for the weight that determines the line bundle $L_{\mu}^* \otimes L_{\lambda}$, and

$$p(\xi) := \#\{\alpha \in \Delta^+; (\xi + \rho, \alpha) < 0\}.$$

A direct application of the Borel-Weil-Bott theorem to the weight ξ proves the following proposition, which characterizes some spaces of vertex operators relative to the flag manifold $X = G_{\mathbb{C}}/Q$.

Proposition 9. *If $\xi + \rho$ is either singular or $p \neq p(\xi)$, then the space of vertex operators $H^p(X, \mathcal{H}om(\mathcal{O}(L_{\mu}), \mathcal{O}(L_{\lambda})))$ is zero. Otherwise, the representation of G in that space is the irreducible representation with highest weight is $w(\xi + \rho) - \rho$, where w is the element of the Weyl group such that $w(\xi + \rho)$ is dominant.*

As $\mathcal{O}(L_{\mu})$ is a locally free \mathcal{O} -module, the space of strings from the brane $\mathcal{O}(L_{\mu})$ to $\mathcal{O}(L_{\lambda})$ with ghost number k is

$$\mathcal{E}xt^k(\mathcal{O}(L_{\mu}), \mathcal{O}(L_{\lambda})) = H^k(X, \mathcal{H}om(\mathcal{O}(L_{\mu}), \mathcal{O}(L_{\lambda}))).$$

Thus, taking into account the proposition, the Weyl dimension formula permits us to determine the dimension of these spaces.

2.3. Correlation Functions

Let us assume that X is a projective Calabi-Yau variety of dimension n . For $j = 0, \dots, k$, let \mathcal{F}_j be a general brane. Given $a_j \in H^{q_j}(X, \mathcal{E}xt^{p_j}(\mathcal{F}_{j-1}, \mathcal{F}_j))$, when $\mathcal{F}_k = \mathcal{F}_0$ and $\sum(q_j + p_j) = n$, one can define the correlation function $\langle a_1 \dots a_k \rangle$ of the vertex operators a_j 's (see [29]).

If the \mathcal{F}_j are locally free sheaves, i.e. $\mathcal{F}_j = \mathcal{O}(V_j)$, with V_j holomorphic vector bundle, the correlation function can be calculated as follows. Given

$$a_j \in H^{q_j}(X, \mathcal{H}om(\mathcal{O}(V_{j-1}), \mathcal{O}(V_j))) = H_{\bar{\partial}}^{0, q_j}(X, \mathcal{H}om(V_{j-1}, V_j))$$

a_j will be the class of a form $\psi^j \otimes \omega^j$, with ψ^j a holomorphic section of the space $\mathcal{H}om(V_{j-1}, V_j)$ and ω^j a differential form on X of type $(0, q_j)$. Then [1, 29]

$$\langle a_1 \dots a_k \rangle = \int_X \text{tr}(\psi^k \circ \dots \circ \psi^1) \omega^k \wedge \dots \wedge \omega^1 \wedge \Omega \quad (20)$$

where Ω is an fixed n -holomorphic form on X which vanishes nowhere.

Next, let us assume that the \mathcal{O} -modules \mathcal{F}_j are G -equivariant, and we denote by λ^j the isomorphism that defines the G -structure on \mathcal{F}_j . Given $g \in G$, by (17)

$$\langle g \cdot a_1 \dots g \cdot a_k \rangle = \int_X \text{tr}(g \cdot \psi^k \circ \dots \circ g \cdot \psi^1) L_{g^{-1}}^* \omega^k \wedge \dots \wedge L_{g^{-1}}^* \omega^1 \wedge \Omega. \quad (21)$$

From (15) together with the cocycle condition, it follows

$$(g \cdot \psi^j)_x \circ (g \cdot \psi^{j-1})_x = \lambda_{g, g^{-1}x}^j \circ \psi_{g^{-1}x}^j \circ \psi_{g^{-1}x}^{j-1} \circ \lambda_{g^{-1}x}^{j-2}.$$

As $\lambda^0 = \lambda^k$

$$\begin{aligned} \text{tr}(g \cdot \psi^k \circ \dots \circ g \cdot \psi^1)(x) &= \text{tr}(\lambda_{g, g^{-1}x}^0 \circ \psi_{g^{-1}x}^k \circ \dots \circ \psi_{g^{-1}x}^1 \circ \lambda_{g^{-1}x}^0) \\ &= \text{tr}(\psi^k \circ \dots \circ \psi^1)(g^{-1} \cdot x). \end{aligned}$$

Since the G action on X is analytic, $L_{g^{-1}}^* \Omega = \Omega$; so, the integrals (20) and (21) are equal, that is

$$\langle g \cdot a_1 \dots g \cdot a_k \rangle = \langle a_1 \dots a_k \rangle. \quad (22)$$

Thus, we have the following proposition.

