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INVERSION OF DOUBLE-COVERING MAP $\mathrm{SPIN}(N) \to \mathrm{SO}(N, \mathbb{R})$ FOR $N \leq 6$

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Abstract. This work provides an algorithmic procedure for finding the pair of elements in the spin group which map to a given matrix in the special orthogonal group of order five or six. This is achieved by first solving the problem when the special orthogonal matrix is a Givens rotation, and then exploiting the fact that the covering maps are group homomorphisms and that any special orthogonal matrix can be explicitly decomposed into a product of Givens rotations. For this purpose systems of quadratic equations in several variables have to be solved symbolically. The resulting solution display a transparent dependency on the entries of the Givens matrices.

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1. Introduction

The double covering maps from the spin groups to the orthogonal groups are very useful in practice, [9–11]. For instance, the covering map from the unit quaternions to $\mathrm{SO}(3,\mathbb{R})$ plays a vital role in robotics, gaming and animation etc., [12, 16]. The unit quaternion representing a matrix in $\mathrm{SO}(3,\mathbb{R})$ is not only a reduction in storage cost, but also provides immediate information such as the axis of rotation. Similarly, the covering by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ of $\mathrm{SO}(4,\mathbb{R})$ is known to be useful in fields such as nanophotonics and switched electrical networks, [1, 4, 14, 17].

For instance, in [4], the analysis of photonic circuits enabled by four directional analogues of lossless mirrors (called four-port couplers) was facilitated by the covering $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4,\mathbb{R})$. Each four-port coupler splits an input signal into a reflected, transmitted, right and left components. Such a coupler can be represented by a matrix in $\mathrm{SO}(4,\mathbb{R})$, [4]. The entries of this matrix play a decisive role in the input-output behavior of the photonic circuit. Thus, it is desirable that instead of using the sixteen entries of each coupler matrix, one uses a pair of unit quaternions which amount to eight real parameters satisfying two constraints simpler than the constraints satisfied by the sixteen original parameters. In a similar fashion, representing a matrix in $\mathrm{SO}(5,\mathbb{R})$ by a matrix in $\mathrm{Sp}(4)$ amounts to a reduction from 25 real parameters satisfying 15 quadratic constraints to 16 real parameters satisfying 6 quadratic constraints.

The control of switched lossless electrical networks lead to control problems on the orthogonal groups which can be “lifted" to analogous control problems on the spin group, [8]. One reason to study systems evolving on orthogonal groups via systems on the corresponding spin group is simply because the latter have been studied more intensively. Thus $\mathrm{SU}(4)$, the spin group of $\mathrm{SO}(6,\mathbb{R})$ has been analyzed in great detail due to its relevance in quantum computing, [2].

Thus, in all these applications inverting the double covering map, in closed form when possible is important because it provides a more economical description of the orthogonal matrix being analyzed. For $n = 3$, this inversion seems to be folklore and involves the solution of simple quadratic equations albeit in several variables. For dimension four, a trick reduces the question to the $n = 3$ case, [4]. This inversion was used in the photonic circuitry problem alluded to above.

The nanophotonics application also motivates this inversion when $n = 6$ since it seems experimentally feasible that the four-port coupler can be enhanced into a lossless “mirror" which splits an incoming signal into six independent directions.

The purpose of this work is, therefore, to explicitly invert the covering map for $n = 5, 6$. Unlike the $n = 3$ case it is forbiddingly complicated to use directly the polynomial equations provided by the covering map for a generic special orthogonal matrix X, for this purpose. In this work, this obstacle is circumvented by solving the inversion problem when X is a Givens rotation and then using the fact that an element of $\mathrm{SO}(n, \mathbb{R})$ can constructively be decomposed into a product of Givens rotations. The polynomial systems corresponding to the inversion of covering map when the target matrix X is a Givens rotations are still quite intricate. Moreover, the system of equations changes with the location of the sole nontrivial 2×2 principal submatrix in a Givens rotation. Nevertheless, in this work we demonstrate that these equations can be solved in closed form. We emphasize that the solution of the inversion in closed form is a desideratum in the applications alluded to above. Furthermore, as shall be presently seen, the dependence of the elements in the spin group mapping to a Givens rotation X, on the entries in X, is very transparent than would be the case for an arbitrary special orthogonal X. Indeed, the double angle relation between the spin group and the orthogonal group in dimension three is equally visible for $n = 5, 6$ when the target is a Givens rotation matrix.

The balance of this paper is organized as follows. In Section 2 we invert the $\mathrm{Sp}(4) \to \mathrm{SO}(5, \mathbb{R})$ map. In section 3, the $\mathrm{SU}(4) \to \mathrm{SO}(6, \mathbb{R})$ map is inverted. Both sections first detail the polynomial systems that describe the forward map. Then the solution when the target is a Givens rotation, is explicitly described. In each section we describe the solution to only one Givens factor. In Section 4 illustrative examples of these inversion procedures are provided. The analysis of the remaining factors (nine for $\mathrm{SO}(5, \mathbb{R})$ and fourteen for $\mathrm{SO}(6, \mathbb{R})$) is deferred to two Appendices.

This paper uses Clifford algebras and a modicum of the technique of Gröbner bases. For the former all details may be found in [13, 15]. However, all that is needed to understand the results here are that the covering maps are determined by sending an element G of the spin group to $\Phi_n(G)$, where $\Phi_n(G)$ is the matrix of the linear map

$$X \to GXG^*$$

with respect to a concrete basis of matrices (called one-vectors) $\{X_1, \ldots, X_n\}$, $n = 5$ or $n = 6$. Here G^* stands for the Hermitian conjugate of G. These bases will be spelled out later in this work. Details of how precisely these bases are produced may be found in [6]. For Gröbner bases the reader is referred to [3], for instance.

2. Inversion of the Double Covering Map from $\mathrm{SO}(5,\mathbb{R})$ to $\mathrm{Sp}(4)$

Let us commence with a set of one-vectors for $\mathrm{Cl}(0,5)$:

$$F_1 = -\mathrm{Id}_2 \otimes \mathrm{i}\sigma_z = \begin{pmatrix} -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{pmatrix}, \quad F_2 = -\sigma_x \otimes (\mathrm{i}\sigma_y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$F_3 = \mathrm{Id}_2 \otimes (\mathrm{i}\sigma_y) = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad F_4 = \mathrm{i}\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}$$

$$F_5 = \sigma_z \otimes (-\mathrm{i}\sigma_y) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Here $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices and $\otimes$ stands for the Kronecker product.

Remark 1. *This basis of one-vectors is conjugate to a set of one-vectors used in our earlier work,* [6] *via the matrix $M_{1\otimes q}$, where $q = \frac{1}{\sqrt{2}}(1-k)$ and $M_{1\otimes q}$, for a quaternion q, is the real 4×4 matrix of the linear map which sends a quaternion x to $x\bar{q}$. The basis used in* [6] *is what naturally results when using usual iterative constructions in the theory of Clifford algebras,* [13, 15]. *However, this necessitated working with a nonstandard version of the group $\mathrm{Sp}(4)$. The basis described above results in the standard version of $\mathrm{Sp}(4)$. The main advantage of the basis being used here is the greater familiarity of the standard representation of $\mathrm{Sp}(4)$. With respect to this set, Clifford conjugation is given by $X^{cc} = X^*$ and the grade involution is given by $X^{gr} = G^T\bar{X}G$, where $G = J_4$, where $J_4 = \begin{bmatrix} 0_2 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0_2 \end{bmatrix}$. This follows from the results in* [6] *and also can be checked by direct calculation. The spin group then contains all X such that $X^{cc}X = X^*X = \mathrm{Id}_4$, and $X^{gr} = J_4^T\bar{X}J_4 = X$. Together these conditions are equivalent to the condition that $X \in \mathrm{Sp}(4)$ Thus the advantage of using this set of* one-vectors *is that the spin group is now the standard unitary symplectic group,* $\mathrm{Sp}(4)$.

