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ON THE GENERALIZED f-BIHARMONIC MAPS AND STRESS f-BIENERGY TENSOR

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Abstract. In this paper, we investigate some properties for generalized f-harmonic and f-biharmonic maps between two Riemannian manifolds. In particular we present some new properties for the generalized stress f-energy tensor and the divergence of the generalized stress f-bienergy.

1. Introduction

Consider a smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds $M = (M^m, g)$ and $N = (N^n, h)$ and $f : M \times N \longrightarrow (0, +\infty)$ is a smooth positive function, then the f-energy functional of φ is defined by

$$E_f(\varphi) = \frac{1}{2} \int_M f(x, \varphi(x)) \, |\mathrm{d}_x \varphi|^2 \, v_g$$

(or over any compact subset $K \subset M$).

A map is called f-harmonic if it is a critical point of the $E_f(\varphi)$. In terms of Euler-Lagrange equation, φ is harmonic if the f-tension field of φ

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}^M f_\varphi) - e(\varphi)(\mathrm{grad}^N f) \circ \varphi.$$

The f-bienergy functional of φ is defined as

$$E_{2,f}(\varphi) = \frac{1}{2} \int_M |\tau_f(\varphi)|^2 \, v_g.$$

A map is called f-biharmonic if it is a critical point of the f-bienergy functional.

The f-harmonic and f-biharmonic concept is a natural generalization of harmonic maps (Eells and Sampson [8]), and biharmonic maps (Jiang [9]).

In mathematical physics, f-harmonic maps, are related to the equations of the motion of a continuous system of spins (see [6]) and the gradient Ricci-soliton structure (see [12]).

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In this paper, we investigate some properties for generalized f-harmonic and f-biharmonic maps between two Riemannian manifolds. In particular we present some new properties for the generalized stress f-energy tensor (Theorem 4) and the divergence of the generalized stress f-bienergy (Theorem 8).

2. f-Harmonic Maps

Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds and let $f : M \times N \longrightarrow (0, +\infty)$ be a smooth positive function.

In particular, if $\varphi : M \longrightarrow N$ has no critical points, i.e., $|\mathrm{d}_x\varphi| \neq 0$, then harmonic maps, p-harmonic maps and F-harmonic map are f-harmonic map with $f = 1$, $f = |\mathrm{d}\varphi|^{p-2}$ and $f = F'(\frac{|\mathrm{d}\varphi|^2}{2})$ respectively.

Let $f_1 : M \longrightarrow (0, \infty)$ be a smooth positive function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then $\tau_f(\varphi) = \tau_{f_1}(\varphi) = f_1\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}^M f_1)$. And $\varphi : M \longrightarrow N$ is f-harmonic if and only if it is f_1-harmonic (see [11]).

A map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is f-harmonic if it satisfies the system of differential equation

$$g^{ij}\Big(\frac{\partial f}{\partial x^i} + \frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial f}{\partial y^\alpha}\Big)\frac{\partial \varphi^\delta}{\partial x^j} - \frac{1}{2}g^{ij}\frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j}h_{\alpha\beta}h^{\gamma\delta}\frac{\partial f}{\partial y^\gamma} = 0$$

for all $\delta = 1, ..., n$.

The identity map $\mathrm{Id} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is f-harmonic if it satisfies the system of differential equation

$$\frac{\partial f}{\partial x^i} + \frac{2-m}{2}\frac{\partial f}{\partial y^i} = 0$$

for all $i = 1, ..., m$, where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ is a smooth positive function.

Theorem 1. *Let $\varphi : M \longrightarrow N$ is a smooth map of two Riemannian manifolds and let $i : N \hookrightarrow P$ be the inclusion map of a submanifold. Then φ is f-harmonic if and only if $\tau_f(i \circ \varphi)$ is normal to N, where $f \in C^\infty(M \times P)$ is a smooth positive function.*

Proof: The f-tension field of the composition $i \circ \varphi : M \longrightarrow P$ is given by

$$\tau_f(i \circ \varphi) = f_{i\circ\varphi}\tau(i \circ \varphi) + \mathrm{d}i(\mathrm{d}\varphi(\mathrm{grad}^M f_{i\circ\varphi})) - e(i \circ \varphi)(\mathrm{grad}^P f) \circ i \circ \varphi$$

where $f_{i\circ\varphi} : M \longrightarrow (0, \infty)$ defined by

$$f_{i\circ\varphi}(x) = f(x, i(\varphi(x))) = f(x, \varphi(x)) = f_\varphi(x)$$

for all $x \in M$. Thus

$$\tau_f(i \circ \varphi) = f_\varphi \tau(i \circ \varphi) + \mathrm{d}i(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)) - e(i \circ \varphi)(\mathrm{grad}^P f) \circ i \circ \varphi.$$

The tension field of the composition $i \circ \varphi$ is given by

$$\tau(i \circ \varphi) = \mathrm{d}i(\tau(\varphi)) + \mathrm{trace}\nabla \mathrm{d}i(\mathrm{d}\varphi, \mathrm{d}\varphi)$$

and

$$\begin{aligned}\tau_f(i \circ \varphi) &= f_\varphi \mathrm{d}i(\tau(\varphi)) + f_\varphi \mathrm{trace}\nabla \mathrm{d}i(\mathrm{d}\varphi, \mathrm{d}\varphi) \\ &\quad + \mathrm{d}i(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)) - e(i \circ \varphi)(\mathrm{grad}^P f) \circ i \circ \varphi.\end{aligned}$$

Since the energy density of $i \circ \varphi$ is $e(\varphi)$

$$(\mathrm{grad}^P f) \circ i \circ \varphi = \mathrm{d}i(\mathrm{grad}^N f) \circ \varphi + (\mathrm{grad}^P f)^\perp \circ i \circ \varphi \tag{1}$$

where $(\mathrm{grad}^P f)^\perp(\varphi(x)) \in (T_{\varphi(x)}N)^\perp$, for all $x \in M$, we obtain

$$\begin{aligned}\tau_f(i \circ \varphi) &= f_\varphi \mathrm{d}i(\tau(\varphi)) + f_\varphi \mathrm{trace}\nabla \mathrm{d}i(\mathrm{d}\varphi, \mathrm{d}\varphi) + \mathrm{d}i(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)) \\ &\quad - e(\varphi)\mathrm{d}i(\mathrm{grad}^N f) \circ \varphi - e(\varphi)(\mathrm{grad}^P f)^\perp \circ i \circ \varphi \\ &= \mathrm{d}i(\tau_f(\varphi)) + f_\varphi \mathrm{trace}\nabla \mathrm{d}i(\mathrm{d}\varphi, \mathrm{d}\varphi) - e(\varphi)(\mathrm{grad}^P f)^\perp \circ i \circ \varphi.\end{aligned}$$

Therefore

$$\tau_f(i \circ \varphi) - \mathrm{d}i(\tau_f(\varphi)) = f_\varphi \mathrm{trace}\nabla \mathrm{d}i(\mathrm{d}\varphi, \mathrm{d}\varphi) - e(\varphi)(\mathrm{grad}^P f)^\perp \circ i \circ \varphi \tag{2}$$

is normal to N. ■

In particular, if $f = 1$, let $\varphi : M \longrightarrow N$ is a smooth map of two Riemannian manifolds and let $i : N \hookrightarrow P$ be the inclusion map of a submanifold. Then φ is harmonic if and only if $\tau(i \circ \varphi)$ is normal to N (see [1]).