Proposition 10. *Let \mathcal{F} and \mathcal{G} be G -equivariant locally free \mathcal{O} -modules, then the correlation functions associated to vertex operators for strings between \mathcal{F} to \mathcal{G} are G -invariant.*

3. Cohomology of Equivariant Branes

Because of the equivariance, the branes in $D(\mathcal{Coh}^G)$ admit a lift to branes on the homotopy quotient of X , and so one can define the corresponding equivariant cohomology for these objects. In this section, we will consider that equivariant cohomology.

Let G be a compact Lie group. By the Peter-Weyl theorem G is a closed subgroup of $GL(N, \mathbb{R})$ for some N . The limit as $n \rightarrow \infty$ of Stiefel manifold of N -frames in \mathbb{R}^{N+n} is a smooth model for the universal G -bundle. We will adopt this model and denote by $EG \xrightarrow{q} BG := EG/G$ the corresponding G -fibration.

We put $\bar{X} := (EG \times X)/G$ the homotopy quotient of X by G and set τ and π for the projections $X \xleftarrow{\tau} EG \times X \xrightarrow{\pi} \bar{X}$. Let $X \rightarrow \bar{X} = EG \times_G X \xrightarrow{\nu} BG$ be the fibration constructed from the action of G on X , and let p denote the composition $EG \times X \xrightarrow{proj} EG \xrightarrow{q} BG$. One has the following commutative diagram

$$\begin{array}{ccccc} \bar{X} & \xleftarrow{\pi} & EG \times X & \xrightarrow{\tau} & X \\ & & \nu \searrow & & \downarrow p \\ & & & & BG \end{array}$$

Let \mathcal{H} be a G -equivariant coherent sheaf on X , then the inverse image $\tilde{\mathcal{H}} := \tau^*\mathcal{H}$ is G -equivariant coherent sheaf on $EG \times X$. Denoting by \tilde{L}_g the obvious multiplication by $g \in G$ in $EG \times X$, we put $\tilde{\lambda}_g$ by the isomorphism $\tilde{\lambda}_g : \tilde{\mathcal{H}} \rightarrow (\tilde{L}_{g^{-1}})_*\tilde{\mathcal{H}}$ determined by the equivariance of $\tilde{\mathcal{H}}$ (see Proposition 1).

On the other hand, if V is an open set of \bar{X} , $\pi_*\tilde{\mathcal{H}}(V) = \tilde{\mathcal{H}}(W)$, where W is the G -invariant subset $\pi^{-1}(V)$. Thus, there is a G -action on the sheaf $\pi_*\tilde{\mathcal{H}}$ defined on $\pi_*\tilde{\mathcal{H}}(V)$ by the isomorphism $\tilde{\lambda}_{g,W}$. We put $\bar{\mathcal{H}}$ for denoting the subsheaf of $\pi_*\tilde{\mathcal{H}}$ defined by G -invariant sections of $\pi_*\tilde{\mathcal{H}}$. Then the sheaves $\tau^*\mathcal{H}$ and $\pi^*\bar{\mathcal{H}}$ are isomorphic (see [4, page 3]) and denote by h the isomorphism $h : \tau^*\mathcal{H} \rightarrow \pi^*\bar{\mathcal{H}}$.

For any open not empty subset $U \subset BG$, one has $\tau p^{-1}(U) = X$ and $\nu^{-1}(U) = \pi p^{-1}(U)$, and therefore

$$\begin{aligned} \Gamma(\nu^{-1}(U), \tilde{\mathcal{H}}) &= \bar{\mathcal{H}}(\pi p^{-1}(U)) = \pi^*\bar{\mathcal{H}}(p^{-1}(U)) \\ &\simeq \tau^*\mathcal{H}(p^{-1}(U)) = \Gamma(X, \mathcal{H}) \end{aligned} \tag{23}$$

where the isomorphism is determined by h . We have proved the following proposition.