Let $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, for some $A, B, C, D \in M(2, \mathbb{C})$, satisfy $J_4^T \bar{X} J_4 = X$. In matrix form, this is

$$\begin{bmatrix} 0_2 & -\mathrm{Id}_2 \\ \mathrm{Id}_2 & 0_2 \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} 0_2 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \text{so} \qquad X = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}.$$

Then the condition $X^* X = \mathrm{Id}_4$ is equivalent to

$$\begin{bmatrix} A^* & -B^T \\ B^* & A^T \end{bmatrix} \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix} = \mathrm{Id}_4, \qquad \text{so} \qquad \begin{bmatrix} A^*A + B^T\bar{B} & A^*B - B^T\bar{A} \\ B^*A - A^T\bar{B} & B^*B + A^T\bar{A} \end{bmatrix} = \mathrm{Id}_4.$$

This in turn is equivalent to two distinct equations

$$A^*A + B^T\bar{B} = \mathrm{Id}_2, \qquad A^*B - B^T\bar{A} = 0_2.$$

If we set $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ for some $a_j = x_j + \mathrm{i}y_j$, $b_j = z_j + \mathrm{i}w_j$, $b_j, a_j \in \mathbb{C}$, $j = 1, 2, 3, 4$, then a calculation shows that these last two conditions are equivalent to the following polynomial system in the x_k, y_k, z_k, w_k

$$\begin{aligned} x_1^2 + y_1^2 + x_3^2 + y_3^2 + z_1^2 + w_1^2 + z_3^2 + w_3^2 &= 1 \\ x_2^2 + y_2^2 + x_4^2 + y_4^2 + z_2^2 + w_2^2 + z_4^2 + w_4^2 &= 1 \\ x_1x_2 + y_1y_2 + x_3x_4 + y_3y_4 + z_1z_2 + w_1w_2 + z_3z_4 + w_3w_4 &= 0 \\ x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 - z_1w_2 + z_2w_1 - z_3w_4 + z_4w_3 &= 0 \\ x_1z_2 + y_1w_2 + x_3z_4 + y_3w_4 - x_2z_1 - y_2w_1 - x_4z_3 - y_4w_3 &= 0 \\ x_1w_2 - y_1z_2 + x_3w_4 - y_3z_4 - x_2w_1 + y_2z_1 - x_4w_3 + y_4z_3 &= 0. \end{aligned} \tag{1}$$

Thus any element $G \in \mathrm{Spin}(0, 5)$ is determined by 16 real numbers satisfying the system of equations (1).

Now we proceed to find the matrix of the linear map $F \to GFG^*$ acting on $F \in V$ by running through $F = F_j$ for $j = 1, 2, 3, 4, 5$ and writing the result as a linear combination of the F_j. In other words, we have a description of the double covering map $\Phi_5 : \mathrm{Sp}(4) \to \mathrm{SO}(5, \mathbb{R})$.

Theorem 2. *Let $G \in \mathrm{Spin}(0, 5) = \mathrm{Sp}(4)$ have the form $G = \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix}$ where $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ for some $a_j = x_j + \mathrm{i}y_j, b_j = z_j + \mathrm{i}w_j \in \mathbb{C}$, $j = 1, 2, 3, 4$. The double covering map $\Phi_5 : \mathrm{Sp}(4) \to \mathrm{SO}(5, \mathbb{R})$ is given by*

$\Phi(G) = (a_{i,j})$, *where*

$$\begin{aligned}
a_{1,1} &= x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 + w_1^2 - w_2^2 \\
a_{2,1} &= x_1w_3 - x_2w_4 - x_3w_1 + x_4w_2 - y_3z_1 + y_4z_2 + y_1z_3 - y_2z_4 \\
a_{3,1} &= -x_1x_3 + x_2x_4 - y_1y_3 + y_2y_4 - z_1z_3 + z_2z_4 - w_1w_3 + w_2w_4 \\
a_{4,1} &= -x_1z_3 + x_2z_4 + x_3z_1 - x_4z_2 + y_1w_3 - y_2w_4 - y_3w_1 + y_4w_2 \\
a_{5,1} &= x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2 + z_1w_3 - z_2w_4 - z_3w_1 + z_4w_2 \\
a_{1,2} &= 2(-x_1w_2 + x_2w_1 + y_1z_2 - y_2z_1) \\
a_{2,2} &= x_1x_4 - x_2x_3 - y_1y_4 + y_2y_3 - z_1z_4 + z_2z_3 + w_1w_4 - w_2w_3 \\
a_{3,2} &= x_1w_4 - x_2w_3 + x_3w_2 - x_4w_1 - y_1z_4 + y_2z_3 - y_3z_2 + y_4z_1 \\
a_{4,2} &= x_1y_4 - x_2y_3 - x_3y_2 + x_4y_1 - z_1w_4 + z_2w_3 + z_3w_2 - z_4w_1 \\
a_{5,2} &= x_1z_4 - x_2z_3 - x_3z_2 + x_4z_1 + y_1w_4 - y_2w_3 - y_3w_2 + y_4w_1 \\
a_{1,3} &= 2(x_3x_4 + y_3y_4 + z_3z_4 + w_3w_4) \\
a_{2,3} &= -x_1w_4 - x_2w_3 + x_3w_2 + x_4w_1 - y_1z_4 - y_2z_3 + y_3z_2 + y_4z_1 \\
a_{3,3} &= x_1x_4 + x_2x_3 + y_1y_4 + y_2y_3 + z_1z_4 + z_2z_3 + w_1w_4 + w_2w_3 \\
a_{4,3} &= x_1z_4 + x_2z_3 - x_3z_2 - x_4z_1 - y_1w_4 - y_2w_3 + y_3w_2 + y_4w_1 \\
a_{5,3} &= -x_1y_4 - x_2y_3 + x_3y_2 + x_4y_1 - z_1w_4 - z_2w_3 + z_3w_2 + z_4w_1 \\
a_{1,4} &= 2(-x_3z_4 + x_4z_3 - y_3w_4 + y_4w_3) \\
a_{2,4} &= -x_1y_4 + x_2y_3 + x_3y_2 - x_4y_1 - z_1w_4 + z_2w_3 + z_3w_2 - z_4w_1 \\
a_{3,4} &= -x_1z_4 + x_2z_3 - x_3z_2 + x_4z_1 - y_1w_4 + y_2w_3 - y_3w_2 + y_4w_1 \\
a_{4,4} &= x_1x_4 - x_2x_3 - y_1y_4 + y_2y_3 + z_1z_4 - z_2z_3 - w_1w_4 + w_2w_3 \\
a_{5,4} &= x_1w_4 - x_2w_3 - x_3w_2 + x_4w_1 - y_1z_4 + y_2z_3 + y_3z_2 - y_4z_1 \\
a_{1,5} &= 2(x_3y_4 - x_4y_3 - z_3w_4 + z_4w_3) \\
a_{2,5} &= -x_1z_4 + x_2z_3 + x_3z_2 - x_4z_1 + y_1w_4 - y_2w_3 - y_3w_2 + y_4w_1 \\
a_{3,5} &= x_1y_4 - x_2y_3 + x_3y_2 - x_4y_1 - z_1w_4 + z_2w_3 - z_3w_2 + z_4w_1 \\
a_{4,5} &= -x_1w_4 + x_2w_3 + x_3w_2 - x_4w_1 - y_1z_4 + y_2z_3 + y_3z_2 - y_4z_1 \\
a_{5,5} &= x_1x_4 - x_2x_3 + y_1y_4 - y_2y_3 - z_1z_4 + z_2z_3 - w_1w_4 + w_2w_3.
\end{aligned}$$