3. f-Biharmonic Maps

Let $f : M \times N \longrightarrow (0, +\infty)$ be a smooth positive function. An f-biharmonic map $\varphi : (M, g) \longrightarrow (N, h)$ is the critical point of the f-bienergy functional

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M |\tau_f(\varphi)|^2 \, v_g. \tag{3}$$

In terms of Euler-Lagrange equation, φ is f-biharmonic if and only if the f-bitension field of φ is given by (see [5])

$$\begin{aligned}\tau_{2,f}(\varphi) &= -f_\varphi \mathrm{trace}_g R^N(\tau_f(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi - \mathrm{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ &\quad + e(\varphi)(\nabla^N_{\tau_f(\varphi)} \mathrm{grad}^N f) \circ \varphi - \mathrm{d}\varphi(\mathrm{grad}^M \tau_f(\varphi)(f)) \\ &\quad - \tau_f(\varphi)(f)\tau(\varphi) + \langle \nabla^\varphi \tau_f(\varphi), \mathrm{d}\varphi \rangle (\mathrm{grad}^N f) \circ \varphi = 0\end{aligned} \tag{4}$$

where $\langle\,,\,\rangle$ is the inner product on $T^*M \otimes \varphi^{-1}TN$ and R^N is the curvature tensor of N.

In particular, if $f = 1$, we have

$$\tau_{2,f}(\varphi) = \tau_2(\varphi) = -\mathrm{trace}_g R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi - \mathrm{trace}_g(\nabla^\varphi)^2\tau(\varphi) \quad (5)$$

is the natural bi-tension field of φ.

Let $f_1 : M \longrightarrow (0, \infty)$ be a smooth positive function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then

$$\begin{aligned}\tau_{2,f}(\varphi) = \tau_{2,f_1}(\varphi) = & -f_1 \mathrm{trace}_g R^N(\tau_{f_1}(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi \\ & -\mathrm{trace}_g \nabla^\varphi f_1 \nabla^\varphi \tau_{f_1}(\varphi)\end{aligned} \quad (6)$$

where $\tau_{f_1}(\varphi) = f_1\tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}^M f_1)$ (see [11]). By applying the similar techniques as [3] we can derive the following theorem.

Theorem 2. *If $\varphi : (M, g) \longrightarrow (N, h)$ is a f-biharmonic map from a compact Riemannian manifold M into a Riemannian manifold N with non-positive curvature satisfying*

$$\begin{aligned} f_\varphi \mathrm{trace}_g \nabla^\varphi\nabla^\varphi\tau_f(\varphi) & - \mathrm{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ & + e(\varphi)(\nabla^N_{\tau_f(\varphi)}\mathrm{grad}^N f) \circ \varphi - \mathrm{d}\varphi(\mathrm{grad}^M \tau_f(\varphi)(f)) \\ & - \tau_f(\varphi)(f)\tau(\varphi) + \langle\nabla^\varphi\tau_f(\varphi), \mathrm{d}\varphi\rangle(\mathrm{grad}^N f) \circ \varphi \geq 0 \end{aligned} \quad (7)$$

$$(\mathrm{grad}^N f) \circ \varphi \geq 0 \quad (8)$$

then φ is f-harmonic.

Proof: Since $\varphi : (M, g) \longrightarrow (N, h)$ is f-biharmonic it follows from (4) that

$$\begin{aligned} - f_\varphi \mathrm{trace}_g R^N(\tau_f(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi & - \mathrm{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi \tau_f(\varphi) \\ & + e(\varphi)(\nabla^N_{\tau_f(\varphi)}\mathrm{grad}^N f) \circ \varphi - \mathrm{d}\varphi(\mathrm{grad}^M \tau_f(\varphi)(f)) \\ & - \tau_f(\varphi)(f)\tau(\varphi) + \langle\nabla^\varphi\tau_f(\varphi), \mathrm{d}\varphi\rangle(\mathrm{grad}^N f) \circ \varphi = 0. \end{aligned} \quad (9)$$

Fix a point $x \in M$ and let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g on M, such that $\nabla^M_{e_i} e_j = 0$, at x for all $i, j = 1, ..., m$.

A calculation at x

$$\begin{aligned} \frac{1}{2} f_\varphi \Delta(|\tau_f(\varphi)|^2) & = \frac{1}{2} f_\varphi e_i(e_i(h(\tau_f(\varphi), \tau_f(\varphi)))) = f_\varphi e_i(h(\nabla^\varphi_{e_i}\tau_f(\varphi), \tau_f(\varphi))) \\ & = f_\varphi h(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_f(\varphi), \tau_f(\varphi)) + f_\varphi h(\nabla^\varphi_{e_i}\tau_f(\varphi), \nabla^\varphi_{e_i}\tau_f(\varphi)) \end{aligned}$$

and taking into account (9), we have

$$\begin{aligned}\frac{1}{2}f_\varphi\Delta(|\tau_f(\varphi)|^2) = {} & f_\varphi h(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_f(\varphi),\tau_f(\varphi)) + f_\varphi h(\nabla^\varphi_{e_i}\tau_f(\varphi),\nabla^\varphi_{e_i}\tau_f(\varphi)) \\ & - f_\varphi h(\mathrm{trace}_g R^N(\tau_f(\varphi),\mathrm{d}\varphi)\mathrm{d}\varphi,\tau_f(\varphi)) \\ & - h(\mathrm{trace}_g\nabla^\varphi f_\varphi\nabla^\varphi\tau_f(\varphi),\tau_f(\varphi)) \\ & + e(\varphi)h((\nabla^N_{\tau_f(\varphi)}\mathrm{grad}^N f)\circ\varphi,\tau_f(\varphi)) \\ & - h(\mathrm{d}\varphi(\mathrm{grad}^M\tau_f(\varphi)(f)),\tau_f(\varphi)) \\ & - \tau_f(\varphi)(f)h(\tau(\varphi),\tau_f(\varphi)) \\ & + \langle\nabla^\varphi\tau_f(\varphi),\mathrm{d}\varphi\rangle h((\mathrm{grad}^N f)\circ\varphi,\tau_f(\varphi)).\end{aligned}$$