Proposition 11. *Given \mathcal{H} a G -equivariant coherent sheaf on X , it determines a sheaf $\bar{\mathcal{H}}$ on \bar{X} and an isomorphism $\tau^*\mathcal{H} \simeq \pi^*\bar{\mathcal{H}}$. Furthermore, $\nu_*\bar{\mathcal{H}}$ is the constant sheaf $\Gamma(X, \mathcal{H})$.*

Given \mathcal{F} an object of $D(\mathcal{Coh}^G)$, i.e., a G -equivariant brane, according to the preceding remarks, it defines a object $\bar{\mathcal{F}}$ of the derived category $D(\mathcal{Sh}(\bar{X}))$, of abelian sheaves on \bar{X} , and an isomorphism f of the category of $D(\mathcal{Sh}(EG \times X))$ from $\tau^*(\mathcal{F})$ to $\pi^*(\bar{\mathcal{F}})$. Then the triple $\mathbf{F} = (\mathcal{F}, \bar{\mathcal{F}}, f)$ is an object of the equivariant derived category $D_G(X)$, defined in [4].

If \mathbf{F} is an object as above, let $\bar{\mathcal{I}}^\bullet$ and \mathcal{I}^\bullet be complexes of injectives on \bar{X} and on X that represent $\bar{\mathcal{F}}$ and \mathcal{F} , respectively. By Proposition 11

$$R^q\nu_*\bar{\mathcal{F}} = H^q(\nu_*(\bar{\mathcal{I}}^\bullet)) = H^q(\Gamma(X, \mathcal{I}^\bullet)) = H^q(X, \mathcal{F}).$$

Hence, $R^q\nu_*\bar{\mathcal{F}}$ is the constant sheaf on BG defined by $H^q(X, \mathcal{F})$.

The equivariant cohomology $H_G(X, \mathbf{F})$ of $\mathbf{F} \in D_G(X)$, is by definition the cohomology $H(\bar{X}, \bar{\mathcal{F}})$; that is, the cohomology $H(BG, R\nu_*\bar{\mathcal{F}})$.

Proposition 12. *The spectral sequence $E_2^{pq} = H^p(BG) \otimes H^q(X, \mathcal{F})$ abuts to $H_G^{p+q}(X, \mathbf{F})$.*

Proof: Since $R^q\nu_*\bar{\mathcal{F}}$ is the constant sheaf $H^q(X, \mathcal{F})$, the E_2 term of the Leray-Serre spectral sequence associated to the fibration $X \rightarrow \bar{X} \rightarrow BG$ is

$$E_2^{pq} = H^p(BG, R^q\nu_*(\bar{\mathcal{F}})) = H^p(BG) \otimes H^q(X, \mathcal{F})$$

and the sequence abuts to $H^{p+q}(\bar{X}, \bar{\mathcal{F}}) = H_G^{p+q}(X, \mathbf{F})$. ■

Corollary 13. *If $H^q(X, \mathcal{F}) = 0$ for q odd, then*

$$H_G^{p+q}(X, \mathbf{F}) \simeq H^p(BG) \otimes H^q(X, \mathcal{F}).$$

Proof: As the cohomology $H^p(BG)$ vanishes when p is odd, then the differential operators of spectral sequence $E_2^{pq} = H^p(BG) \otimes H^q(X, \mathcal{F})$ vanish and spectral sequence collapses. ■

We assume that G is the *torus* $T = (U(1))^k$. The T -equivariant cohomology with complex coefficients of a point $H_T(\text{pt}; \mathbb{C}) = H(BT; \mathbb{C})$ can be identified to the algebra $\mathbb{C}[t_{\mathbb{C}}^*]$, of polynomials on the complexification $t_{\mathbb{C}}$ of the Lie algebra of T .

We denote by Ξ the multiplicative subset of $\mathbb{C}[t_{\mathbb{C}}^*]$ consisting of the non-zero polynomials, and let S denote the fixed point set of the T -action. From the localization

theorem [13, Section 6] one deduces that the restriction map defines an isomorphism

$$H_T(X; \mathbf{F})_{\Xi} \rightarrow H_T(S; \mathbf{F})_{\Xi}$$

between the corresponding localization modules.

