DEFECTS IN FOUR-DIMENSIONAL CONTINUA: A PARADIGM FOR THE EXPANSION OF THE UNIVERSE?

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Abstract. The presence of defects in material continua is known to produce internal permanent strained states. Extending the theory of defects to four dimensions and allowing for the appropriate signature, it is possible to apply these concepts to space-time. In this case a defect would induce a non-trivial metric tensor, which can be interpreted as a gravitational field. The image of a defect in space-time can be applied to the description of the Big Bang. A review of the four-dimensional generalisation of defects and an application to the expansion of the universe will be presented.

1. Introduction

The correspondence between the space-time description typical of the general relativity theory (GR) and the geometrical properties of continua has remote roots in the ether theories of the XIXth century (see some interesting references to 1839 Mac Cullagh theory in a review by A. Unzicker [23]). More specifically a formal link between moving dislocations and special relativity was pointed out by Frank [4] in 1949, then variously discussed by a number of other authors (cited in Section 2.1 of [23]). It is the very geometrization of space-time which immediately suggests a correspondence with material continua, their metric properties, and the theory of elasticity. This long known analogy has been, and is now and then, revived, but has never been taken too seriously and/or used as a constitutive theory of space-time. There are of course philosophical reasons for this mistrust, in a description of our universe basically dualistic (space-time on one side, matter/energy on the other), where the attribute of "reality", whatever it is, is easily assigned to matter/energy and rather ambiguously recognized for space-time. Even within the framework of relativity it is in practice hardly accepted the idea that time (apart from signature) is really like the other dimensions of space and that space-time with its metric properties is something more than a simple conceptual

artifact. In the present work I shall review the essentials of an interpretation of space-time as a real, though peculiar, continuum where defects play a fundamental role and I will draw some cosmological conclusions from this approach. I am in fact trying to be consistent and to seriously take space-time as an existing real entity, most in the sense of what Einstein said in an address delivered on May 5th, 1920, at the University of Leyden: "... *according to the general theory of relativity space is endowed with physical qualities*; *in this sense, therefore, there exists an ether. ... But this ether may not be thought of as endowed with the quality characteristic of ponderable media, as consisting of parts which may be tracked through time ...*" [3]. There is a specific motivation to try and explore again the present description of space-time. Since nine years or so the observation of cosmic phenomena has evidenced what has reasonably been interpreted as an accelerated expansion of the universe. This behaviour has been initially recognized considering the apparent magnitude of type Ia supernovae [14, 16] (SnIa). SnIa's are a special type of supernovae which are commonly thought to be originated from white dwarfs in binary systems, with an implosion mechanism based on the reaching of the Chandrasekhar mass limit [6]. This mechanism leads to a more or less fixed absolute luminosity which makes SnIa's allegedly good standard candles (see [1,8]). The fact that the observed luminosity of such supernovae appears to be systematically smaller than what expected from their cosmic redshift, suggests the idea of an accelerated expansion. The discovery of the acceleration has stimulated an intense and vast theoretical effort in order to explain the unexpected behaviour. In general the attempts of finding the reasons for the acceleration are based on some mechanism able to produce a negative pressure on space-time, which is mostly attributed to some "dark" (i.e., otherwise unseen) energy component present in the universe. This dark energy ranges from the simplest (and most effective) cosmological constant, uselessly introduced by Einstein in order to avoid the whole expansion of the universe, to more sophisticated variants of some exotic energy fluid endowed with a non-standard equation of state. Other attempts, instead of directly introducing dark actors, concentrate in looking for heuristic forms of the space-time Lagrangian, other than the standard Einstein-Hilbert one. The approach which is outlined here tries rather to build on the analogy with known, even though enlarged and extended, properties of continua. As we shall see the results are interesting and promising.