By (7), $f > 0$ and $R^N \leq 0$ one obtains

$$\Delta(|\tau_f(\varphi)|^2) \geq 0. \tag{10}$$

By the Green's theorem

$$\int_M \Delta(|\tau_f(\varphi)|^2)v_g = 0 \tag{11}$$

and (10), we have also

$$\Delta(|\tau_f(\varphi)|^2) = 0 \tag{12}$$

and finally

$$\nabla^\varphi_{e_i}\tau_f(\varphi) = 0, \qquad i = 1,...,m. \tag{13}$$

Now using the identity

$$\begin{aligned}\mathrm{div}(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi)) = {} & |\tau_f(\varphi)|^2 + e(\varphi)h((\mathrm{grad}^N f)\circ\varphi,\tau_f(\varphi)) \\ & + f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\tau_f(\varphi))\end{aligned}$$

and after integration, we conclude

$$\int_M |\tau_f(\varphi)|^2\, v_g = -\int_M e(\varphi)h((\mathrm{grad}^N f)\circ\varphi,\tau_f(\varphi))\, v_g. \tag{14}$$

By (8) and (14), we obtain

$$\int_M |\tau_f(\varphi)|^2\, v_g \leq 0$$

then, $\tau_f(\varphi) = 0$, i.e. φ is f-harmonic. ■

In particular, if $f = 1$. A smooth map $\varphi : (M, g) \longrightarrow (N, h)$ from a compact Riemannian manifold M into a Riemannian manifold N with non-positive curvature is biharmonic if and only if it is harmonic (see [9]).

Let $f_1 : M \longrightarrow (0, \infty)$ be a smooth positive function, $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$. If $\varphi : (M, g) \longrightarrow (N, h)$ is a f_1-biharmonic map from a compact Riemannian manifold M into a Riemannian manifold N with non-positive curvature satisfying

$$f_1 \mathrm{trace}_g \nabla^\varphi \nabla^\varphi \tau_{f_1}(\varphi) - \mathrm{trace}_g \nabla^\varphi f_1 \nabla^\varphi \tau_{f_1}(\varphi) \geq 0$$

then φ is f_1-harmonic (see [3]).

Theorem 3 ([3]) *If $\tau_f(\varphi)$ is a Jacobi field for a smooth map $\varphi : M \longrightarrow N$ of two Riemannian manifolds, and $\psi : N \longrightarrow P$ is a totally geodesic map of two Riemannian manifolds, satisfying*

$$e(\varphi)\, \mathrm{d}\psi(\mathrm{grad}^N f) \circ \varphi = e(\psi \circ \varphi)(\mathrm{grad}^P \tilde{f}) \circ \psi \circ \varphi \tag{15}$$

then $\tau_{\tilde{f}}(\psi \circ \varphi)$ is a Jacobi field. Where $\tilde{f} \in C^\infty(M \times P)$ be a smooth positive function and $f \in C^\infty(M \times N)$ defined by $f(x, y) = \tilde{f}(x, \psi(y))$ for all $(x, y) \in M \times N$.

Proof: The $\tilde{f}$-tension field of the composition $\psi \circ \varphi : M \longrightarrow P$ is given by

$$\tau_{\tilde{f}}(\psi \circ \varphi) = \tilde{f}_{\psi\circ\varphi} \tau(\psi \circ \varphi) + \mathrm{d}\psi(\mathrm{d}\varphi(\mathrm{grad}^M \tilde{f}_{\psi\circ\varphi})) - e(\psi \circ \varphi)(\mathrm{grad}^P \tilde{f}) \circ \psi \circ \varphi$$

where $\tilde{f}_{\psi\circ\varphi} : M \longrightarrow (0, \infty)$ given by

$$\tilde{f}_{\psi\circ\varphi}(x) = \tilde{f}(x, \psi(\varphi(x))) = f(x, \varphi(x)) = f_\varphi(x)$$

for all $x \in M$. So that

$$\tau_{\tilde{f}}(\psi \circ \varphi) = f_\varphi \tau(\psi \circ \varphi) + \mathrm{d}\psi(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)) - e(\psi \circ \varphi)(\mathrm{grad}^P \tilde{f}) \circ \psi \circ \varphi.$$

The tension field of the composition $\psi \circ \varphi$ is given by

$$\tau(\psi \circ \varphi) = \mathrm{d}\psi(\tau(\varphi)) + \mathrm{trace}\, \nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi)$$

(see [1]). So

$$\begin{aligned}\tau_{\tilde{f}}(\psi \circ \varphi) &= f_\varphi \mathrm{d}\psi(\tau(\varphi)) + f_\varphi \mathrm{trace}\, \nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi) \\ &\quad + \mathrm{d}\psi(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)) - e(\psi \circ \varphi)(\mathrm{grad}^P \tilde{f}) \circ \psi \circ \varphi.\end{aligned}$$

Let $\tau_f(\varphi) = f_\varphi \tau(\varphi) + \mathrm{d}\varphi(\mathrm{grad}^M f_\varphi) - e(\varphi)(\mathrm{grad}^N f) \circ \varphi$ the f-tension field of φ, then we have

$$\begin{aligned}\tau_{\tilde{f}}(\psi \circ \varphi) &= \mathrm{d}\psi(\tau_f(\varphi)) + f_\varphi \mathrm{trace}\, \nabla \mathrm{d}\psi(\mathrm{d}\varphi, \mathrm{d}\varphi) \\ &\quad + e(\varphi)\mathrm{d}\psi(\mathrm{grad}^N f) \circ \varphi - e(\psi \circ \varphi)(\mathrm{grad}^P \tilde{f}) \circ \psi \circ \varphi.\end{aligned}$$