As the action of T on S is trivial, $ET \times_T S = BT \times S$. Thus, for the particular case of complex coefficients

$$H_T(S; \mathbf{F}) \simeq H(BT) \otimes_{\mathbb{C}} H(S; \mathcal{F}). \quad (24)$$

Hence

$$H_T(X; \mathbf{F})_{\Xi} \simeq \mathbb{C}(t_{\mathbb{C}}^*) \otimes_{\mathbb{C}} H(S; \mathcal{F}) \quad (25)$$

in which $\mathbb{C}(t_{\mathbb{C}}^*)$ is the field of rational functions on $t_{\mathbb{C}}$. In other words, $H_T(X; \mathbf{F})_{\Xi}$ is the result of the extension of scalars in $H(S; \mathcal{F})$ from \mathbb{C} to $\mathbb{C}(t_{\mathbb{C}}^*)$.

Proposition 14. *Given \mathcal{F} a brane which belongs to \mathfrak{Coh}^T , if the fixed point set for the T action is $\{x_1, \dots, x_r\}$, then*

$$H_T(X, \mathbf{F}) \simeq \bigoplus_{i=1}^r (\mathbb{C}(t_{\mathbb{C}}^*) \otimes \mathcal{F}_{x_i}).$$

Corollary 15. *Under the hypotheses of Proposition 14, if \mathcal{G} is other object of \mathfrak{Coh}^T such that $\bigoplus_{i=1}^r (\mathbb{C}(t_{\mathbb{C}}^*) \otimes \mathcal{F}_{x_i})$ and $\bigoplus_{i=1}^r (\mathbb{C}(t_{\mathbb{C}}^*) \otimes \mathcal{G}_{x_i})$ are not isomorphic as $\mathbb{C}(t_{\mathbb{C}}^*)$ -vector spaces, then \mathcal{F} and \mathcal{G} are inequivalent T -equivariant branes.*

3.1. Equivariant Charges

The charge of a brane \mathcal{F} which is a locally free \mathcal{O} -module is an element of the cohomology of X defined from certain characteristic classes of X and \mathcal{F} [1, 19, 23]. When \mathcal{F} is a coherent sheaf, to define the charge it is necessary to pass to a locally free resolution of \mathcal{F} . The existence of such resolutions is a well-known fact if X is a smooth variety [10]. This property permits to extent the definition to objects of the Grothendieck group of X , when X is a variety. In this section, we study the equivariant versions of that process.

The triangulated category $D(\mathfrak{Coh}^G)$, of G -equivariant branes on X , has associated the corresponding Grothendieck group $K(D(\mathfrak{Coh}^G))$. By $K^G(X)$ we denote the Grothendieck group of the abelian category \mathfrak{Coh}^G . The map

$$[\mathcal{F}] \in K(D(\mathfrak{Coh}^G)) \mapsto \sum_i (-1)^i [H^i(\mathcal{F})] \in K^G(X) \quad (26)$$

where $[Z]$ the equivalence class of the object Z , is an isomorphism of abelian groups.

Let $K'^G(X)$ be the Grothendieck group of G -equivariant locally free \mathcal{O} -modules on X [26, 27]. In particular, $K'^G(\text{pt})$ is the ring $R(G)$ of virtual representations of G . The tensor product defines a ring structure on $K'^G(X)$, and the tensor product of locally free sheaves by coherent sheaves gives to $K^G(X)$ the structure of module on the ring $K'^G(X)$.

Given a G -equivariant brane, it seems natural to assign it a G -equivariant charge. When the brane is a locally free sheaf, that assignation can be carried out by means of the respective G -equivariant characteristic classes. The resulting charge will be an element of the equivariant cohomology $H_G(X)$.

As in the non equivariant setting, an appropriate choice of the characteristic classes will permit to extent the definition to the objects of $K'^G(X)$. To define an equivariant charge for branes in \mathfrak{Coh}^G , it is necessary to consider the cases for which the Grothendieck groups $K'^G(X)$ and $K^G(X)$ are isomorphic. In this situation, the equivariant charges are defined for arbitrary G -equivariant branes through the isomorphism (26).

One says that X has the G -equivariant resolution property if any G -equivariant coherent \mathcal{O} -module is the quotient of a locally free G -module. In this case, the natural homomorphism $K'^G(X) \rightarrow K^G(X)$ between the Grothendieck groups is an isomorphism.

Thomason [28] proved the G -equivariant resolution property for actions of linear algebraic groups which act on smooth varieties. According with this result, from now on in this subsection, we assume that:

- The Kähler manifold X is an algebraic manifold, that is, X admits a complex analytic embedding as a closed submanifold of $\mathbb{C}\mathbb{P}^N$, for some N .
- G is a linear algebraic group.
- The action of G on X is algebraic.