2. Elasticity in N Dimensions

Suppose that we have a featureless material continuum. Perfect homogeneity is assumed. In the view of a physicist, and assuming by default that all appropriate mathematical conditions are fulfilled, it is natural to associate with the continuum an Euclidean appropriately dimensioned manifold with the related geometry. Each point in the continuum will naturally be labelled by Cartesian coordinates (or any other sound coordinate system). If now, in the case of a real continuum, we consider a boundary and apply to it a set of forces globally in equilibrium, what happens is that our manifold will be stretched, or, in other terms, each point within the chosen boundary will be moved away from its original position. If we remove the applied forces, we expect the induced strain to be nullified bringing each point back to its former rest position. We know this is essentially a simple pictorial description of elasticity. In terms of coordinates of a given point, labelling the unstrained situation by means of ξ's and the strained one by means of x's, we may write

$$x^\mu = \xi^\mu + u^\mu \tag{1}$$

ranging the μ apex from 1 to N (dimension of the manifold). The u^μ's are the components of the displacement N-vector leading from the original unstrained to the final strained position. In practice the elastic deformation is described by giving a peculiar displacement vector field. The displacement field will in general not be uniform, otherwise we would have a rigid translation neutralizable by a simple coordinate transformation. The coordinates u^μ's may be expressed equally well in terms of the x's (intrinsic coordinates) or of the ξ's (extrinsic coordinates). So far we may use also that x's are differentiable functions of ξ's, i.e.,

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^\nu}\,\mathrm{d}\xi^\nu. \tag{2}$$

In practice we may compare two distinct manifolds, the unstrained or reference one and the strained or natural one [12], whose points are in one to one correspondence. Comparing corresponding lengths in the two manifolds leads to the definition of the *strain tensor* [7]

$$\varepsilon_{\mu\nu} = \frac{1}{2}\left(\frac{\partial u_\mu}{\partial \xi^\nu} + \frac{\partial u_\nu}{\partial \xi^\mu} + \eta_{\alpha\beta}\frac{\partial u^\alpha}{\partial \xi^\mu}\frac{\partial u^\beta}{\partial \xi^\nu} \right) \tag{3}$$

which enters the metric tensor of the strained manifold expressed in extrinsic coordinates

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\varepsilon_{\mu\nu}. \tag{4}$$

In equations (3) and (4) $\eta_{\mu\nu}$ means a component of the metric tensor of the reference manifold. As previously written we should expect this to correspond to an Euclidean geometry, however, in order to apply our approach to space-time, we shall allow for a Minkowski geometry. Actually the problem of the origin of the Lorentzian signature from a fully symmetric N-dimesional manifold is an open one. We know that we may formally go from the Euclidean to the Minkowskian geometry introducing an imaginary coordinate that will act as time (Wick's rotation), however no physically based mechanism for that has been found until now.

According to the assumptions made so far and to equation (2) we expect also

$$\eta_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu}. \tag{5}$$

De SaintVenant's integrability condition for equation (5) is

$$R_{\alpha\beta\mu\nu} \equiv 0 \tag{6}$$

where $R_{\alpha\beta\mu\nu}$ is a generic element of the *Riemann tensor* for the natural manifold. When condition (6) is satisfied, globally defined $x^\mu \left(\xi^1, \xi^2, \dots \right)$'s exist and in the same time the strain of the natural manifold cannot be perceived from inside, or, which is the same, the metric in intrinsic coordinates always turns to be globally Euclidean (Minkowskian).

All this is typical of pure elasticity and the corresponding deformations are not seen from within the deformed medium.

3. Defects

The scenario outlined in the previous section becomes richer when the notion of *defect* is introduced. A defect may be reduced to its essentials generalizing equation (1) to the case of a singular displacement field. Differently stated, we may substitute equation (2) with

$$\mathrm{d}x^\mu = \Phi^\mu_\nu \, \mathrm{d}\xi^\nu$$

where now $\Phi^\mu_\nu \, \mathrm{d}\xi^\nu$ is a non-integrable vector-valued one-form.

A complete classification of defects exists according to the peculiarities of Φ^μ_ν. Consider for instance a closed path in the manifold such that

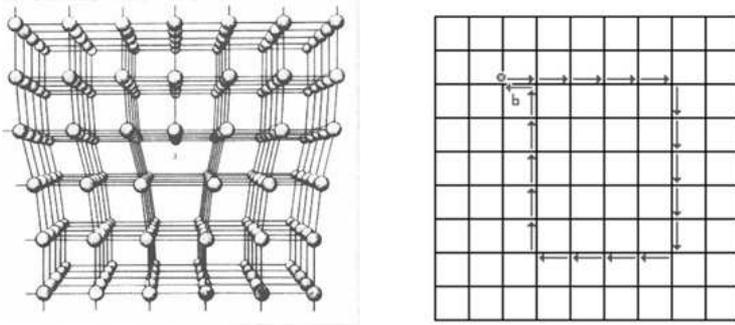$$\oint \Phi^\mu_\nu \, \mathrm{d}x^\nu = -b^\mu \neq 0. \tag{7}$$

Figure 1. Schematic representation of a dislocation. The pictorial view on the left is apparently applied to a crystal, but this is absolutely not needed. On the right a closed path in the real manifold is transposed to the reference manifold evidencing the Burgers vector.