Proof: A typical element of V is

$$\sum_{k=1}^{5} a_k F_k = \begin{bmatrix} -a_1\mathrm{i} & a_3\mathrm{i} - a_5 & 0 & -a_2 - a_4\mathrm{i} \\ a_5 + a_3\mathrm{i} & a_1\mathrm{i} & a_2 + a_4\mathrm{i} & 0 \\ 0 & a_4\mathrm{i} - a_2 & -a_1\mathrm{i} & a_5 + a_3\mathrm{i} \\ a_2 - a_4\mathrm{i} & 0 & a_3\mathrm{i} - a_5 & a_1\mathrm{i} \end{bmatrix}.$$

We need to compute GF_jG^* for each $j = 1, 2, 3, 4, 5$ and represent the result as an element of V. For $j = 1$, we have

$$\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix} F_1 \begin{bmatrix} A^* & -B^T \\ B^* & A^T \end{bmatrix} = \begin{bmatrix} -a_{1,1}\mathrm{i} & a_{3,1}\mathrm{i} - a_{5,1} & 0 & -a_{2,1} - a_{4,1}\mathrm{i} \\ a_{5,1} + a_{3,1}\mathrm{i} & a_{1,1}\mathrm{i} & a_{2,1} + a_{4,1}\mathrm{i} & 0 \\ 0 & a_{4,1}\mathrm{i} - a_{2,1} & -a_{1,1}\mathrm{i} & a_{5,1} + a_{3,1}\mathrm{i} \\ a_{2,1} - a_{4,1}\mathrm{i} & 0 & a_{3,1}\mathrm{i} - a_{5,1} & a_{1,1}\mathrm{i} \end{bmatrix}$$

where

$$\begin{aligned}
a_{1,1} &= x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 + w_1^2 - w_2^2 \\
a_{2,1} &= x_1w_3 - x_2w_4 - x_3w_1 + x_4w_2 - y_3z_1 + y_4z_2 + y_1z_3 - y_2z_4 \\
a_{3,1} &= -x_1x_3 + x_2x_4 - y_1y_3 + y_2y_4 - z_1z_3 + z_2z_4 - w_1w_3 + w_2w_4 \\
a_{4,1} &= -x_1z_3 + x_2z_4 + x_3z_1 - x_4z_2 + y_1w_3 - y_2w_4 - y_3w_1 + y_4w_2 \\
a_{5,1} &= x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2 + z_1w_3 - z_2w_4 - z_3w_1 + z_4w_2.
\end{aligned}$$

Analogously, we can find the remaining $a_{i,j}$, $j = 2, 3, 4, 5$, by direct computation of GF_jG^*. ■

Remark 3. *The result of computing GF_jG^* is, in fact*

$$\begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix} F_j \begin{bmatrix} A^* & -B^T \\ B^* & A^T \end{bmatrix} = \begin{bmatrix} -a_{1,j}\mathrm{i} & a_{3,j}\mathrm{i} - a_{5,j} & 0 & -a_{2,j} - a_{4,j}\mathrm{i} \\ a_{5,j} + a_{3,j}\mathrm{i} & \tilde{a}_{1,j}\mathrm{i} & a_{2,j} + a_{4,j}\mathrm{i} & 0 \\ 0 & a_{4,j}\mathrm{i} - a_{2,j} & -a_{1,j}\mathrm{i} & a_{5,j} + a_{3,j}\mathrm{i} \\ a_{2,j} - a_{4,j}\mathrm{i} & 0 & a_{3,j}\mathrm{i} - a_{5,j} & \tilde{a}_{1,j}\mathrm{i} \end{bmatrix}$$

where

$$\begin{aligned}
a_{1,1} &= x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 + w_1^2 - w_2^2 \\
\tilde{a}_{1,1} &= -x_3^2 + x_4^2 - y_3^2 + y_4^2 - z_3^2 + z_4^2 - w_3^2 + w_4^2 \\
a_{1,2} &= 2(-x_1w_2 + x_2w_1 - y_2z_1 + y_1z_2) \\
\tilde{a}_{1,2} &= 2(x_3w_4 - x_4w_3 + y_4z_3 - y_3z_4) \\
a_{1,3} &= 2(-x_1x_2 - y_1y_2 - z_1z_2 - w_1w_2) \\
\tilde{a}_{1,3} &= 2(x_3x_4 + y_3y_4 + z_3z_4 + w_3w_4) \\
a_{1,4} &= 2(x_1z_2 - x_2z_1 + y_1w_2 - y_2w_1) \\
\tilde{a}_{1,4} &= 2(-x_3z_4 + x_4z_3 - y_3w_4 + y_4w_3) \\
a_{1,5} &= 2(-x_1y_2 + x_2y_1 + z_1w_2 - z_2w_1) \\
\tilde{a}_{1,5} &= 2(x_3y_4 - x_4y_3 - z_3w_4 + z_4w_3).
\end{aligned}$$

The desired condition, viz., $GF_jG^* \in V$, requires $a_{1,j} = \tilde{a}_{1,j}$ for $j = 1,2,3,4,5$. This follows from the assumption that $G \in \mathrm{Sp}(4)$, implying (1) *is satisfied. Consider the first two equations in* (1)

$$x_1^2 + y_1^2 + x_3^2 + y_3^2 + z_1^2 + w_1^2 + z_3^2 + w_3^2 = 1$$
$$x_2^2 + y_2^2 + x_4^2 + y_4^2 + z_2^2 + w_2^2 + z_4^2 + w_4^2 = 1$$

then

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2 + w_1^2 - w_2^2 = -x_3^2 + x_4^2 - y_3^2 + y_4^2 - z_3^2 + z_4^2 - w_3^2 + w_4^2$$

so $a_{1,1} = \tilde{a}_{1,1}$. Similarly, the sixth equation implies $a_{1,2} = \tilde{a}_{1,2}$, the third implies $a_{1,3} = \tilde{a}_{1,3}$, the fifth implies $a_{1,4} = \tilde{a}_{1,4}$, and the fourth implies $a_{1,5} = \tilde{a}_{1,5}$. As a cautionary note, it is worth observing that while $G \in \mathrm{Sp}(4)$ is sufficient to ensure $GF_jG^* \in V$ for all $j = 1,2,3,4,5$, it is not a necessary condition.

Now we would like to compute the preimage of this covering map, which will consist of two matrices in $\mathrm{Sp}(4)$, of any matrix in $\mathrm{SO}(5,\mathbb{R})$. It follows from the theory of spin groups that the two matrices must be negatives of one another. Direct inversion of the covering map seems to be a daunting task. To ameliorate this we exploit the fact that $\mathrm{SO}(5,\mathbb{R})$ is generated by Givens rotations about the 10 coordinate planes in $\mathbb{R}^5$.Thus a Givens rotation R_{ij} is the identity matrix except in the principal 2×2 submatrix located in rows and columns indexed by $\{i,j\}$. This principal submatrix is $\begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix}$. Thus, for instance, $R_{2,5}$ is given by

$$R_{2,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & c\theta & 0 & 0 & s\theta \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -s\theta & 0 & 0 & c\theta \end{pmatrix}.$$

For brevity we label the 10 desired Givens rotations as follows

Definition 4. $R_1 = R_{1,2}$, $R_2 = R_{2,3}$, $R_3 = R_{3,4}$, $R_4 = R_{4,5}$, $R_5 = R_{1,3}$, $R_6 = R_{2,4}$, $R_7 = R_{3,5}$, $R_8 = R_{1,4}$, $R_9 = R_{2,5}$, $R_{10} = R_{1,5}$.