Under these assumptions the smooth algebraic variety X has the G -resolution property and, hence, the homomorphism $K'^G(X) \rightarrow K^G(X)$ between the algebraic Grothendieck groups is an isomorphism.

By the GAGA principle, given a G -equivariant coherent \mathcal{O} -module \mathcal{F} , it is algebraic and coherent with respect the algebraic structure. So, \mathcal{F} is the quotient of an algebraic locally free G -equivariant sheaf \mathcal{E}_0 on X . Consequently, it is possible to construct a resolution

$$0 \rightarrow \mathcal{E}_m \rightarrow \mathcal{E}_{m-1} \rightarrow \cdots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad (27)$$

consisting of G -equivariant locally free sheaves and $m \leq \dim X$ [28].

Given an algebraic G -equivariant vector bundle V on X of rank r , by the equivariant splitting principle [11, page 315], V has r Chern roots: $x_1 \dots, x_r$, such that the equivariant Chern class $c_j^G(V) = \sigma_j(x_1, \dots, x_r)$, with σ_j the j th elementary symmetric function.

One defines the equivariant Chern character of V by the following element of the equivariant cohomology of X

$$\mathrm{ch}^G(V) = \sum_1^r e^{x_i} \in H_G(X, \mathbb{Q}).$$

The equivariant Chern character is “additive” with respect the exact sequences of equivariant vector bundles, that is, if $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ is an exact sequence, then $\mathrm{ch}^G(V) = \mathrm{ch}^G(V') + \mathrm{ch}^G(V'')$. Hence, it admits an extension to the algebraic Grothendieck group $K^G(X) \simeq K^G(X)$ of G equivariant coherent sheaves.

In particular, given a G -equivariant brane $\mathcal{F} \in \mathcal{Coh}^G$, the complex \mathcal{F} is quasi-isomorphic to a complex \mathcal{E}_\bullet , where \mathcal{E}_i is the sheaf of sections of a G -equivariant holomorphic vector bundle V_i . Then we define

$$\mathrm{ch}^G(\mathcal{F}) := \sum_i (-1)^i \mathrm{ch}^G(V_i).$$

The constant map $p : X \rightarrow \mathrm{pt}$ gives rise to a pushforward homomorphism $p_* : K(D(\mathcal{Coh}^G)) \rightarrow R(G)$ which maps the class $[\mathcal{F}]$ to the virtual representation

$$\sum_i (-1)^i R^i p_* \mathcal{F} = \sum_i (-1)^i H^i(X, \mathcal{F})$$

where the representations on the cohomologies are the ones of Proposition 3.

The Chern character does not commutes with the pushforward by proper maps [10, page 280], [6] and the equivariant one neither [8]. If $V \rightarrow X$ is an equivariant vector bundle on X , then the non commutativity of the Chern character with the pushforward p_* is expressed by the statement of the equivariant Hirzebruch-Riemann-Roch theorem

$$\mathrm{ch}(p_*(V)) = p_*(\mathrm{ch}^G(V) \mathrm{td}^G(X))$$

where $\mathrm{td}^G(X)$ is the G -equivariant Todd of the bundle TX , that can be defined by means of the equivariant splitting principle [11, page 317]. The equivariant Hirzebruch-Riemann-Roch allows us to express the pushforward $\mathrm{ch}(p_*(V))$ in terms of equivariant characteristic classes.

Let \mathcal{F} be a general brane of $D(\mathcal{Coh}^G)$. The previous remarks lead us to associate with \mathcal{F} the following equivariant cohomology class

$$Q^G(\mathcal{F}) := \text{ch}^G(\mathcal{F}) \text{td}^G(X) \quad (28)$$

that can be considered as an *equivariant charge* of the brane \mathcal{F} .

Different equivariant charges can be defined by means of other G -equivariant forms on X , for example $\sqrt{\text{td}^G(X)}$, etc.

Brane on an equivariant subvariety. We will consider the natural brane defined by a subvariety of the algebraic variety X . Let \mathcal{I} be an ideal sheaf of \mathcal{O} . We put

$$Z = \text{supp}(\mathcal{O}/\mathcal{I}).$$

Here Z is an analytic subvariety of X and $\mathcal{F} := \mathcal{O}/\mathcal{I}$ is a coherent sheaf on X , that can be considered as the structure sheaf of Z .