The quantity b^μ produced by the integration in (7) is a component of the Burgers N-vector which expresses the fact that a closed contour in the reference manifold does not correspond to a closed one in the natural manifold, and vice-versa. Burgers vector measures the size of the non-closure. If the defect which is the origin of this behaviour is thought to be localized in the manifold, then we are considering a linear (or edge) defect whose direction is given by the Burgers vector. Figure 1 shows a typical dislocation in a crystal. Using a lattice makes the graph clearer but is not necessary; a closed contour encircling the edge of the singularity corresponds to an open path in the reference manifold, as it can be seen on the right, where also the Burgers vector is drawn.

The integral in (7) may be transformed by means of Stoke's theorem, becoming

$$b^\mu = - \iint \mathfrak{T}^\mu_{\alpha\beta} \, \mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha.$$

Now the integration is over the oriented surface enclosed in the former integration path. $\mathfrak{T}^\mu_{\alpha\beta}$ is a dislocation density and corresponds (as being anti-symmetric in α and β) to the torsion tensor of the manifold.

Another well known type of edge defect is obtained when condition (6) is not fulfilled. In this case parallelly transporting a vector n along a closed contour ends with a rotated vector with respect to the initial one: we have a disclination. It is indeed

$$\delta n^\nu = \oint \mathrm{d}n^\nu = - \iint R^\nu_{\mu\alpha\beta} n^\mu \, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \neq 0.$$

The curvature tensor is now interpreted as a disclination density.

When considering space-time we have a four-dimensional manifold with Lorentzian signature. What has been written concerning defects still holds and it is remarkable that curvature (then gravity) can be read as a consequence of the presence of defects in the manifold. In the case of space-time edge defects can be qualified in terms of the Poincaré group. In fact a general deformation of the continuum may be thought as a combination of a translation and a local rotation. If r is the N-vector localizing a point in a given manifold and a given reference frame, the new position after the deformation has been applied may be written [15]

$$r' = T(r) + \Lambda(r)r.$$

$T(r)$ respectively $\Lambda(r)$ correspond to local translation and Lorentz transformation operators. Within this approach the presence of a defect is expressed in terms of the soldering one form, which introduces the singular behaviour of the displacement field in the typical line element (then the metric tensor)

$$\omega = \mathrm{d}x + \Gamma^T + \Gamma^L x, \qquad \mathrm{d}s^2 = \eta_{\mu\nu}\omega^\mu \otimes \omega^\nu$$

and Γ^T respectively Γ^L represent the translation and the Lorentz connection. By this method ten separate types of edge defects of space-time are found [15].

In order to complete the analogy between continuous media and Riemannian manifolds we may recall that, at least in the linear elasticity theory, there is a rather simple proportionality law between strains and stresses, which is the general form of Hooke's law

$$\sigma^{\mu\nu} = C^{\mu\nu}_{\alpha\beta}\varepsilon^{\alpha\beta} \tag{8}$$

where $C_{\mu\nu\alpha\beta}$ is the *elastic modulus tensor*, peculiar to any given material continuum.

We may think to generalize (8) to any number of dimensions, and, even more, to space-time, although in this case the interpretation of the stress tensor σ is not at all obvious. This generalization may be useful when looking for appropriate Lagrangians describing the state of a given manifold, with or without defects. By the way, in an isotropic medium (which could be the case of space-time) the elastic modulus tensor assumes the simple form

$$C_{\alpha\beta\gamma\delta} = \lambda\eta_{\alpha\beta}\eta_{\gamma\delta} + \mu\left(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}\right)$$

depending on two parameters only: the *Lamé coefficients* λ and μ. The Hooke's law becomes

$$\sigma^{\mu\nu} = \lambda\eta^{\mu\nu}\varepsilon + 2\mu\varepsilon^{\mu\nu} \tag{9}$$