Remark 5. *Given an element $X \in \mathrm{SO}(5,\mathbb{R})$, one can constructively and routinely factorize it into a product of Givens rotations. We omit the details, which may be found in* [5], *for instance. In fact, each $G \in \mathrm{SO}(5,\mathbb{R})$ can be expressed using only a smaller subset of these Givens rotations $\{R_1,\ldots,R_{10}\}$. For instance, it suffices to use R_1, R_2, R_3, R_4 (with a factor of each type used possibly more than once). However, it may be better to use the remaining rotations depending on the structure of the given X.*

Table 1. Matrices G_i, $i = 1, 2, 3, ..., 10$.

$$G_1 = \begin{pmatrix} \widehat{c} & 0 & 0 & -\mathrm{i}\widehat{s} \\ 0 & \widehat{c} & -\mathrm{i}\widehat{s} & 0 \\ 0 & -\mathrm{i}\widehat{s} & \widehat{s} & 0 \\ -\mathrm{i}\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}, \; G_5 = \begin{pmatrix} \widehat{c} & -\widehat{s} & 0 & 0 \\ \widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}, \; G_9 = \begin{pmatrix} \widehat{c} & 0 & -\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\widehat{s} \\ \widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \widehat{s} & 0 & \widehat{c} \end{pmatrix}$$

$$G_2 = \begin{pmatrix} \widehat{c} & 0 & \mathrm{i}\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\mathrm{i}\widehat{s} \\ \mathrm{i}\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & -\mathrm{i}\widehat{s} & 0 & \widehat{c} \end{pmatrix}, \qquad G_6 = \begin{pmatrix} z_2 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_1 \end{pmatrix}, \; G_{10} = \begin{pmatrix} \widehat{c} & -\mathrm{i}\widehat{s} & 0 & 0 \\ -\mathrm{i}\widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & \mathrm{i}\widehat{s} \\ 0 & 0 & \mathrm{i}\widehat{s} & \widehat{c} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} \widehat{c} & 0 & \widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\widehat{s} \\ -\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \widehat{s} & 0 & \widehat{c} \end{pmatrix}, \qquad G_7 = \begin{pmatrix} z_2 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix},$$

$$G_4 = \begin{pmatrix} \widehat{c} & 0 & -\mathrm{i}\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\mathrm{i}\widehat{s} \\ -\mathrm{i}\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & -\mathrm{i}\widehat{s} & 0 & \widehat{c} \end{pmatrix}, \; G_8 = \begin{pmatrix} \widehat{c} & 0 & 0 & \widehat{s} \\ 0 & \widehat{c} & \widehat{s} & 0 \\ 0 & -\widehat{s} & \widehat{c} & 0 \\ -\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix},$$

$$\widehat{c} = c\frac{\theta}{2}, \qquad \widehat{s} = s\frac{\theta}{2}, \qquad z_1 = \widehat{c} + \widehat{s}\mathrm{i}, \qquad z_2 = \widehat{c} - \widehat{s}\mathrm{i}.$$

Suppose an arbitrary rotation $R \in \mathrm{SO}(5, \mathbb{R})$ is generated by $\prod_{k=1}^{L} R^k$, with each R^k one of $R_i = R_i(\theta_i), i \in \{1, \ldots, 10\}$ for some angles $\theta_i \in [0, 2\pi)$, $i = 1, 2, ..., 10$. Let G be such that $\Phi_5(\pm G) = R$ and G_i be such that $\Phi(\pm G_i) = R_i$ for each $i = 1, 2, ..., 10$. Let $G^k \in \mathrm{Sp}(4)$ be such that $\Phi_5(G^k) = R^k$. Since each R^k is some $\pm R_i, i = 1, \ldots, 10$, it follows that each G^k is some $\pm G_i, i = 1, \ldots, 10$. Then, using the fact that Φ_5 is a group homomorphism, it is easily seen that $\pm G$ is a product of some $\pm G_i$, and if G corresponds to one choice of signs amongst the G_i, then $-G$ corresponds to the other choice of signs amongst the G_i .

This permits us to characterize the preimage of any $R \in \mathrm{SO}(5, \mathbb{R})$ as a product of the preimages of the 10 generators given above. This, as we shall see presently, is a far more amenable task.

Theorem 6. *Let $R \in \mathrm{SO}(5, \mathbb{R})$. The preimage of $R = \prod_{k=1}^{L} R^k$ under Φ_5 is $\{\pm G\}$, where $G = \prod_{k=1}^{L} G^k$, with $G^k \in \{G_1, \ldots, G_{10}\}$ for G_i given in* Table 1

Proof: $G_1 \in \mathrm{Sp}(4)$ must satisfy $\Phi(G_1) = R_1$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} \end{pmatrix} = \begin{pmatrix} c\theta & s\theta & 0 & 0 & 0 \\ -s\theta & c\theta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $c = c\theta$ and $s = s\theta$. Then we can construct a system of equations by setting the expressions for $a_{i,j}$ given in Theorem 2 equal to the corresponding element of R_1, along with the six equations in (1) governing elements of $\mathrm{Sp}(4)$ and the new equation $s^2 + c^2 = 1$. In total, this gives 32 quadratic equations in 18 real variables. Order the variables first by taking the real and imaginary components of each element of A, moving left to right and top to bottom ($x_1 > y_1 > x_2 > y_2 > x_3 > y_3 > x_4 > y_4$), followed by the real and imaginary components of the elements of B in the same order, followed finally by s and c. Then usage of the degree reverse lexicographical order produces a Gröbner basis and the attendant system of equations becomes: $x_2 = x_3 = y_1 = y_2 = y_3 = y_4 = z_1 = z_2 = z_3 = z_4 = w_1 = w_4 = 0$, and

$$\begin{array}{llll} (a) & c + 2w_3^2 - 1 = 0, & (b) \quad s + 2w_3x_4 = 0, & (c) \; w_3 + cw_3 + sx_4 = 0, \\ (d) & c^2 + s^2 - 1 = 0, & (e) \; sw_3 + x_4 - cx_4 = 0, & (f) \quad c - 2x_4^2 + 1 = 0. \\ & x_1 - x_4 = 0, & \quad w_2 - w_3 = 0. & \end{array}$$

From equation (f), $x_4 = \pm\left|c\frac{\theta}{2}\right|$. Similarly, equation (a) implies $w_3 = \pm\left|s\frac{\theta}{2}\right|$. However, equation (b) indicates that the signs chosen for x_4 and w_3 are not independent of each other. Let

$$x_4 = c\frac{\theta}{2} = \begin{cases} +\left|c\frac{\theta}{2}\right| & \theta \in [0,\pi) \\ -\left|c\frac{\theta}{2}\right| & \theta \in [\pi, 2\pi). \end{cases}$$