Locally, on an open U , the sheaf \mathcal{I} will be generated by the holomorphic functions f_1, \dots, f_r . We will assume that each function is G -invariant. Hence, Z is a G -invariant subvariety of X and \mathcal{O}/\mathcal{I} is an equivariant coherent sheaf.

Let us assume that f_1, \dots, f_r is a regular sequence of functions and let e_1, \dots, e_r be the canonical basis of \mathbb{C}^r . We put

$$\mathcal{E}_k := \mathcal{O} \otimes_{\mathbb{C}} (\wedge^k \mathbb{C}^r).$$

Then $\mathcal{E}_k \simeq \mathcal{O}^{\oplus \tilde{k}}$, where $\tilde{k} := \binom{r}{k}$. The known Koszul complex [14, page 687] is the following G -equivariant locally free resolution of \mathcal{O}/\mathcal{I}

$$0 \rightarrow \mathcal{E}_r \xrightarrow{\partial} \mathcal{E}_{r-1} \rightarrow \dots \rightarrow \mathcal{E}_1 \xrightarrow{\partial} \mathcal{E}_0 = \mathcal{O} \xrightarrow{\text{proj}} \mathcal{F} \rightarrow 0 \quad (29)$$

where

$$\partial(h \otimes e_{j_1} \wedge \dots \wedge e_{j_k}) = \sum_i (-1)^i h f_i e_{j_1} \wedge \dots \wedge \hat{e}_{j_i} \wedge \dots \wedge e_{j_k}.$$

Since the f_i are G -invariant, the operator ∂ is equivariant.

We can use the equivariant locally free resolution (29) to define $\text{ch}^G(\mathcal{O}/\mathcal{I})$

$$\text{ch}^G(\mathcal{O}/\mathcal{I}) = \sum_{k=0}^r (-1)^k \text{ch}^G(\mathcal{E}_k) = \left(\sum_{k=0}^r (-1)^k \tilde{k} \right) \text{ch}^G(\mathcal{O}).$$

Since $\sum_{k=0}^r (-1)^k \tilde{k} = (-1 + 1)^r = 0$, we have the following proposition

Proposition 16. *The G -equivariant charge of the brane \mathcal{O}/\mathcal{I} is zero.*

Index of the Dirac operator. In some particular cases, the charge has a natural interpretation in terms of the index of an elliptic operator. The exterior bundle Λ^*T^*X of X with the connection induced by the Levi-Civita connection and the standard Clifford multiplication is a Dirac bundle (see [22, page 114]). This bundle has associated the corresponding Dirac operator D . If G acts as a group of isometries of X , then D is a G -operator [22, page 211], i.e., D is G -equivariant.

Let us assume that G is compact and that V is a G -brane consisting of a locally free sheaf. The compactness of G allows us to average over the group for obtaining G -invariant metrics on X and G -invariant connections on V . On the other hand, the tensor product of $(\Lambda^*T^*X) \otimes V$ is a Dirac bundle (see [22, page 122]) and the corresponding Dirac operator D is also G -equivariant, by the G -invariance of the metric and the connection. As D is elliptic, $\ker D$ and $\operatorname{coker} D$ are representations of G of finite dimension, since X is compact. For $g \in G$ the virtual character $\chi(D)(g)$ of D at g is defined by

$$\chi(D)(g) = \operatorname{trace}(g|_{\ker D}) - \operatorname{trace}(g|_{\operatorname{coker} D}).$$

By the equivariant index theorem [3, Chapter 8], we have the proposition.

Proposition 17. *In a neighborhood of $0 \in \mathfrak{g} := \operatorname{Lie}(G)$*

$$\chi(D) \circ \exp = \int_X Q^G(V). \quad (30)$$

The value of $\chi(D)(\exp(\xi))$, for $\xi \in \mathfrak{g}$, can be calculated by the localization formula in equivariant cohomology. The result is the Atiyah-Segal-Singer fixed point formula [3, 22]. Next, we will evaluate the integral (30) when X is a toric variety.