where $\varepsilon = \varepsilon^\alpha_\alpha$ is the trace of the strain tensor.
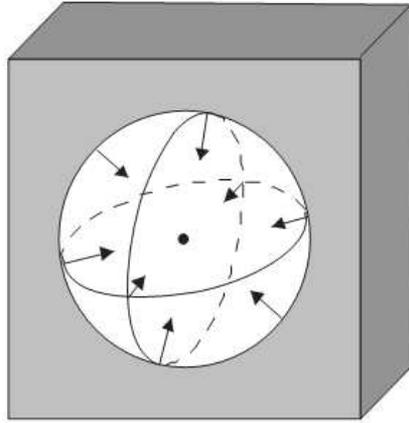
Figure 2. A point defect obtained by contraction towards the center of an initial hollow sphere.

3.1. A Point Defect

Edge defects are not the only possibility. A simple and interesting case is the one of a point defect, which is graphically schematized in Figure 2. The imagined process goes back to Vito Volterra [24], who studied plastic deformations and defects at the beginning of the XX century. We may think of digging out of a continuum a sphere of the material (whatever it is), then close the hole left behind, by pulling inwardly on the walls.

In order to avoid the problems of singularities, Volterra applied his ideal process outside a fixed reference surface in the medium surrounding the defect, but we may think to go on until the center. Of course we will produce a singularity (the actual point defect) and induce everywhere a spherically symmetric strained state. The displacement vector field is easily written

$$u = (\psi(r), 0, 0, \dots). \tag{10}$$

The only non-zero component is of course the radial one and it will depend on the distance r from the center (the defect) only.

From (10) it is also easy to find the induced strain tensor. Let us specialize to a four-dimensional manifold. The "radial" coordinate, with Lorentzian signature, is indeed time (τ, measured in meters). Let us then use polar coordinates (arbitrary origin for three-space) for the space submanifold. The non-zero components of

the strain tensor will be

$$\varepsilon_{00} = \frac{1}{2} \left[2\frac{\mathrm{d}\psi}{\mathrm{d}\tau} + \left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 \right]$$

$$\varepsilon_{rr} = \frac{\psi^2}{2}$$

$$\varepsilon_{\theta\theta} = \frac{\psi^2}{2} r^2 \tag{11}$$

$$\varepsilon_{\phi\phi} = \frac{\psi^2}{2} r^2 \sin^2 \theta.$$

Using equation (4) we are now able to write the typical line element in the strained manifold

$$\mathrm{d}s^2 = \left(1 + \frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)^2 \mathrm{d}\tau^2 - \left(1 - \psi^2\right) \left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right).$$

The presence of the defect is expressed by the discontinuity of the derivative of ψ in the origin, whereas $\psi(0)$ is finite and measures the "size" of the defect. Excluding the origin, it is possible to redefine time choosing

$$\mathrm{d}t = \left(1 + \frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right) \mathrm{d}\tau$$

which gives

$$t = \tau + \psi + T_0.$$

In the origin ($t = \tau = 0$) it is

$$\psi(0) = -T_0.$$

This change in the time coordinate, transforms the line element into

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t) \left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right) \tag{12}$$

with

$$a^2(t) = 1 - \psi^2(t). \tag{13}$$

In (12) we immediately recognize a Robertson-Walker (RW) line element, which is not a surprise since RW's is the most general line element for the assumed symmetry. However here we have established a correspondence between the RW metric and the presence of a defect in the origin, treating the manifold as a material continuum.
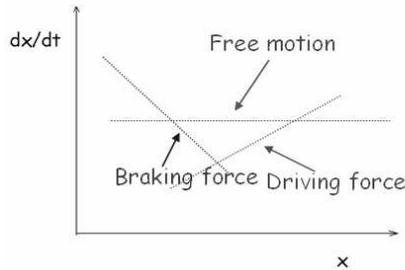
Figure 3. Phase space of a point particle interacting with an isotropic homogeneous medium. Changing x into a, the scale factor of a RW universe, the phase space stays unchanged.

4. A Lagrangian for Space-Time

The correspondence found in the previous section between the metric of a continuum with a point defect and the one of a RW universe is suggestive, however in order to treat a universe with matter an appropriate Lagrangian is needed. As it is well known, there is no general recipe for building Lagrangians, so we may proceed in a more or less formal way or try to look for analogies with already known situations.