Since ψ is totally geodesic and by (15), we have $\tau_{\tilde{f}}(\psi\circ\varphi) = \mathrm{d}\psi(\tau_f(\varphi))$. Fix a point $x\in M$ and let $\{e_i\}_i$ be an orthonormal frame on M, such that $\nabla^M_{e_i}e_j = 0$, at x for all i,j. The calculating at x

$$\begin{aligned}\mathrm{trace}R^P(\tau_{\tilde{f}}(\psi\circ\varphi), \mathrm{d}(\psi\circ\varphi))\mathrm{d}(\psi\circ\varphi)&\\ = R^P(\mathrm{d}\psi(\tau_f(\varphi)), \mathrm{d}\psi(\mathrm{d}\varphi(e_i)))\mathrm{d}\psi(\mathrm{d}\varphi(e_i))&\\ = \mathrm{d}\psi(R^N(\tau_f(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i)).&\end{aligned} \tag{16}$$

$$\mathrm{trace}(\nabla^{\psi\circ\varphi})^2\tau_{\tilde{f}}(\psi\circ\varphi) = \nabla^{\psi\circ\varphi}_{e_i}\nabla^{\psi\circ\varphi}_{e_i}\mathrm{d}\psi(\tau_f(\varphi)) \tag{17}$$

we derive

$$\begin{aligned}\nabla^{\psi\circ\varphi}_{e_i}\mathrm{d}\psi(\tau_f(\varphi)) &= \nabla^{\psi\circ\varphi}_{e_i}\mathrm{d}\psi(\tau_f(\varphi)) = \nabla^{\psi}_{\mathrm{d}\varphi(e_i)}\mathrm{d}\psi(\tau_f(\varphi))\\ &= \nabla\mathrm{d}\psi(\mathrm{d}\varphi(e_i), \tau_f(\varphi)) + \mathrm{d}\psi(\nabla^{\varphi}_{e_i}\tau_f(\varphi)) = \mathrm{d}\psi(\nabla^{\varphi}_{e_i}\tau_f(\varphi))\end{aligned}$$

since ψ is totally geodesic. By (17) and (18), we have

$$\mathrm{trace}(\nabla^{\psi\circ\varphi})^2\tau_{\tilde{f}}(\psi\circ\varphi) = \nabla^{\psi\circ\varphi}_{e_i}\mathrm{d}\psi(\nabla^{\varphi}_{e_i}\tau_f(\varphi)) = \mathrm{d}\psi(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}\tau_f(\varphi)). \tag{18}$$

By (16) and (18) we obtain

$$\begin{aligned}&\mathrm{trace}\,R^P(\tau_{\tilde{f}}(\psi\circ\varphi), \mathrm{d}(\psi\circ\varphi))\mathrm{d}(\psi\circ\varphi) + \mathrm{trace}\,(\nabla^{\psi\circ\varphi})^2\tau_{\tilde{f}}(\psi\circ\varphi)\\ &\quad = \mathrm{d}\psi\big(\mathrm{trace}R^N(\tau_f(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi + \mathrm{trace}\,(\nabla^{\varphi})^2\tau_f(\varphi)\big).\end{aligned}$$

Consequently, if $\tau_f(\varphi)$ is a Jacobi field, then $\tau_{\tilde{f}}(\psi\circ\varphi)$ is a Jacobi field. ■

In particular, if $\tilde{f} = 1$, let $\varphi : M \longrightarrow N$ is a biharmonic map between two Riemannian manifolds and $\psi : N \longrightarrow P$ is totally geodesic between two Riemannian manifolds, then $\psi\circ\varphi$ is a biharmonic map (Chiang and Sun [4]).

Let $f_1 : M \longrightarrow (0,\infty)$ be a smooth positive function, $\tilde{f}(x,z) = f_1(x)$ for all $(x,z)\in M\times P$. If $\tau_{f_1}(\varphi)$ is a Jacobi field for a smooth map $\varphi : M \longrightarrow N$ of two Riemannian manifolds, and $\psi : N \longrightarrow P$ is a totally geodesic map of two Riemannian manifolds, then $\tau_{f_1}(\psi\circ\varphi)$ is a Jacobi field (Chiang [3]).

4. Stress f-Energy Tensors

Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds and let $f\in C^\infty(M\times N)$ be a smooth positive function. Consider a smooth one-parameter variation of the metric g, i.e. a smooth family of metrics (g_t) $(-\varepsilon <$

$t < \varepsilon$) such that $g_0 = g$. Write $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$, then $\delta g \in \Gamma(\odot^2 T^* M)$ is a symmetric two-covariant tensor field on M. Consider the f-energy functional

$$E_f(\varphi) = \frac{1}{2}\int_M f(x,\varphi(x))\,|\mathrm{d}\varphi|^2\, v_g.$$

Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{ij}(t,x)\,\mathrm{d}x^i\,\mathrm{d}x^j$. We now compute

$$\frac{\mathrm{d}}{\mathrm{d}t}E_f(\varphi)\Big|_{t=0} = \frac{1}{2}\int_M f(x,\varphi(x))\,\delta(|\mathrm{d}\varphi|^2)v_g + \frac{1}{2}\int_M f(x,\varphi(x))\,|\mathrm{d}\varphi|^2\delta(v_{g_t}).$$

Since

$$\delta(v_{g_t}) = \frac{1}{2}\langle g, \delta g\rangle v_g \tag{19}$$

and $\varphi^* h$ is the pull-back of the metric h, then

$$\delta\Big(\frac{|\mathrm{d}\varphi|^2}{2}\Big) = -\frac{1}{2}\langle \varphi^* h, \delta g\rangle \tag{20}$$

where $\langle\, , \rangle$ is the induced Riemannian metric on $\otimes^2 T^* M$. So one ends with

$$\frac{\mathrm{d}}{\mathrm{d}t}E_f(\varphi)\Big|_{t=0} = \frac{1}{2}\int_M \langle f_\varphi\, e(\varphi)\, g - f_\varphi\, \varphi^* h, \delta g\rangle\, v_g$$

where $f_\varphi : M \longrightarrow (0,\infty)$ is a smooth function defined by $f_\varphi(x) = f(x,\varphi(x))$ for all $x \in M$ and $e(\varphi) = \frac{1}{2}|\mathrm{d}\varphi|^2$ is the energy density of φ. Then the stress f-energy tensor of the smooth map $\varphi : (M,g) \longrightarrow (N,h)$ is defined by

$$S_f(\varphi) = f_\varphi\, e(\varphi)\, g - f_\varphi\, \varphi^* h. \tag{21}$$

In particular, if $f = 1$. Then the stress f-energy tensor of the smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds is given by

$$S_f(\varphi) = S(\varphi) = e(\varphi)\, g - \varphi^* h.$$

Let $f_1 : M \longrightarrow (0,\infty)$ be a smooth positive function if $f(x,y) = f_1(x)$ for all $(x,y) \in M \times N$. Then the stress f-energy tensor of the smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds given by (see [11])

$$S_f(\varphi) = S_{f_1}(\varphi) = f_1\, e(\varphi)\, g - f_1\, \varphi^* h.$$

Theorem 4. *Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds. Then*

$$\operatorname{div}\, S_f(\varphi) = -h(\tau_f(\varphi), \mathrm{d}\varphi) + e(\varphi)\big(\mathrm{d}f_\varphi - h((\operatorname{grad}^N f)\circ\varphi, \mathrm{d}\varphi)\big).$$