If $\theta \in [0,\pi)$, then $s\frac{\theta}{2} \geq 0$, $c\frac{\theta}{2} \geq 0$ and $s\theta \geq 0$. Equation (b) thus becomes $s\theta = -2w_3 \cos\frac{\theta}{2}\cdot$ Hence, $w_3 = -s\frac{\theta}{2}\cdot$ If $\theta \in [\pi, 2\pi)$, then $s\theta \leq 0$ and $c\frac{\theta}{2} \leq 0$, but still $s\frac{\theta}{2} \geq 0$. Then $s\theta = -2w_3c\frac{\theta}{2}$, and therefore $w_3 = -s\frac{\theta}{2}\cdot$ So, for any $\theta \in [0, 2\pi)$, if $x_4 = c$ then $w_3 = s\frac{\theta}{2}\cdot$ If $x_4 = -c\frac{\theta}{2}$, then for $\theta \in [0,\pi)$, we have $s\theta = -2w_3(-c\frac{\theta}{2})$, hence $w_3 = s\frac{\theta}{2}$. While if $\theta \in [\pi, 2\pi)$, from equation $s\theta = -2w_3(-c\frac{\theta}{2})$, we have $w_3 = s\frac{\theta}{2}\cdot$ So, if $x_4 = -c\frac{\theta}{2}$, then $w_3 = s\frac{\theta}{2}\cdot$ In future computations, we will omit these steps and simply write $x_4 = \pm c\frac{\theta}{2}$ and $w_3 = \mp s\frac{\theta}{2}\cdot$ Since $x_1 = x_4$ and $w_2 = w_3$, we have the two explicit solutions: $x_1 = x_4 = c\frac{\theta}{2}$ and $w_2 = w_3 = -s\frac{\theta}{2}$; or $x_1 = x_4 = -c\frac{\theta}{2}$ and $w_2 = w_3 = s\frac{\theta}{2}\cdot$

Equations (c) and (e) are consistent with these solutions: $\mp s\frac{\theta}{2} \mp c\theta s\frac{\theta}{2} \pm s\theta c\frac{\theta}{2} = 0$ is equivalent to $s\frac{\theta}{2} = -c\theta s\frac{\theta}{2} + s\theta c\frac{\theta}{2}$, and $\pm s\theta s\frac{\theta}{2} \mp c\frac{\theta}{2} \pm c\theta c\frac{\theta}{2} = 0$ is equivalent to $c\frac{\theta}{2} = c\theta c\frac{\theta}{2} + s\theta s\frac{\theta}{2}$, which are true for any $\theta \in [0, 2\pi)$. Of course, equation (d) is true by our definition of c and s . Finally we substitute our solutions into

$$G_1 = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

to arrive at

$$\Phi^{-1}(R_1) = \left\{ \pm \begin{pmatrix} c\frac{\theta}{2} & 0 & 0 & -\mathrm{i}s\frac{\theta}{2} \\ 0 & c\frac{\theta}{2} & -\mathrm{i}s\frac{\theta}{2} & 0 \\ 0 & -\mathrm{i}s\frac{\theta}{2} & c\frac{\theta}{2} & 0 \\ -\mathrm{i}s\frac{\theta}{2} & 0 & 0 & c\frac{\theta}{2} \end{pmatrix} \right\}.$$

We may choose either matrix in this set to act as G_1 . For $i = 2, 3, ..., 10$, the computations are analogous. We defer these to Appendix 1. ■

3. Inversion of the Double Covering Map from $\mathrm{SO}(6, \mathbb{R})$ to $\mathrm{SU}(4)$

Following [6] we construct $\mathrm{Cl}(0, 6)$ from the following set of one-vectors

$$F_1 = -\mathrm{Id}_2 \otimes (\mathrm{i}\sigma_y) \otimes \sigma_x, \quad F_2 = -\mathrm{i}\sigma_y \otimes \sigma_z \otimes \sigma_x, \quad F_3 = -\sigma_x \otimes (\mathrm{i}\sigma_y) \otimes \sigma_z$$
$$F_4 = \mathrm{i}\sigma_y \otimes \sigma_x \otimes \sigma_x, \qquad F_5 = \mathrm{i}\sigma_y \otimes \mathrm{Id}_2 \otimes \sigma_z, \quad F_6 = -\sigma_z \otimes (\mathrm{i}\sigma_y) \otimes \sigma_z.$$

As in [6], this yields $\mathrm{Spin}(0, 6)$ as $\theta_{\mathbb{C}}(\mathrm{SU}(4))$, where $\theta_{\mathbb{C}}(X)$ is the embedding of a matrix $X \in M(n, \mathbf{C})$ into a real $2n \times 2n$ matrix defined by first setting $\theta_C(z) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ for a complex scalar $z = x + \mathrm{i}y$. We then define $\theta_{\mathbb{C}}(M) = (\theta_{\mathbb{C}}(m_{ij}))$, i.e., $\theta_{\mathbb{C}}(M)$ is a $n \times n$ block matrix, with the (i, j)th block equal to the 2×2 real matrix $\theta_{\mathbb{C}}(m_{ij})$.

$\Theta_{\mathbb{C}}$ is an algebra isomorphism onto its image.

Suppose $G \in \theta(\mathrm{SU}(4))$ has the form $G = \theta_{\mathbb{C}}(Z)$ for $Z = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_5 & z_6 & z_7 & z_8 \\ z_9 & z_{10} & z_{11} & z_{12} \\ z_{13} & z_{14} & z_{15} & z_{16} \end{bmatrix}$ where $z_j = x_j + \mathrm{i}y_j$ for some $x_j, y_j \in \mathbb{R}$, $j = 1, 2, ..., 16$. Then the conditions $Z^*Z = \mathrm{Id}_4$ (equivalently, $G^TG = \mathrm{Id}_8$) and $\det(Z) = 1$ are equivalent to the following equations

$$
\begin{aligned}
x_1^2 + x_{13}^2 + x_5^2 + x_9^2 + y_1^2 + y_{13}^2 + y_5^2 + y_9^2 &= 1 \\
x_{10}^2 + x_{14}^2 + x_2^2 + x_6^2 + y_{10}^2 + y_{14}^2 + y_2^2 + y_6^2 &= 1 \\
x_{11}^2 + x_{15}^2 + x_3^2 + x_7^2 + y_{11}^2 + y_{15}^2 + y_3^2 + y_7^2 &= 1 \\
x_{12}^2 + x_{16}^2 + x_4^2 + x_8^2 + y_{12}^2 + y_{16}^2 + y_4^2 + y_8^2 &= 1 \\
x_{13}x_{14} + x_1x_2 + x_5x_6 + x_{10}x_9 + y_{13}y_{14} + y_1y_2 + y_5y_6 + y_{10}y_9 &= 0 \\
-x_2y_1 + x_9y_{10} - x_{14}y_{13} + x_{13}y_{14} + x_1y_2 - x_6y_5 + x_5y_6 - x_{10}y_9 &= 0 \\
x_{13}x_{15} + x_1x_3 + x_5x_7 + x_{11}x_9 + y_{13}y_{15} + y_1y_3 + y_5y_7 + y_{11}y_9 &= 0 \\
-x_3y_1 + x_9y_{11} - x_{15}y_{13} + x_{13}y_{15} + x_1y_3 - x_7y_5 + x_5y_7 - x_{11}y_9 &= 0 \\
x_{13}x_{16} + x_1x_4 + x_5x_8 + x_{12}x_9 + y_{13}y_{16} + y_1y_4 + y_5y_8 + y_{12}y_9 &= 0 \quad (2) \\
-x_4y_1 + x_9y_{12} - x_{16}y_{13} + x_{13}y_{16} + x_1y_4 - x_8y_5 + x_5y_8 - x_{12}y_9 &= 0 \\
x_{10}x_{11} + x_{14}x_{15} + x_2x_3 + x_6x_7 + y_{10}y_{11} + y_{14}y_{15} + y_2y_3 + y_6y_7 &= 0 \\
-x_{11}y_{10} + x_{10}y_{11} - x_{15}y_{14} + x_{14}y_{15} - x_3y_2 + x_2y_3 - x_7y_6 + x_6y_7 &= 0 \\
x_{10}x_{12} + x_{14}x_{16} + x_2x_4 + x_6x_8 + y_{10}y_{12} + y_{14}y_{16} + y_2y_4 + y_6y_8 &= 0 \\
-x_{12}y_{10} + x_{10}y_{12} - x_{16}y_{14} + x_{14}y_{16} - x_4y_2 + x_2y_4 - x_8y_6 + x_6y_8 &= 0 \\
x_{11}x_{12} + x_{15}x_{16} + x_3x_4 + x_7x_8 + y_{11}y_{12} + y_{15}y_{16} + y_3y_4 + y_7y_8 &= 0 \\
-x_{12}y_{11} + x_{11}y_{12} - x_{16}y_{15} + x_{15}y_{16} - x_4y_3 + x_3y_4 - x_8y_7 + x_7y_8 &= 0 \\
\Re(\det Z) &= 1 \\
\Im(\det Z) &= 0.
\end{aligned}
$$

Due to their length, we refrain from explicitly writing the last two equations as polynomials in x_j, y_j here and instead defer them to Appendix 2.