Toric varieties. Let Σ be a fan in $N = \mathbb{Z}^r$, such that each cone in Σ is generated by a subset of a basis of \mathbb{Z}^r . We will denote by X the smooth toric variety defined by Σ [7, 9, 24]. We put $M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and T for the torus

$$T = N \otimes \mathbb{C}^\times = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times).$$

We denote by X^T fixed point set for the T -action. For $x \in X^T$, let $\nu_{i,x} \in 2\pi(\mathbb{Z})^n$, $i = 1, \dots, n$, be the weights of the isotropy representation of T on the tangent space $T_x X$. The fixed points of the T -action are in bijective correspondence with the n -dimensional cones in Σ [7, §3.2]. If the point x is associated with the cone σ , then the

$$\omega_{i,x} = \frac{\nu_{i,x}}{2\pi} \quad (31)$$

are the generators of $\sigma^\vee \cap M$, where σ^\vee is the dual cone of σ .

Let V be a T -equivariant holomorphic vector bundle on X of range m . The equivariant splitting principle together with the fact that T acts trivially on X^T , permit us to express the restriction of V to X^T as a direct sum of T -equivariant line bundles

$$V|_{X^T} = \bigoplus_{j=1}^m L_j$$

where action of T on $V|_{X^T}$ will be defined by m weights φ_j . Thus, the T -equivariant Chern class of L_j is given by (see [11])

$$c_1^T(L_j) = c_1(L_j) + \frac{1}{2\pi}\varphi_j. \quad (32)$$

The T -equivariant Chern character of $V|_{X^T}$ is

$$\text{ch}^T(V|_{X^T}) = \sum_{j=1}^m \exp(c_1^T(L_j)). \quad (33)$$

Proposition 18. *Let X be a toric manifold and V be a holomorphic T -equivariant vector bundle on X . Denoting by $\{\varphi_{j,x}\}_{j=1,\dots,m}$ the weights of the representation of T on the fibre of V at a fixed point x , then (30) is equal to*

$$\sum_{x \in X^T} \left(\sum_{j=1}^m e^{\frac{1}{2\pi}\varphi_{j,x}} \right) \prod_{i=1}^n \left(1 - e^{-\omega_{i,x}} \right)^{-1} \quad (34)$$

where X^T is the set of fixed points of X for the T -action and the $\omega_{i,x}$ are defined in (31).

Proof: The localization theorem in equivariant cohomology [17, §10.9] allows us to calculate the value (30) as a sum of contributions of the connected components of X^T . As X^T is discrete, the localization formula adopts the following form

$$\chi(D) \circ \exp = (2\pi)^n \sum_{x \in X^T} \frac{Q^G(V)(x)}{\prod_i \nu_{i,x}} \quad (35)$$

where the $\nu_{i,x}$ are the weights of the isotropy representation of T at the fixed point x . From (32) and (33), it follows

$$\text{ch}^T(V)|_x = \sum_{j=1}^m e^{\frac{1}{2\pi}\varphi_{j,x}}.$$

Similarly (see [22, page 230])

$$\mathrm{td}^T(TX)|_x = \prod_{i=1}^n \omega_{i,x} \left(1 - e^{-\omega_{i,x}}\right)^{-1}.$$

The proposition follows from (28) together with (31). ■

Note that the contribution of the manifold X to (34) is encoded in the n -dimensional cones of the fan Σ .

4. Appendix

In this section, we will prove Proposition 1 and Proposition 2. Let (\mathcal{H}, λ) be an object of \mathfrak{Mod}^G and $g \in G$. With the notations introduced in Subsection 2.1, $b^*\mathcal{H}$ is the sheaf associated with the presheaf

$$W \mapsto \mathcal{P}(W) := \mathcal{O}_{G \times X}(W) \otimes_{b^{-1}\mathcal{O}(W)} b^{-1}\mathcal{H}(W).$$

With W being an open set of $G \times X$. Given an open subset $U \subset X$ and $g \in G$, we put $U_g := \{g\} \times U \subset G \times X$. Identifying $\mathcal{O}_{G \times X}(U_g)$ with $\mathcal{O}(U)$, one has the following isomorphism of $\mathcal{O}(U)$ -modules

$$\mathcal{P}(U_g) = \mathcal{O}_{G \times X}(U_g) \otimes_{\mathcal{O}(U)} \mathcal{H}(U) \rightarrow \mathcal{H}(U), \quad \tilde{f} \otimes \tau \mapsto f\tau \quad (36)$$

with $f(x) = \tilde{f}(g, x)$.