Proof: Calculating in a normal frame at x we have

$$S_f(\varphi)(e_i,e_j)=\frac{1}{2}f_\varphi h(\mathrm{d}\varphi(e_k),\mathrm{d}\varphi(e_k))\delta_{ij}-f_\varphi h(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j))$$

and further on

$$\begin{aligned}(\operatorname{div} S_f(\varphi))(e_j) &= e_i(S_f(\varphi)(e_i,e_j))=\frac{1}{2}e_i(f_\varphi)h(\mathrm{d}\varphi(e_k),\mathrm{d}\varphi(e_k))\delta_{ij}\\ &\quad+\frac{1}{2}f_\varphi e_i(h(\mathrm{d}\varphi(e_k),\mathrm{d}\varphi(e_k)))\delta_{ij}\\ &\quad-e_i(f_\varphi)h(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j))-f_\varphi e_i(h(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j)))\\ &= e(\varphi)\mathrm{d}f_\varphi(e_j)+f_\varphi h(\nabla^\varphi_{e_j}\mathrm{d}\varphi(e_k),\mathrm{d}\varphi(e_k))\\ &\quad-h(\mathrm{d}\varphi(\operatorname{grad}^M f_\varphi),\mathrm{d}\varphi(e_j))-f_\varphi h(\tau(\varphi),\mathrm{d}\varphi(e_j))\\ &\quad-f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j)).\end{aligned}$$

Since the second fundamental is symmetry we obtain

$$(\operatorname{div} S_f(\varphi))(e_j)=e(\varphi)\mathrm{d}f_\varphi(e_j)-h(\mathrm{d}\varphi(\operatorname{grad}^M f_\varphi),\mathrm{d}\varphi(e_j))-f_\varphi h(\tau(\varphi),\mathrm{d}\varphi(e_j)).$$

Let $\tau_f(\varphi)=f_\varphi\tau(\varphi)+\mathrm{d}\varphi(\operatorname{grad}^M f_\varphi)-e(\varphi)(\operatorname{grad}^N f)\circ\varphi$ be the f-tension field of φ, then we have

$$\begin{aligned}(\operatorname{div} S_f(\varphi))(e_j)=e(\varphi)\mathrm{d}f_\varphi(e_j)-h(\tau_f(\varphi),\mathrm{d}\varphi(e_j))\\ -e(\varphi)h((\operatorname{grad}^N f)\circ\varphi,\mathrm{d}\varphi(e_j)).\end{aligned}$$

■

In particular, if $f=1$. Let $\varphi:(M,g)\longrightarrow(N,h)$ be a smooth map between two Riemannian manifolds. Then

$$\operatorname{div} S_f(\varphi)=\operatorname{div} S(\varphi)=-h(\tau(\varphi),\mathrm{d}\varphi).$$

Let $f_1:M\longrightarrow(0,\infty)$ be a smooth positive function if $f(x,y)=f_1(x)$ for all $(x,y)\in M\times N$. Let $\varphi:(M,g)\longrightarrow(N,h)$ be a smooth map between two Riemannian manifolds. Then

$$\operatorname{div} S_f(\varphi)=\operatorname{div} S_{f_1}(\varphi)=-h(\tau_{f_1}(\varphi),\mathrm{d}\varphi)+e(\varphi)\mathrm{d}f_1.$$

5. Stress f-Bienergy Tensors

Let $\varphi:(M,g)\longrightarrow(N,h)$ be a smooth map between two Riemannian manifolds and let $f\in C^\infty(M\times N)$ be a smooth positive function. Consider a smooth one-parameter variation (g_t) of the metric g. Write $\delta = \frac{\partial}{\partial t}\Big|_{t=0}$. Consider the f-bienergy functional

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M |\tau_f(\varphi)|^2 v_g.$$

Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{ij}(t,x)\,\mathrm{d}x^i\,\mathrm{d}x^j$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{2,f}(\varphi)\Big|_{t=0} = \frac{1}{2}\int_M \delta(|\tau_f(\varphi)|^2)v_g + \frac{1}{2}\int_M |\tau_f(\varphi)|^2\delta(v_{g_t}).$$

The calculation of the first term breaks down in three lemmas.

Lemma 5. *The vector field* $\xi = \big(\mathrm{div}(\delta g)\big)^\sharp - \frac{1}{2}\mathrm{grad}^M\big(\mathrm{trace}(\delta g)\big)$ *satisfies*

$$\begin{aligned}\delta(|\tau_f(\varphi)|^2) \;=\; & -2f_\varphi\langle h(\nabla\mathrm{d}\varphi,\tau_f(\varphi)),\delta g\rangle - 2f_\varphi h(\mathrm{d}\varphi(\xi),\tau_f(\varphi))\\ & -2\langle \mathrm{d}f_\varphi \odot h(\mathrm{d}\varphi,\tau_f(\varphi)),\delta g\rangle + \tau_f(\varphi)(f)\langle\varphi^* h,\delta g\rangle.\end{aligned}$$

Proof: In local coordinates (x^i) on M and (y^α) on N it is true that

$$\delta(|\tau_f(\varphi)|^2) = \delta\big(\tau_f(\varphi)^\alpha\tau_f(\varphi)^\beta h_{\alpha\beta}\big) = 2\,\delta(\tau_f(\varphi)^\alpha)\tau_f(\varphi)^\beta h_{\alpha\beta}.$$

Now we have also

$$\delta(\tau_f(\varphi)^\alpha) = \delta\big(f_\varphi\tau(\varphi)^\alpha + \theta^\alpha - \eta^\alpha\big) = f_\varphi\,\delta(\tau(\varphi)^\alpha) + \delta(\theta^\alpha) - \delta(\eta^\alpha) \tag{22}$$

where

$$\tau(\varphi)^\alpha = g^{ij}\Big(\varphi^\alpha_{i,j} + {}^N\Gamma^\alpha_{\mu\sigma}\varphi^\mu_i\varphi^\sigma_j - {}^M\Gamma^k_{ij}\varphi^\alpha_k\Big)$$

$$\theta^\alpha = g^{ij}(f_\varphi)_i\varphi^\alpha_j, \qquad \eta^\alpha = e(\varphi)h^{\alpha\mu}f_\mu.$$

By [10] we have

$$\delta(\tau(\varphi)^\alpha) \;=\; -g^{ai}g^{bj}\delta(g_{ab})(\nabla\mathrm{d}\varphi)^\alpha_{ij} - \xi^k\varphi^\alpha_k$$

then the first term in the right-hand side of (22) is

$$f_\varphi\,\delta(\tau(\varphi)^\alpha) = -f_\varphi g^{ai}g^{bj}\delta(g_{ab})(\nabla\mathrm{d}\varphi)^\alpha_{ij} - f_\varphi\xi^k\varphi^\alpha_k.$$