Remark 7. *However, it is emphasized here that these last two equations are very much needed, since* $\det(\theta_C(Z))$ *is always* 1*, even for a* Z *which is merely unitary but not special unitary. Thus* $\det(G) = 1$ *is inequivalent to* $\det(Z) = 1$.

Now let us write explicitly the equations describing the double covering map $\Phi_6 : \theta_{\mathbb{C}}(\mathrm{SU}(4)) \to \mathrm{SO}(6, \mathbb{R})$.

Theorem 8. *The double covering map* $\Phi_6 : \theta_{\mathbb{C}}(\mathrm{SU}(4)) \to \mathrm{SO}(6, \mathbb{R})$ *is given by* $\Phi_6(G) = (a_{i,j})$*, where*

$$
\begin{aligned}
a_{1,1} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 + y_2y_5 - y_1y_6 + y_4y_7 - y_3y_8 \\
a_{2,1} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 - y_1y_{10} - y_{12}y_3 + y_{11}y_4 + y_2y_9 \\
a_{3,1} &= x_{14}y_1 - x_2y_{13} + x_1y_{14} - x_4y_{15} + x_3y_{16} - x_{13}y_2 + x_{16}y_3 - x_{15}y_4
\end{aligned}
$$

$$
\begin{aligned}
a_{4,1} &= -x_1x_{14} + x_{13}x_2 - x_{16}x_3 + x_{15}x_4 + y_1y_{14} - y_{13}y_2 + y_{16}y_3 - y_{15}y_4 \\
a_{5,1} &= -x_{10}y_1 - x_1y_{10} + x_4y_{11} - x_3y_{12} + x_9y_2 - x_{12}y_3 + x_{11}y_4 + x_2y_9 \\
a_{6,1} &= x_6y_1 - x_5y_2 + x_8y_3 - x_7y_4 - x_2y_5 + x_1y_6 - x_4y_7 + x_3y_8 \\
a_{1,2} &= -x_3x_5 + x_4x_6 + x_1x_7 - x_2x_8 + y_3y_5 - y_4y_6 - y_1y_7 + y_2y_8 \\
a_{2,2} &= x_1x_{11} - x_{12}x_2 + x_{10}x_4 - x_3x_9 - y_1y_{11} + y_{12}y_2 - y_{10}y_4 + y_3y_9 \\
a_{3,2} &= x_{15}y_1 - x_3y_{13} + x_4y_{14} + x_1y_{15} - x_2y_{16} - x_{16}y_2 - x_{13}y_3 + x_{14}y_4 \\
a_{4,2} &= -x_1x_{15} + x_{16}x_2 + x_{13}x_3 - x_{14}x_4 + y_1y_{15} - y_{16}y_2 - y_{13}y_3 + y_{14}y_4 \\
a_{5,2} &= -x_{11}y_1 - x_4y_{10} - x_1y_{11} + x_2y_{12} + x_{12}y_2 + x_9y_3 - x_{10}y_4 + x_3y_9 \\
a_{6,2} &= x_7y_1 - x_8y_2 - x_5y_3 + x_6y_4 - x_3y_5 + x_4y_6 + x_1y_7 - x_2y_8 \\
a_{1,3} &= -x_8y_1 + x_7y_2 - x_6y_3 + x_5y_4 + x_4y_5 - x_3y_6 + x_2y_7 - x_1y_8 \\
a_{2,3} &= -x_{12}y_1 - x_3y_{10} + x_2y_{11} - x_1y_{12} + x_{11}y_2 - x_{10}y_3 + x_9y_4 + x_4y_9 \\
a_{3,3} &= x_1x_{16} - x_{15}x_2 + x_{14}x_3 - x_{13}x_4 - y_1y_{16} + y_{15}y_2 - y_{14}y_3 + y_{13}y_4 \\
a_{4,3} &= x_{16}y_1 - x_4y_{13} + x_3y_{14} - x_2y_{15} + x_1y_{16} - x_{15}y_2 + x_{14}y_3 - x_{13}y_4 \\
a_{5,3} &= -x_1x_{12} + x_{11}x_2 - x_{10}x_3 + x_4x_9 + y_1y_{12} - y_{11}y_2 + y_{10}y_3 - y_4y_9 \\
a_{6,3} &= -x_4x_5 + x_3x_6 - x_2x_7 + x_1x_8 + y_4y_5 - y_3y_6 + y_2y_7 - y_1y_8 \\
a_{1,4} &= x_4x_5 + x_3x_6 - x_2x_7 - x_1x_8 - y_4y_5 - y_3y_6 + y_2y_7 + y_1y_8 \\
a_{2,4} &= -x_1x_{12} - x_{11}x_2 + x_{10}x_3 + x_4x_9 + y_1y_{12} + y_{11}y_2 - y_{10}y_3 - y_4y_9 \\
a_{3,4} &= -x_{16}y_1 + x_4y_{13} + x_3y_{14} - x_2y_{15} - x_1y_{16} - x_{15}y_2 + x_{14}y_3 + x_{13}y_4 \\
a_{4,4} &= x_1x_{16} + x_{15}x_2 - x_{14}x_3 - x_{13}x_4 - y_1y_{16} - y_{15}y_2 + y_{14}y_3 + y_{13}y_4 \\
a_{5,4} &= x_{12}y_1 - x_3y_{10} + x_2y_{11} + x_1y_{12} + x_{11}y_2 - x_{10}y_3 - x_9y_4 - x_4y_9 \\
a_{6,4} &= -x_8y_1 - x_7y_2 + x_6y_3 + x_5y_4 + x_4y_5 + x_3y_6 - x_2y_7 - x_1y_8 \\
a_{1,5} &= x_7y_1 + x_8y_2 - x_5y_3 - x_6y_4 - x_3y_5 - x_4y_6 + x_1y_7 + x_2y_8 \\
a_{2,5} &= x_{11}y_1 - x_4y_{10} + x_1y_{11} + x_2y_{12} + x_{12}y_2 - x_9y_3 - x_{10}y_4 - x_3y_9 \\
a_{3,5} &= -x_1x_{15} - x_{16}x_2 + x_{13}x_3 + x_{14}x_4 + y_1y_{15} + y_{16}y_2 - y_{13}y_3 - y_{14}y_4 \\
a_{4,5} &= -x_{15}y_1 + x_3y_{13} + x_4y_{14} - x_1y_{15} - x_2y_{16} - x_{16}y_2 + x_{13}y_3 + x_{14}y_4 \\
a_{5,5} &= x_1x_{11} + x_{12}x_2 - x_{10}x_4 - x_3x_9 - y_1y_{11} - y_{12}y_2 + y_{10}y_4 + y_3y_9 \\
a_{6,5} &= x_3x_5 + x_4x_6 - x_1x_7 - x_2x_8 - y_3y_5 - y_4y_6 + y_1y_7 + y_2y_8 \\
a_{1,6} &= -x_6y_1 + x_5y_2 + x_8y_3 - x_7y_4 + x_2y_5 - x_1y_6 - x_4y_7 + x_3y_8 \\
a_{2,6} &= -x_{10}y_1 - x_1y_{10} - x_4y_{11} + x_3y_{12} + x_9y_2 + x_{12}y_3 - x_{11}y_4 + x_2y_9 \\
a_{3,6} &= x_1x_{14} - x_{13}x_2 - x_{16}x_3 + x_{15}x_4 - y_1y_{14} + y_{13}y_2 + y_{16}y_3 - y_{15}y_4 \\
a_{4,6} &= x_{14}y_1 - x_2y_{13} + x_1y_{14} + x_4y_{15} - x_3y_{16} - x_{13}y_2 - x_{16}y_3 + x_{15}y_4 \\
a_{5,6} &= -x_1x_{10} + x_{12}x_3 - x_{11}x_4 + x_2x_9 + y_1y_{10} - y_{12}y_3 + y_{11}y_4 - y_2y_9 \\
a_{6,6} &= -x_2x_5 + x_1x_6 + x_4x_7 - x_3x_8 + y_2y_5 - y_1y_6 - y_4y_7 + y_3y_8.
\end{aligned}
$$