There is indeed a very simple analogy we may consider. It is synthesised in Figure 3. The phase space of a point particle interacting with an homogeneous isotropic medium is simply bidimensional: the motion of the particle can only be straight and the relevant parameters are just the position x and speed dx/dt of the particle. If we now consider a RW universe we see that its phase space is also bidimensional, so it suffices to change the position of the particle with the scale factor of the universe, a, and the speed with the rate of change of a, $\dot{a}$, and Figure 3 is converted into the phase space of the universe. In this case the free motion corresponds to an inertial expansion (linear increase of a) and the effect of a braking force is the equivalent of a decelerated expansion while a driving force corresponds to accelerated expansion.

Let us then examine the system composed of a point particle and an homogeneous isotropic medium. There exists a simple situation in which viscous motion can be described by means of the action integral [2, 19]

$$S = \frac{m}{2} \int \mathrm{e}^{(\alpha t + \beta x)/m} \dot{x}^2 \, \mathrm{d}t. \tag{14}$$

The Lagrangian in (14) is rather naive, but it can be recast in a relativistic invariant form, as

$$S = m \int e^{\gamma \cdot \gamma} \, ds = m \int e^{\eta_{\mu\nu}\gamma^\mu x^\nu} \, ds \tag{15}$$

where the exponent contains the scalar given by the internal product of two four-vectors, $\gamma = (\alpha, \beta, \beta, \beta)$ (whose components are the "viscous" coefficients of the medium) and $r = (t, x, y, z)$, which corresponds to the position vector of the particle in space-time and ds is the line element of the world-line of the particle. The interaction with the medium is here described by a modification, or, to say better, an extension of the usual relativistic free-particle Lagrangian.

Exploiting the analogy between the phase spaces, we may conjecture from (15) an action integral for space-time as such [20]

$$S = \int e^{-g_{\mu\nu}\gamma^\mu\gamma^\nu} R \sqrt{-g} \, d^4x. \tag{16}$$

The exponent in (16) is the simplest scalar we can build combining the configuration "coordinates" of our manifold (the components of the metric tensor) with a four-vector. The sign has been chosen with hindsight, once the interesting consequences of this choice have been worked out. The rest is the traditional Einstein-Hilbert action. The vector γ will be non-trivial, i.e., a function of the coordinates, whenever defects are present in the manifold.

The Accelerated Expansion

We may study the implications of (16) in the case of a RW symmetry, which is induced by a point defect[1]. The effective Lagrangian density is then

$$L = e^{-\chi^2} \left(a\ddot{a} + \dot{a}^2\right) a \tag{17}$$

where χ is the time component of the four-vector γ (the only non-zero component, because of the symmetry). If we introduce for γ the typical condition for incompressibility in the elasticity theory, that would now be $\nabla_\mu \gamma^\mu = 0$, we obtain

$$\chi \propto \frac{1}{a^3} \tag{18}$$

a sort of a four-dimensional Coulomb's law.

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[1]Actually the same general results hold also in the case that the defect is represented by any given space-like hypersurface.
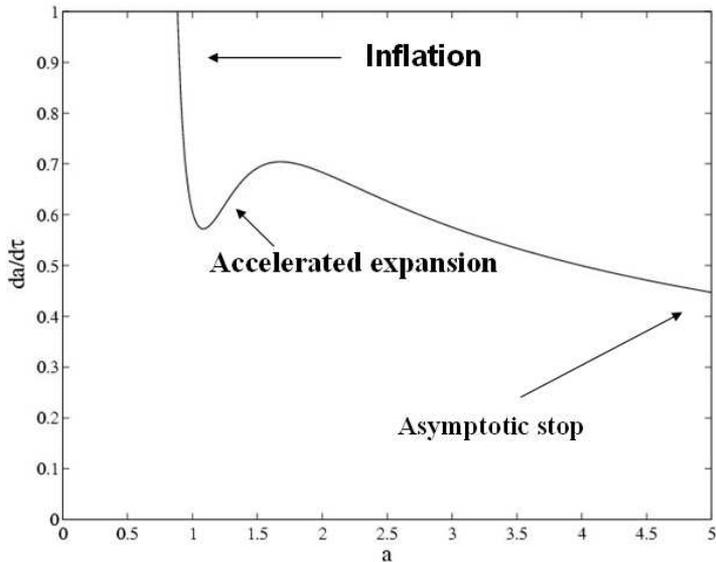
Figure 4. Expansion rate of a RW empty space-time with a defect in the origin. The behaviour includes an initial inflationary era, followed by an accelerated expansion era, and finally by a decreasing expansion rate leading asymptotically to a stop.