The second term on the right-hand side of (22) is

$$\delta(\theta^\alpha) = \delta(g^{ij})(f_\varphi)_i\varphi^\alpha_j.$$

By (20) the third term on the right-hand side of (22) is

$$-\delta(\eta^\alpha) = -\delta(e(\varphi))\, h^{\alpha\mu} f_\mu = \frac{1}{2}\langle \varphi^* h, \delta g\rangle h^{\alpha\mu} f_\mu.$$

Finally we have

$$\begin{aligned}\delta(\tau_f(\varphi)^\alpha) = -f_\varphi g^{ai} g^{bj}\delta(g_{ab})(\nabla \mathrm{d}\varphi)^\alpha_{ij} - f_\varphi \xi^k \varphi^\alpha_k \\ + \delta(g^{ij})(f_\varphi)_i \varphi^\alpha_i + \frac{1}{2}\langle \varphi^* h, \delta g\rangle h^{\alpha\mu} f_\mu\end{aligned}$$

and

$$\begin{aligned}\delta(|\tau_f(\varphi)|^2) &= -2 f_\varphi g^{ai} g^{bj}\delta(g_{ab})(\nabla \mathrm{d}\varphi)^\alpha_{ij}\tau_f(\varphi)^\beta h_{\alpha\beta} - 2 f_\varphi \xi^k \varphi^\alpha_k \tau_f(\varphi)^\beta h_{\alpha\beta} \\ &\quad + 2\delta(g^{ij})(f_\varphi)_i \varphi^\alpha_j \tau_f(\varphi)^\beta h_{\alpha\beta} + \langle \varphi^* h, \delta g\rangle h^{\alpha\mu} f_\mu \tau_f(\varphi)^\beta h_{\alpha\beta} \\ &= -2 f_\varphi \langle h(\nabla \mathrm{d}\varphi, \tau_f(\varphi)), \delta g\rangle - 2 f_\varphi h(\mathrm{d}\varphi(\xi), \tau_f(\varphi)) \\ &\quad -2\langle \mathrm{d}f_\varphi \odot h(\mathrm{d}\varphi, \tau_f(\varphi)), \delta g\rangle + \tau_f(\varphi)(f)\langle \varphi^* h, \delta g\rangle.\end{aligned}$$

■

Lemma 6.

$$\begin{aligned}\int_M f_\varphi h(\mathrm{d}\varphi(\xi), \tau_f(\varphi)) v_g &= \int_M \langle -\operatorname{sym}\big(\nabla f_\varphi h(\mathrm{d}\varphi, \tau_f(\varphi))\big) \\ &\quad + \frac{1}{2}\operatorname{div}\big(f_\varphi h(\mathrm{d}\varphi, \tau_f(\varphi))^\sharp\big) g, \delta g\rangle\, v_g.\end{aligned}$$

Proof: Let $\omega = f_\varphi\, h(\mathrm{d}\varphi, \tau_f(\varphi))$, by the definition of ξ

$$\int_M \omega(\xi) v_g = \int_M \omega\big((\operatorname{div}(\delta g))^\sharp\big) v_g - \frac{1}{2}\int_M \omega\big(\operatorname{grad}(\operatorname{trace}(\delta g))\big) v_g \tag{23}$$

the first term on the right-hand side of (23) is

$$\begin{aligned}\int_M \omega\big((\operatorname{div}(\delta g))^\sharp\big) v_g &= \int_M g(\omega^\sharp, (\operatorname{div}(\delta g))^\sharp) v_g \\ &= \int_M g^*(\omega, \operatorname{div}(\delta g)) v_g.\end{aligned}$$

(Here g^* denote the Riemannian metric on T^*M).
On the other hand, if $\sigma \in \Gamma(\otimes^2 T^*M)$ and $C(\omega, \sigma) = \omega^i \sigma_{ij}\mathrm{d}x^j$, we have

$$g^*(\omega, \operatorname{div}\sigma) = \operatorname{div}(C(\omega,\sigma)^\sharp) - \langle \operatorname{sym}(\nabla\omega), \sigma\rangle. \tag{24}$$

Applying (24) to $\sigma = \delta g$, we have

$$\int_M \omega\big((\operatorname{div}(\delta g))^\sharp\big) v_g = -\int_M \langle \operatorname{sym}(\nabla\omega), \delta g\rangle.$$

If $\lambda \in C^\infty(M)$, we have

$$\omega(\operatorname{grad}^M \lambda) = g^*(\omega, d\lambda). \tag{25}$$

Applying (25) to $\lambda = \operatorname{trace}(\delta g)$, then the second term on the right-hand side of (23) is

$$\begin{aligned} -\frac{1}{2}\int_M \omega\big(\operatorname{grad}(\operatorname{trace}(\delta g))\big) v_g &= -\frac{1}{2}\int_M g^*(\omega, \mathrm{d}(\operatorname{trace}(\delta g))) v_g \\ &= -\frac{1}{2}\int_M g(\omega^\sharp, \operatorname{grad}^M(\operatorname{trace}(\delta g))) v_g \\ &= \frac{1}{2}\int_M \operatorname{trace}(\delta g)\operatorname{div}(\omega^\sharp) v_g \\ &= \frac{1}{2}\int_M \langle \operatorname{div}(\omega^\sharp) g, \delta g\rangle v_g. \end{aligned}$$

■

Theorem 7. *Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map and let $\{g_t\}$ be a one parameter variation of g. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{2,f}(\varphi)\Big|_{t=0} = \frac{1}{2}\int_M \langle S_{2,f}(\varphi), \delta g\rangle\, v_g$$

where $S_{2,f}(\varphi) \in \Gamma(\odot^2 T^*M)$ is given by

$$\begin{aligned} S_{2,f}(\varphi)(X,Y) &= -\frac{1}{2}|\tau_f(\varphi)|^2 g(X,Y) - f_\varphi \langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle g(X,Y) \\ &\quad + f_\varphi h(\mathrm{d}\varphi(X), \nabla^\varphi_Y \tau_f(\varphi)) + f_\varphi h(\mathrm{d}\varphi(Y), \nabla^\varphi_X \tau_f(\varphi)) \\ &\quad - \tau_f(\varphi)(f)\big(e(\varphi) g(X,Y) - h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\big). \end{aligned}$$