Proof: A typical element of $V = \mathrm{Span}_{\mathbb{R}}\{F_j;\ j = 1, 2, ..., 6\}$ is

$$\sum_{k=1}^{6} a_k F_k = \begin{bmatrix} 0 & 0 & -a_6 & -a_1 & a_5 & -a_2 & -a_3 & a_4 \\ 0 & 0 & -a_1 & a_6 & -a_2 & -a_5 & a_4 & a_3 \\ a_6 & a_1 & 0 & 0 & a_3 & a_4 & a_5 & a_2 \\ a_1 & -a_6 & 0 & 0 & a_4 & -a_3 & a_2 & -a_5 \\ -a_5 & a_2 & -a_3 & -a_4 & 0 & 0 & a_6 & -a_1 \\ a_2 & a_5 & -a_4 & a_3 & 0 & 0 & -a_1 & -a_6 \\ a_3 & -a_4 & -a_5 & -a_2 & -a_6 & a_1 & 0 & 0 \\ -a_4 & -a_3 & -a_2 & a_5 & a_1 & a_6 & 0 & 0 \end{bmatrix}.$$

Let $G \in \theta_{\mathbb{C}}(\mathrm{SU}(4))$. We need to compute GF_jG^T for each $j = 1, 2, ..., 6$ and represent the result as an element of V. Upon multiplying out GF_1G^T, for example, we have

$$\begin{aligned}
a_{1,1} = -(GF_1G^T)_{1,4} &= -x_2x_5 + x_1x_6 - x_4x_7 + x_3x_8 + y_2y_5 - y_1y_6 \\
&\quad + y_4y_7 - y_3y_8 \\
a_{2,1} = -(GF_1G^T)_{1,6} &= x_1x_{10} + x_{12}x_3 - x_{11}x_4 - x_2x_9 - y_1y_{10} - y_{12}y_3 \\
&\quad + y_{11}y_4 + y_2y_9 \\
a_{3,1} = -(GF_1G^T)_{1,7} &= x_{14}y_1 - x_2y_{13} + x_1y_{14} - x_4y_{15} + x_3y_{16} - x_{13}y_2 \\
&\quad + x_{16}y_3 - x_{15}y_4 \\
a_{4,1} = \ (GF_1G^T)_{1,8} &= -x_1x_{14} + x_{13}x_2 - x_{16}x_3 + x_{15}x_4 + y_1y_{14} - y_{13}y_2 \\
&\quad + y_{16}y_3 - y_{15}y_4 \\
a_{5,1} = \ (GF_1G^T)_{1,5} &= -x_{10}y_1 - x_1y_{10} + x_4y_{11} - x_3y_{12} + x_9y_2 - x_{12}y_3 \\
&\quad + x_{11}y_4 + x_2y_9 \\
a_{6,1} = -(GF_1G^T)_{1,3} &= x_6y_1 - x_5y_2 + x_8y_3 - x_7y_4 - x_2y_5 + x_1y_6 \\
&\quad - x_4y_7 + x_3y_8.
\end{aligned}$$

Analogously, directly computing GF_jG^T for $j = 2, 3, ..., 6$ provides the rest of the terms listed above. ■

As in the previous section we employ Givens rotations to constructively factor any element of $\mathrm{SO}(6, \mathbb{R})$ as the product of the following 15 matrices, representing rotations on each of the 15 coordinate planes in $\mathbb{R}^6$. As before one need employ only a strictly smaller subset of them, but it may be expedient to avail of all of them. The 15 Givens rotations are labelled for convenience as follows

Table 2. Matrices G_i, for $i = 1, 2, 3, ..., 15$.

$$G_1 = \begin{pmatrix} \widehat{c} & 0 & 0 & \widehat{s} \\ 0 & \widehat{c} & \widehat{s} & 0 \\ 0 & -\widehat{s} & \widehat{c} & 0 \\ -\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}, \quad G_6 = \begin{pmatrix} \widehat{c} & 0 & -\mathrm{i}\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\mathrm{i}\widehat{s} \\ -\mathrm{i}\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & -\mathrm{i}\widehat{s} & 0 & \widehat{c} \end{pmatrix}, \quad G_{11} = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} \widehat{c} & \mathrm{i}\widehat{s} & 0 & 0 \\ \mathrm{i}\widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\mathrm{i}\widehat{s} \\ 0 & 0 & -\mathrm{i}\widehat{s} & \widehat{c} \end{pmatrix}, \quad G_7 = \begin{pmatrix} \widehat{c} & -\widehat{s} & 0 & 0 \\ \widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\widehat{s} \\ 0 & 0 & \widehat{s} & \widehat{c} \end{pmatrix}, \quad G_{12} = \begin{pmatrix} \widehat{c} & 0 & -\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\widehat{s} \\ \widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \widehat{s} & 0 & \widehat{c} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} z_2 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}, \quad G_8 = \begin{pmatrix} \widehat{c} & -\widehat{s} & 0 & 0 \\ \widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & \widehat{s} \\ 0 & 0 & -\widehat{s} & \widehat{c} \end{pmatrix}, \quad G_{13} = \begin{pmatrix} \widehat{c} & 0 & 0 & -\mathrm{i}\widehat{s} \\ 0 & \widehat{c} & \mathrm{i}\widehat{s} & 0 \\ 0 & \mathrm{i}\widehat{s} & \widehat{c} & 0 \\ -\mathrm{i}\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$$

$$G_4 = \begin{pmatrix} \widehat{c} & -\mathrm{i}\widehat{s} & 0 & 0 \\ -\mathrm{i}\widehat{s} & \widehat{c} & 0 & 0 \\ 0 & 0 & \widehat{c} & -\mathrm{i}\widehat{s} \\ 0 & 0 & -\mathrm{i}\widehat{s} & \widehat{c} \end{pmatrix}, \quad G_9 = \begin{pmatrix} \widehat{c} & 0 & -\mathrm{i}\widehat{s} & 0 \\ 0 & \widehat{c} & 0 & \mathrm{i}\widehat{s} \\ -\mathrm{i}\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \mathrm{i}\widehat{s} & 0 & \widehat{c} \end{pmatrix}, \quad G_{14} = \begin{pmatrix} \widehat{c} & 0 & 0 & -\mathrm{i}\widehat{s} \\ 0 & \widehat{c} & -\mathrm{i}\widehat{s} & 0 \\ 0 & -\mathrm{i}\widehat{s} & \widehat{c} & 0 \\ -\mathrm{i}\widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}$$

$$G_5 = \begin{pmatrix} \widehat{c} & 0 & 0 & -\widehat{s} \\ 0 & \widehat{c} & \widehat{s} & 0 \\ 0 & -\widehat{s} & \widehat{c} & 0 \\ \widehat{s} & 0 & 0 & \widehat{c} \end{pmatrix}, \quad G_{10} = \begin{pmatrix} \widehat{c} & 0 & \widehat{s} & 0 \\ 0 & \widehat{c} & 0 & -\widehat{s} \\ -\widehat{s} & 0 & \widehat{c} & 0 \\ 0 & \widehat{s} & 0 & \widehat{c} \end{pmatrix}, \quad G_{15} = \begin{pmatrix} z_2 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_1 \end{pmatrix}$$

$$\widehat{c} = c\frac{\theta}{2}, \qquad \widehat{s} = s\frac{\theta}{2}, \qquad z_1 = \widehat{c} + \mathrm{i}\widehat{s}, \qquad z_2 = \widehat{c} - \mathrm{i}\widehat{s}.$$