From (17) and (18) it is possible to deduce $\dot{a}$ as a function of a, which is shown in Figure 4 (see [20]). Remarkably, the expansion rate starts from an infinite value, giving rise to an initial inflationary era, then an accelerated expansion epoch follows. Finally the expansion starts again to slow down asymptotically reaching zero.

This result is interesting in view of the observed accelerated expansion of the universe, furthermore displaying a much reassuring asymptotic behaviour, and giving, as a free gift, also inflation. However it must be reminded that we are dealing with the empty space time only, so what we are deducing is the pure effect of the defect at the origin. Matter can be introduced in the Lagrangian in the usual way, i.e., adding appropriate terms minimally coupled to the geometry.

This theory, which we have called Cosmic Defect (CD) theory, has indeed been applied to the fit of the SnIa luminosity data, giving good and encouraging results [21].

5. The "Elastic" Approach

Pursuing literally the elastic analogy we may try another approach to the definition of the Lagrangian. We may treat the strain induced by the presence of a defect as a field *in* the space-time rather then a property *of* the space-time. A weakness of this approach is in the fact that we already know, for instance, that attempts to treat gravity as a field in the space-time usually fail. Despite this, let us see how cosmology would look like. We must consider the elastic energy density associated with a strained state, which would be $w = \frac{1}{2}\sigma_{\alpha\beta}\varepsilon^{\alpha\beta}$. The corresponding action integral for the (empty) space-time is then

$$S = \int \left(R - \frac{\kappa}{2}\sigma_{\mu\nu}\varepsilon^{\mu\nu} \right) \sqrt{-g}\mathrm{d}^4x. \tag{19}$$

A series of comments regarding the action (19) are in order. When introducing a field in a Lagrangian we expect to have both potential and dynamical terms; the latter are here apparently missing. However we should remember that the strain tensor, according to (3), does indeed contain (as well as $\sigma_{\mu\nu}$ does) the displacement vector u and its first derivatives. The curvature scalar, in turn, contains up to third order derivatives of u, because of (4). This is why, in a sense, we can interpret R as being the "dynamical" term in (19) so that the structure of the Lagrangian is formally the equivalent of the difference between dynamical and potential terms.

Another remark is that what in (19) appears to be the usual minimal coupling between the field (the elastic field) and the geometry is actually more complicated because the field is also included in the metric tensor, so that the coupling goes up to higher order terms than with other fields. Besides this we will also see in a moment that the coupling constant κ is absorbed into other parameters typical of space-time as a continuum.

Limiting our considerations to the linear elasticity case[2], adding the hypothesis of a pointlike defect in an homogeneous isotropic space-time, which means RW symmetry, Hooke's law takes the form (9) and finally (19) becomes

$$S = \int \left(R - \frac{\lambda}{2}\varepsilon^2 - \mu\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} \right) \sqrt{-g}\mathrm{d}^4x \tag{20}$$

with

$$\varepsilon = \frac{\dot{\psi}}{2}\left(2 + \dot{\psi} \right) - \frac{3}{2}\frac{\psi^2}{1 - \psi^2}, \qquad \varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = \frac{\dot{\psi}^2}{2}\left(2 + \dot{\psi} \right)^2 + \frac{3}{4}\frac{\psi^4}{\left(1 - \psi^2 \right)^2}.$$

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[2]An interesting discussion of the non-linear case may be found in [23].

As already said, the coupling constant κ has been merged with the Lamé coefficients λ and μ. Use has been made also of (13).

From (20) the following fourth order equation for $\dot\psi$ can be obtained.

$$
\begin{aligned}
6\dot\psi^2 \frac{\psi^2}{\sqrt{1-\psi^2}} - \lambda\dot\psi & \left(\frac{\dot\psi}{2}\left(2+\dot\psi\right) - \frac{3}{2}\frac{\psi^2}{1-\psi^2} \right)\left(1+\dot\psi\right)\left(1-\psi^2\right)^{3/2} \\
& - \mu\dot\psi^2\left(2+\dot\psi\right)\left(1+\frac{3}{2}\dot\psi\right)\left(1-\psi^2\right)^{3/2} \\
& + \frac{\lambda}{2}\left(\frac{\dot\psi}{2}\left(2+\dot\psi\right) - \frac{3}{2}\frac{\psi^2}{1-\psi^2} \right)^2 \left(1-\psi^2\right)^{3/2} \\
& + \frac{3}{4}\mu \frac{\psi^4}{\left(1-\psi^2\right)^2}\left(1-\psi^2\right)^{3/2} = W
\end{aligned}
\tag{21}
$$

where W is a constant.