Proof: By Lemma 5 and (20), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_{2,f}(\varphi)\Big|_{t=0} &= \frac{1}{2}\int_M \delta(|\tau_f(\varphi)|^2) v_g + \frac{1}{2}\int_M |\tau_f(\varphi)|^2 \delta(v_{g_t}) \\ &= \frac{1}{2}\int_M \big(-2 f_\varphi \langle h(\nabla \mathrm{d}\varphi, \tau_f(\varphi)), \delta g\rangle - 2 f_\varphi h(\mathrm{d}\varphi(\xi), \tau_f(\varphi)) \\ &\quad - 2\langle \mathrm{d}f_\varphi \odot h(\mathrm{d}\varphi, \tau_f(\varphi)), \delta g\rangle + \tau_f(\varphi)(f)\langle \varphi^* h, \delta g\rangle\big) v_g \\ &\quad + \frac{1}{2}\int_M \langle \frac{1}{2}|\tau_f(\varphi)|^2 g, \delta g\rangle v_g \end{aligned}$$

and by Lemma 6, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{2,f}(\varphi)\Big|_{t=0}= & \ \frac{1}{2}\int_M\big(-2f_\varphi\langle h(\nabla\mathrm{d}\varphi,\tau_f(\varphi)),\delta g\rangle+\langle 2\,\mathrm{sym}\,\big(\nabla f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))\big)\\ & -\mathrm{div}\,\big(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))^\sharp\big)g,\delta g\rangle-2\langle \mathrm{d}f_\varphi\odot h(\mathrm{d}\varphi,\tau_f(\varphi)),\delta g\rangle\\ & +\tau_f(\varphi)(f)\langle\varphi^*h,\delta g\rangle\big)v_g+\frac{1}{2}\int_M\langle\frac{1}{2}|\tau_f(\varphi)|^2g,\delta g\rangle v_g.\end{aligned}$$

So that

$$\begin{aligned}S_{2,f}(\varphi) = & -2f_\varphi h(\nabla\mathrm{d}\varphi,\tau_f(\varphi))+2\,\mathrm{sym}\,\big(\nabla f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))\big)\\ & -\mathrm{div}\,\big(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))^\sharp\big)g-2\mathrm{d}f_\varphi\odot h(\mathrm{d}\varphi,\tau_f(\varphi))\\ & +\tau_f(\varphi)(f)\varphi^*h+\frac{1}{2}|\tau_f(\varphi)|^2g.\end{aligned}$$

For all $X,Y\in\Gamma(TM)$, we have

$$\begin{aligned}2\,\mathrm{sym}\big(\nabla f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))\big)(X,Y) = & \ 2f_\varphi h(\nabla\mathrm{d}\varphi(X,Y),\tau_f(\varphi))\\ & +f_\varphi h(\mathrm{d}\varphi(X),\nabla^\varphi_Y\tau_f(\varphi))\\ & +f_\varphi h(\mathrm{d}\varphi(Y),\nabla^\varphi_X\tau_f(\varphi))\\ & +X(f_\varphi)h(\mathrm{d}\varphi(Y),\tau_f(\varphi))\\ & +Y(f_\varphi)h(\mathrm{d}\varphi(X),\tau_f(\varphi)),\end{aligned}$$

$$\begin{aligned}-2\mathrm{d}f_\varphi\odot h(\mathrm{d}\varphi,\tau_f(\varphi))(X,Y) = & -X(f_\varphi)h(\mathrm{d}\varphi(Y),\tau_f(\varphi))\\ & -Y(f_\varphi)h(\mathrm{d}\varphi(X),\tau_f(\varphi)).\end{aligned}$$

Calculating in a normal frame at x we have

$$\begin{aligned}\mathrm{div}\,\big(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))^\sharp\big) = & \ e_i(g(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))^\sharp,e_i))\\ = & \ e_i(f_\varphi h(\mathrm{d}\varphi(e_i),\tau_f(\varphi)))\\ = & \ e_i(f_\varphi)h(\mathrm{d}\varphi(e_i),\tau_f(\varphi))+f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_i),\tau_f(\varphi))\\ & +f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\tau_f(\varphi))\\ = & \ h(\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi),\tau_f(\varphi))+f_\varphi h(\tau(\varphi),\tau_f(\varphi))\\ & +f_\varphi\langle\mathrm{d}\varphi,\nabla^\varphi\tau_f(\varphi)\rangle.\end{aligned}$$

Let $\tau_f(\varphi)=f_\varphi\tau(\varphi)+\mathrm{d}\varphi(\mathrm{grad}^M f_\varphi)-e(\varphi)(\mathrm{grad}^N f)\circ\varphi$ be the f-tension field of φ. So

$$\mathrm{div}\,\big(f_\varphi h(\mathrm{d}\varphi,\tau_f(\varphi))^\sharp\big) = |\tau_f(\varphi)|^2+e(\varphi)\tau_f(\varphi)(f)+f_\varphi\langle\mathrm{d}\varphi,\nabla^\varphi\tau_f(\varphi)\rangle.$$

■

In particular, if $f = 1$. Then the stress f-bienergy tensor of the smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds is given by (see [10])

$$\begin{aligned} S_{2,f}(\varphi)(X,Y) &= S_2(\varphi)(X,Y) \\ &= -\frac{1}{2}|\tau(\varphi)|^2 g(X,Y) - \langle \mathrm{d}\varphi, \nabla^\varphi \tau(\varphi)\rangle g(X,Y) \\ &\quad + h(\mathrm{d}\varphi(X), \nabla^\varphi_Y \tau(\varphi)) + h(\mathrm{d}\varphi(Y), \nabla^\varphi_X \tau(\varphi)). \end{aligned}$$

Let $f_1 : M \longrightarrow (0,\infty)$ be a smooth positive function if $f(x,y) = f_1(x)$ for all $(x,y) \in M \times N$. Then the stress f-bienergy tensor of the smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds given by

$$\begin{aligned} S_{2,f}(\varphi)(X,Y) &= S_{2,f_1}(\varphi)(X,Y) \\ &= -\frac{1}{2}|\tau_{f_1}(\varphi)|^2 g(X,Y) - f_1\langle \mathrm{d}\varphi, \nabla^\varphi \tau_{f_1}(\varphi)\rangle g(X,Y) \\ &\quad + f_1 h(\mathrm{d}\varphi(X), \nabla^\varphi_Y \tau_{f_1}(\varphi)) + f_1 h(\mathrm{d}\varphi(Y), \nabla^\varphi_X \tau_{f_1}(\varphi)). \end{aligned}$$

Theorem 8. *For any smooth map $\varphi : (M,g) \longrightarrow (N,h)$ between two Riemannian manifolds, we have*

$$\operatorname{div} S_{2,f}(\varphi) = -h(\tau_{2,f}(\varphi), \mathrm{d}\varphi) - \langle \nabla^\varphi \tau_f(\varphi), \mathrm{d}\varphi\rangle \big(\mathrm{d}f_\varphi - h((\mathrm{grad}^N f) \circ \varphi, \mathrm{d}\varphi)\big).$$