Similarly, the sheaf $\mu^*\mathcal{H}$ is associated to the presheaf \mathcal{M} , with

$$W \mapsto \mathcal{M}(W) := \mathcal{O}_{G \times X}(W) \otimes_{\mu^{-1}\mathcal{O}(W)} b\mu^{-1}\mathcal{H}(W).$$

One has the isomorphism of $\mathcal{O}(gU)$ -modules

$$\mathcal{M}(U_g) = \mathcal{O}_{G \times X}(U_g) \otimes_{\mathcal{O}(gU)} \mathcal{H}(gU) \rightarrow \mathcal{H}(gU), \quad \hat{h} \otimes \tau \mapsto h\tau \quad (37)$$

with $h(gx) = \hat{h}(g, x)$.

The restriction to U_g of the morphism of sheaves λ is denoted $\lambda|_{U_g}$

$$\lambda|_{U_g} : b^*\mathcal{H}(U_g) \longrightarrow \mu^*\mathcal{H}(U_g). \quad (38)$$

By the above identifications (36) and (37), $\lambda|_{U_g}$ determines an isomorphism of $\mathcal{O}(U)$ -modules

$$\lambda|_{U_g} : \mathcal{H}(U) \xrightarrow{\sim} \mathcal{H}(gU) \quad (39)$$

where the $\mathcal{O}(U)$ -module structure of $\mathcal{H}(gU)$ is defined through the isomorphism (6). Thus, we have proved Proposition 1.

As a consequence of Proposition 1, there is an isomorphism of \mathcal{O}_x -modules

$$\lambda_{g,x} : \mathcal{H}_x \rightarrow \mathcal{H}_{gx} \quad (40)$$

where the \mathcal{O}_x -structure of \mathcal{H}_{gx} is induced by the isomorphism (6). In this way, the isomorphisms (2) are recovered.

Until now, in this appendix, we have only used the existence of the isomorphism λ . Next, we will exploit the cocycle condition. Both sides of equation (5) are isomorphisms between two objects on the category of $\mathcal{O}_{G \times G \times X}$ -modules. The cocycle condition means the commutativity of the following square

$$\begin{array}{ccc} \mathcal{Z}_1 & \xrightarrow{(m \times 1_X)^*(\lambda)} & \mathcal{Z}_3 \\ p^*(\lambda) \downarrow & & \downarrow 1 \\ \mathcal{Z}_2 & \xrightarrow{(1_G \times \mu)^*(\lambda)} & \mathcal{Z}_3 \end{array} \quad (41)$$

where

$$\begin{aligned} \mathcal{Z}_1 &:= p^*b^*(\mathcal{H}) = (m \times 1_X)^*b^*(\mathcal{H}), & \mathcal{Z}_2 &:= p^*\mu^*(\mathcal{H}) = (1_G \times \mu)^*b^*(\mathcal{H}) \\ \mathcal{Z}_3 &:= (m \times 1_X)^*\mu^*(\mathcal{H}) = (1_G \times \mu)^*\mu^*(\mathcal{H}). \end{aligned}$$

As a consequence of the cocycle condition, we have the following lemma.

Lemma 19. *Let g, h be elements of G and U an open set of X , then*

$$\lambda|_{(gU)_h} \circ \lambda|_{U_g} = \lambda|_{U_{hg}}.$$

In particular, $\lambda|_{X_h} \circ \lambda|_{X_g} = \lambda|_{X_{hg}}$.

Proof: We consider the commutative square (41) and restrict this diagram to $\{h\} \times U_g \subset G \times G \times X$. The restriction of $(m \times 1_X)^*(\lambda)$ is the morphism

$$\mathcal{Z}_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow \mathcal{Z}_3(\{h\} \times U_g) = \mathcal{H}((hg)U)$$

induced by λ . Thus, by (39), the mentioned restriction is $\lambda|_{U_{hg}}$.

The restriction of $p^*(\lambda)$ to $\{h\} \times U_g$

$$\mathcal{Z}_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow \mathcal{Z}_2(\{h\} \times U_g) = \mathcal{H}(gU)$$

is (39).

Finally, we consider the restriction of $(1_G \times \mu)^*(\lambda)$. It is the morphism

$$\mathcal{Z}_2(\{h\} \times U_g) = \mathcal{H}(gU) \longrightarrow \mathcal{Z}_3(\{h\} \times U_g) = \mathcal{H}(h(gU))$$

induced by λ , and according to (39) it is $\lambda|_{(gU)_h}$. Then the lemma follows from the commutativity of (41). \blacksquare

Proposition 2 is a direct consequence the Lemma 19.

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