One More Analogy

Considering how cumbersome equation (21) is, a different approach, still remaining within the elastic framework, may be envisaged [10]. It is a simple suggestive analogy. Let us start from equation (9) in intrinsic coordinates; among the admissible values of the parameters there is also $\lambda = -\mu$. Suppose this is the case for space-time. Then the relation between stress and strain then becomes

$$
\varepsilon_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\varepsilon = \frac{1}{2\mu}\sigma_{\mu\nu}.
\tag{22}
$$

Equation (22) is formally identical to the Einstein equations for the gravitational field, so Madsen's [10] suggestion is to directly identify $\varepsilon_{\mu\nu}$ with $R_{\mu\nu}$. Going on along this play of correspondences, we then use Hooke's law and consider the elastic potential energy density

$$
w = \frac{1}{2}C_{\mu\nu\alpha\beta}\varepsilon^{\mu\nu}\varepsilon^{\alpha\beta} = \frac{\lambda}{2}\varepsilon^2 + \mu\varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta}.
\tag{23}
$$

On the model of (23) we build the space-time potential

$$
\Phi = \frac{1}{2}R^2 - \kappa R_{\mu\nu}R^{\mu\nu}
$$

and use it to write the action integral

$$
\int \left(\frac{1}{2}R^2 - \kappa R_{\mu\nu}R^{\mu\nu} \right)\sqrt{-g}\,\mathrm{d}^4x.
$$

Forgetting for a moment the slippery way followed to obtain it, the final result is a second order theory that could be classified as a special case of an $f(R)$ theory.

6. Generalizing the Cosmic Defect Theory

The version of the CD theory I have outlined in Section 4 is characterized by the presence in the action integral of the exponential factor $\exp\left(-g_{\mu\nu}\gamma^{\mu}\gamma^{\nu}\right)$ which gives rise to an extremely steep expansion in the very early era of the universe. On one side that behaviour is even too fast, on the other in the Lagrangian no explicit evidence of the dynamics of γ appears and, in order to find the functional form of the four-vector, the incompressibility condition $\nabla_{\alpha}\gamma^{\alpha} = 0$ has been introduced.

A possible generalization of (16), which allows to partially release the initial rather strict conditions, is the following action

$$S = \int \mathrm{e}^{-\gamma_{\mu}\gamma^{\nu}} R_{\nu}^{\mu}\sqrt{-g}\mathrm{d}^4x. \tag{24}$$

The exponential term has now to be interpreted in the operatorial sense, and indeed it is the starting point for a series development

$$\mathrm{e}^{-\gamma_{\mu}\gamma^{\nu}} R_{\nu}^{\mu} = R - \gamma^{\alpha}\gamma^{\beta} R_{\alpha\beta} + \cdots . \tag{25}$$

When no defect is present we have $\gamma \equiv 0$ and the usual Hilbert-Einstein action. A defect implies other terms to come into the arena. We should also remember that, due to the properties of the Riemann tensor and to the fact that γ is a four-vector,

$$\gamma^{\beta} R_{\alpha\beta} = \left(\nabla_{\alpha}\nabla_{\nu} - \nabla_{\nu}\nabla_{\alpha}\right)\gamma^{\nu}. \tag{26}$$

Stopping the development in (25) to the first non trivial term and taking into account (26) we have the effective Lagrangian

$$L = \left[R + \nabla_{\nu}\gamma^{\alpha}\nabla_{\alpha}\gamma^{\nu} - \left(\nabla_{\beta}\gamma^{\beta}\right)^2\right]\sqrt{-g}.$$

Introducing the RW symmetry and again using integration by parts in the action integral, we have the corresponding so called point Lagrangian

$$L = 6a\dot{a}^2\left(1 + \chi^2\right) + 6\chi\dot{\chi}a^2\dot{a}. \tag{27}$$

From (27) we get the equations

$$\dot{a}^2 = \frac{W}{a\left(1 + \chi^2\right)}, \qquad \chi a^2\ddot{a} = 0. \tag{28}$$

The solution of (28) is $\dot{a} = V = $ constant (W is also a constant) with $\chi = \sqrt{W/(V^2 a)} - 1$. So this model describes a uniformly expanding space-time. The structure contained in the CD theory (see Figure 4) has disappeared, after adopting (24) and the limited development (25).