Proof: Write $S_{2,f}(\varphi) = T_1 + T_2 + T_3$, where $T_1, T_2, T_3 \in \Gamma(\odot^2 T^* M)$ are defined by

$$\begin{aligned} T_1(X,Y) &= -\frac{1}{2}|\tau_f(\varphi)|^2 g(X,Y) - f_\varphi \langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle g(X,Y) \\ T_2(X,Y) &= f_\varphi h(\mathrm{d}\varphi(X), \nabla^\varphi_Y \tau_f(\varphi)) + f_\varphi h(\mathrm{d}\varphi(Y), \nabla^\varphi_X \tau_f(\varphi)) \\ T_3(X,Y) &= -\tau_f(\varphi)(f)\big(e(\varphi) g(X,Y) - h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))\big). \end{aligned}$$

Calculating in a normal frame at $x \in M$ we have

$$\begin{aligned} (\operatorname{div} T_1)(e_j) &= e_i(T_1(e_i,e_j)) = e_i\big(-\frac{1}{2}|\tau_f(\varphi)|^2 \delta_{ij} - f_\varphi \langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle \delta_{ij}\big) \\ &= -h(\nabla^\varphi_{e_j} \tau_f(\varphi), \tau_f(\varphi)) - e_j(f_\varphi)\langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle \\ &\quad - f_\varphi e_j(\langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle) \\ &= -h(\nabla^\varphi_{e_j} \tau_f(\varphi), \tau_f(\varphi)) - e_j(f_\varphi)\langle \mathrm{d}\varphi, \nabla^\varphi \tau_f(\varphi)\rangle \\ &\quad - f_\varphi h(\nabla^\varphi_{e_j} \mathrm{d}\varphi(e_i), \nabla^\varphi_{e_i} \tau_f(\varphi)) - f_\varphi h(\mathrm{d}\varphi(e_i), \nabla^\varphi_{e_j} \nabla^\varphi_{e_i} \tau_f(\varphi)) \end{aligned}$$

$$\begin{aligned}
(\operatorname{div} T_2)(e_j) &= e_i(T_2(e_i,e_j))\\
&= e_i\big(f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_j}\tau_f(\varphi)) + f_\varphi h(\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi))\big)\\
&= e_i(f_\varphi)h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_j}\tau_f(\varphi)) + f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_j}\tau_f(\varphi))\\
&\quad + f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\nabla^\varphi_{e_j}\tau_f(\varphi)) + e_i(f_\varphi)h(\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi))\big)\\
&\quad + f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi)) + f_\varphi h(\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau_f(\varphi))\\
&= h(\mathrm{d}\varphi(\operatorname{grad}^M f_\varphi),\nabla^\varphi_{e_j}\tau_f(\varphi)) + f_\varphi h(\tau(\varphi),\nabla^\varphi_{e_j}\tau_f(\varphi))\\
&\quad + f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\nabla^\varphi_{e_j}\tau_f(\varphi)) + h(\mathrm{d}\varphi(e_j),\operatorname{trace}_g\nabla^\varphi f_\varphi\nabla^\varphi\tau_f(\varphi))\\
&\quad + f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi))\\
&= h(\tau_f(\varphi),\nabla^\varphi_{e_j}\tau_f(\varphi)) + h(e(\varphi)(\operatorname{grad}^N f)\circ\varphi,\nabla^\varphi_{e_j}\tau_f(\varphi))\\
&\quad + f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\nabla^\varphi_{e_j}\tau_f(\varphi)) + h(\mathrm{d}\varphi(e_j),\operatorname{trace}_g\nabla^\varphi f_\varphi\nabla^\varphi\tau_f(\varphi))\\
&\quad + f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi))\\
&= h(\tau_f(\varphi),\nabla^\varphi_{e_j}\tau_f(\varphi)) + e(\varphi)e_j(\tau_f(\varphi)(f))\\
&\quad - e(\varphi)h(\nabla^\varphi_{\tau_f(\varphi)}\operatorname{grad}^N f,\mathrm{d}\varphi(e_j)) + f_\varphi h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\nabla^\varphi_{e_j}\tau_f(\varphi))\\
&\quad + h(\mathrm{d}\varphi(e_j),\operatorname{trace}_g\nabla^\varphi f_\varphi\nabla^\varphi\tau_f(\varphi)) + f_\varphi h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j),\nabla^\varphi_{e_i}\tau_f(\varphi))
\end{aligned}$$

$$\begin{aligned}
(\operatorname{div} T_3)(e_j) &= e_i(T_3(e_i,e_j))\\
&= e_i\big(-\tau_f(\varphi)(f)e(\varphi)\delta_{ij} + \tau_f(\varphi)(f)h(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j))\big)\\
&= -e_j(\tau_f(\varphi)(f))e(\varphi) - \tau_f(\varphi)(f)e_j(e(\varphi))\\
&\quad + e_i(\tau_f(\varphi)(f))h(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j))\\
&\quad + \tau_f(\varphi)(f)h(\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_j))\\
&\quad + \tau_f(\varphi)(f)h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j))\\
&= -e_j(\tau_f(\varphi)(f))e(\varphi) - \tau_f(\varphi)(f)e_j(e(\varphi))\\
&\quad + h(\mathrm{d}\varphi(\operatorname{grad}^M\tau_f(\varphi)(f)),\mathrm{d}\varphi(e_j)) + \tau_f(\varphi)(f)h(\tau(\varphi),\mathrm{d}\varphi(e_j))\\
&\quad + \tau_f(\varphi)(f)h(\mathrm{d}\varphi(e_i),\nabla^\varphi_{e_i}\mathrm{d}\varphi(e_j)).
\end{aligned}$$

■

In particular, if $f = 1$. Let $\varphi : (M,g) \longrightarrow (N,h)$ be a smooth map between two Riemannian manifolds. Then (see [10])

$$\operatorname{div} S_{2,f}(\varphi) = \operatorname{div} S_2(\varphi) = -h(\tau_2(\varphi),\mathrm{d}\varphi).$$

Let $f_1 : M \longrightarrow (0,\infty)$ be a smooth positive function if $f(x,y) = f_1(x)$ for all $(x,y) \in M\times N$. Let $\varphi : (M,g) \longrightarrow (N,h)$ is a smooth map between two Riemannian manifolds. Then

$$\operatorname{div} S_{2,f}(\varphi) = \operatorname{div} S_{2,f_1}(\varphi) = -h(\tau_{2,f_1}(\varphi),\mathrm{d}\varphi) - \langle\nabla^\varphi\tau_{f_1}(\varphi),\mathrm{d}\varphi\rangle\mathrm{d}f_1$$

and we recover the result in [11].

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