Definition 9. $R_1 = R_{1,2},\ R_2 = R_{2,3},\ R_3 = R_{3,4},\ R_4 = R_{4,5},\ R_5 = R_{5,6},\ R_6 = R_{1,3},\ R_7 = R_{2,4},\ R_8 = R_{3,5},\ R_9 = R_{4,6},\ R_{10} = R_{1,4},\ R_{11} = R_{2,5},\ R_{12} = R_{3,6},\ R_{13} = R_{1,5},\ R_{14} = R_{2,6},\ R_{15} = R_{1,6}$.

Before $R_{i,j}$ represents a rotation in the plane $\{i, j\}$ through an angle θ.

Theorem 10. *Let $R \in \mathrm{SO}(6,\mathbb{R})$. Let $R = \prod_{k=1}^{L} R^k$ be a factorization of R into a product of Givens rotations. Then preimage of R under Φ_6 is $\{\pm G\}$, where $G = \prod_{k=1}^{L} G^k$, with $G^k \in \{G_1, \ldots, G_{15}\}$ for G_i given in* Table 2

Proof: G_1 must satisfy $\Phi_6(G_1) = R_1$, where $\Phi(G_1) = (a_{i,j})$ is as defined in Theorem 8. Let $c = c\theta$ and $s = s\theta$. This provides 36 equations that the 32 variables x_i, y_i, $i = 1, 2, ..., 16$, must satisfy. Since $G_1 \in \mathrm{SU}(4)$, we also must consider the 16 equations in the system (2). Along with the condition $s^2 + c^2 = 1$, we have 53 equations and 34 variables. Again we produce a Gröbner basis for this system. Order the variables $x_1 > y_1 > x_2 > y_2 > ... > x_{16} > y_{16} > s > c$ that is, taking in order the real and imaginary parts of each element of z from left to right, top to bottom, followed finally by s and c. Imposing degree reverse lexicographical order gives the following, where all variables not named below are identically zero

$$\begin{array}{lrcl}
(a) & 1 + c - 2x_{16}^2 & = & 0, \quad x_{10} = x_{13} = -x_7 = -x_4, \qquad -1 + c^2 + s^2 = 0 \\
(b) & -1 + c + 2x_{13}^2 & = & 0, \quad sx_{13} + x_{16} - cx_{16} = 0, \quad x_{13} + cx_{13} + sx_{16} = 0 \\
(c) & s + 2x_{13}x_{16} & = & 0, \qquad\qquad x_{11} = x_{16} = x_6 = x_1.
\end{array}$$

Equations (a) and (b), when considered individually, imply $x_{16} = \pm|c\frac{\theta}{2}|$ and $x_{13} = \pm|s\frac{\theta}{2}|$. However, as in the proof of Theorem 6, if $x_4 = \pm c\frac{\theta}{2}$ then $s\theta = -2w_3(\pm c\frac{\theta}{2})$ implies $w_3 = \mp s\frac{\theta}{2}\cdot$ Explicitly, we have two solutions: $x_1 = x_6 = x_{11} = x_{16} = c\frac{\theta}{2}$ and $-x_4 = -x_7 = x_{10} = x_{13} = -s\frac{\theta}{2}$; or $x_1 = x_6 = x_{11} = x_{16} = -c\frac{\theta}{2}$ and $-x_4 = -x_7 = x_{10} = x_{13} = s\frac{\theta}{2}$. The remaining equations are equivalent to trigonometric identities consistent with these solutions. Therefore

$$\Phi^{-1}(R_1) = \left\{ \pm \begin{bmatrix} c\frac{\theta}{2} & 0 & 0 & s\frac{\theta}{2} \\ 0 & c\frac{\theta}{2} & s\frac{\theta}{2} & 0 \\ 0 & -s\frac{\theta}{2} & c\frac{\theta}{2} & 0 \\ -s\frac{\theta}{2} & 0 & 0 & c\frac{\theta}{2} \end{bmatrix} \right\}.$$

We may choose either matrix in this set to serve as G_1. For $G_2, G_3, ..., G_{15}$, the computations are analogous. We defer these to Appendix 7. ■

4. Illustrative Examples

In this section, the inversion procedures are illustrated through matrices in SO (5) and SO (6), arising from the real representations of the group SO (3). Of course, this is of maximal interest when the representations are irreducible. Though complex unitary representations exist of SO (3) exist in every dimension, *real* irreducible representations exist only in odd dimensions, (see [7, Remark 12.4, page 987]). Therefore we first consider five dimensional representations of SO (3). Then, for the sake of completeness, we consider SO (6) matrices arising from the

the realification of a three dimensional unitary irreducible representation of $\mathrm{SO}\,(3)$. This latter representation will thus be a real six dimensional representation, which however is necessarily reducible.

Let us begin with a folklore $\mathrm{SO}\,(5)$-irreducible real representation of the group $\mathrm{SO}\,(3)$. We define an action of $\mathrm{SO}\,(3)$ on $\mathbb{R}^5$ as follows. We first identify $\mathbb{R}^5$ with the space V of 3×3 real symmetric, traceless matrices. Then the matrices, $V_1, V_2, ..., V_5$, given below, form an orthonormal basis with respect to the trace inner product on V

$$V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad V_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$V_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V_5 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Note that the trace inner product is essentially the Euclidean inner product on $\mathbb{R}^5$, as a simple calculation shows. Define an action of $\mathrm{SO}\,(3)$ on $\mathbb{R}^5$ as follows. For $H \in \mathrm{SO}\,(3)$ and V a real 3×3 traceless, symmetric matrix, define

$$H \cdot V = HVH^T.$$

Clearly this linear action preserves the trace inner product and thus its matrix is represented by 5×5 orthogonal matrices, which can be shown to be special orthogonal as well.

Let $\Phi : \mathrm{SO}\,(3) \to \mathrm{SO}\,(5)$ be the associated homomorphism. Then $\Phi\,(H) = M = (m_{ij}) \in \mathrm{SO}\,(5)$, where $m_{ij} = \mathrm{Tr}\left(GV_jG^TV_i\right)$, $i, j = 1, 2, ..., 5$. This representation is irreducible since, as is well-known (cf. [7, Remark 12.4, p 987]) irreducible $\mathrm{SO}\,(3)$ are all odd dimensional, and thus if it were to be reducible this representation must have a one-dimensional invariant subspace. But there is no $V = \sum_{i=1}^{5} a_i V_i$ such that $H \cdot V$ is proportional, for all H, to V. So it follows that the representation above is irreducible.

For a 3×3 matrix $H = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ we obtain the following entries $m_{ij} = \mathrm{Tr}\left(GV_jG^TV_i\right)$