As a matter of fact (24) does not contain (16) as a special case. If we wish a real generalization of CD we can use a Lagrangian density like

$$
\mathrm{e}^{-\left(\delta_\alpha^\nu \delta_\mu^\beta + \delta_\alpha^\beta \delta_\mu^\nu\right)\gamma^\alpha \gamma_\beta} R_\nu^\mu \sqrt{-g} \tag{29}
$$

which includes the one in (16).

Instead of (25) we now have

$$
\mathrm{e}^{-\left(\delta_\alpha^\nu \delta_\mu^\beta + \delta_\alpha^\beta \delta_\mu^\nu\right)\gamma^\alpha \gamma_\beta} R_\nu^\mu = R\left(1 - \gamma_\mu \gamma^\mu\right) - \gamma^\alpha \gamma^\beta R_{\alpha\beta} + \cdots .
$$

The consequent effective point Lagrangian density (RW symmetry) is

$$
L = 6a\dot{a}^2 - 6\chi\dot{\chi}a^2\dot{a}
$$

however the situation does not really change in the sense that one obtains an expansion $\propto \tau^{2/3}$, typical of a Friedman-Robertson-Walker matter dominated universe, independent from χ.

Evidently the properties of the CD model are contained in the higher order terms of (29).

Of course there are many ways in which one can further generalize the ansatz (29). One can for instance introduce a "potential" term $\gamma^2 = \gamma_\alpha \gamma^\alpha$ in the Lagrangian, considering that, notwithstanding its geometric interpretation, γ is anyway a vector field and its energy content must directly influence curvature. One could also parametrize the Lagrangian

$$
S = \int \left[R\left(1 + \sigma\gamma^2\right) + \lambda\nabla_\alpha\gamma^\beta \nabla_\beta\gamma^\alpha + \mu\left(\nabla_\mu\gamma^\mu\right)^2 + \nu\nabla_\alpha\gamma^\beta \nabla^\alpha\gamma_\beta \right] \sqrt{-g}\, \mathrm{d}^4 x.
$$

However, following the thread of conjectures one looses more and more the contact with the, though fragile, initial physical motivation.

7. Conclusion

We have reviewed here an approach to the description of space-time based on the elastic continuum analogy, integrated with the possible presence of defects, in the

sense of Volterra's description [24]. Once this scheme is adopted we have seen that the approach is not unique. Among other possibility I have privileged the so named Cosmic Defect theory, which proved, simultaneously, to be manageable and to give good results when trying to describe the accelerated expansion of the universe [21].

Actually regarding space-time at the cosmological level a real "forest" of theories exists, mostly based either on the concept of dark energy (from cosmological constant [13] to phantom energy [5]), or on modifications or extensions of General Relativity (from MOND [11] to $f(R)$ theories [18]) and neglecting quantum theories (strings [9] or loop quantum gravity [17]). Most often these numerous approaches belong to what I would call "Lagrangian engineering", i.e., let us somehow change the Lagrangian and see what happens. These attempts can be more or less fortunate and more or less *ad hoc*, but generally rely on rather staggering physical bases, looking for *ex post* justification. It is also often possible to see that apparently different theories and approaches are indeed related to each other and lead, totally or partly, to convergent or coincident results. For example the whole elastic analogy approach is formally a (group of) vector-tensor theory, being based on the displacement vector field and related strain tensor field. We verified that the CD theory also is reducible to a special case of general vector theories [22] and Madsen's conjecture described in Section 5 leads to a sort of second degree $f(R)$ Lagrangian. Furthermore conformal transformations can convert modified or extended gravity theories into GR plus some more or less exotic dark energy fluid (CD is again an example).

In this very jungle I think it is better to have a compass pointing in some direction, rather than moving around blindly in pursuit of a local and ephemeral success. In other words it is better to start from some physical paradigm that suggests where to go and what to look for. This is why, thanks also to the initial positive results, I think the elastic continuum model and the CD theory are a good conceptual framework that deserves further exploration and deepening.

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