JGSP **16** (2009) 1–21

JOURNAL OF
Geometry and Symmetry
in Physics

EXACT NONLINEAR EQUATIONS FOR FLUID FILMS AND PROPER ADAPTATIONS OF CONSERVATION THEOREMS FROM CLASSICAL HYDRODYNAMICS

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Communicated by Gregory Naber

Abstract. We discuss the exact nonlinear equations for the dynamics of fluid films, modeled as a two dimensional manifold. Our main goal is to illustrate the differences and similarities between the fluid film equations and Euler's equations, their classical three dimensional counterpart. Since the geometry of fluid films is fundamentally different – three dimensional velocity field on a two dimensional support with a time varying Riemannian metric – all classical theorems must be properly modified. We offer adaptations of the following theorems: conservation of mass and energy, pointwise conservation of vorticity and Kelvin's circulation theorem. We present proofs of these theorems by employing the calculus of moving surfaces. It is of great interest to develop a simplified model that captures normal deformations of fluid films by assuming that tangential velocities vanish while preserving the exact nonlinear nature of the full system. This cannot be accomplished simply by neglecting the tangential components, for such an attempt leads to internal contradictions. Instead, we modify the initial formulation and present a modified variational approach that leads to a simplified system of equations capable of capturing a broad range of deeply nonlinear effects.

1. Introduction

Fluid dynamics is one of the most developed subjects in classical physics [1], [2] and still one of the most active today. Fluid films have always occupied an important place in hydrodynamics [3], [4] and have recently been receiving a great deal of renewed attention [5–11]. In this paper, we discuss the exact nonlinear equations for fluid films under the influence of generalized surface tension. Laplace's classical model of surface tension figures is a special case.

Historically, the governing equations of fluid dynamics were formulated by analogy with Newton's laws of motion. The force F is postulated as a function of geometry or kinetics. An attempt to do the same for fluid films would be certainly met with nontrivial difficulties, for the concept of force, especially as a primary notion, is elusive [12]. This is often the case when concepts from one branch of physics need to be carried over to another. We therefore pursue an approach based on the Least Action Principle. Significant steps have been taken towards formulating Euler's classical three dimensional equations from a variational point of view. A derivation is given in [13], from where we borrow certain aspects of our analytical methods and combine them with the calculus of moving surfaces, so indispensable when working with deforming interfaces.

The dynamics of fluid films exhibits an intriguing interplay between geometry and physics. Certain elements in the exact equations of motion are purely geometric. For example, mass conservation and the expression for acceleration can be derived strictly from kinematics. Other elements, such as surface tension and van der Waals forces, are less universal and require physical modeling.

We treat the fluid film as a two dimensional manifold. Variation in thickness (which, even in simple experiments, ranges from nanometers to millimeters) is captured by the concept of two dimensional density. The contour boundary of the fluid film is stationary, and the interaction with the ambient air is ignored. The system is derived from the Least Action Principle with the Lagrangian

$$L = \frac{1}{2}\int_S \rho\left(C^2 + |\mathbf{V}|^2\right) \mathrm{d}S - \int_S \rho e\left(\rho\right) \mathrm{d}S \tag{1}$$

where ρ is the two dimensional density of the fluid film, C is the normal velocity, $\mathbf{V}$ is the tangential velocity and $e\left(\rho\right)$ is the internal energy density per unit mass. The choice

$$e\left(\rho\right) = \frac{\sigma}{\rho} \tag{2}$$

results in the classical Laplace model for surface tension.

Many essential features of the newly proposed system are analogous to classical hydrodynamics. Others are fundamentally different. Our main motivation is to identify those features that survive for two dimensional films and those that are completely new. We aim to accomplish two specific goals. First is to prove four essential properties of the proposed exact nonlinear dynamic equations: pointwise conservation of mass, conservation of energy, pointwise conservation of two-dimensional vorticity, and a proper generalization of Kelvin's circulation theorem that states that circulation around a closed material loop is conserved. Our other goal is to derive simplified dynamic equations that incorporate the natural assumption that material particles move in the orthogonal direction to the surface of the film. This mode of motion does not have an analogue in classical fluid dynamics.

The assumption that material particles move normally to the surface of the film appears frequently in literature. The existing infinitesimal linear models [5], [6] are based, in part, on this assumption. Importantly, the proposed *full* system of equations (50) is free of this assumption. As a matter of fact, the full system is *inconsistent* with it. As we discuss below, the proposed exact equations (50) show that finite C will give rise to finite $\mathbf{V}$. It is therefore clear that in order to formulate equations that incorporate the assumption of vanishing $\mathbf{V}$, one must modify the original variational formulation.

We begin by introducing the essential elements from differential geometry and tensor calculus required for the interpretation of the proposed system. We then present the governing equations and discuss their structure. Subsequently, we demonstrate mass conservation, energy conservation, vorticity conservation, and Kelvin's circulation theorem. Finally, we incorporate the nonholonomic $\mathbf{V} = \mathbf{0}$ constraint by means of functional Lagrange multipliers and present the resulting dynamic equations.

2. Differential Geometry Preliminaries

The relationships summarized in this section can be found in standard texts on tensor calculus [14–16].

Suppose that S^α $(\alpha = 1, 2)$ are the surface coordinates on the moving manifold S. S^α can be chosen rather arbitrarily as long as sufficient differentiability is achieved in both space and time. Various choices of coordinates offer certain advantages depending on the problem. For example, Lagrangian coordinates allow tracking material particles, while *normal* coordinates allow the formulation of equations without the use of the $\delta/\delta t$-derivative, provided the ambient Euclidean space is referred to affine coordinates.

Suppose that the ambient Euclidean space is referred to coordinates Z^i. Let $\mathbf{Z}$ be the position vector expressed in these coordinates

$$\mathbf{Z} = \mathbf{Z}(Z). \tag{3}$$

(Note our convention to drop the indices of function arguments. This leads to a slight overloading of the letters Z and S, since those same letters are used to denote volume and surface element in equations (8) and (17). Fortunately, it is always clear from the context in what sense the letter Z is being used.)

Introduce the covariant basis $\mathbf{Z}_i$

$$\mathbf{Z}_i = \frac{\partial \mathbf{Z}(Z)}{\partial Z^i} \tag{4}$$

and the covariant metric tensor Z_{ij}

$$Z_{ij} = \mathbf{Z}_i \cdot \mathbf{Z}_j. \tag{5}$$

The contravariant metric tensor Z^{ij} is defined as the matrix inverse of Z_{ij}

$$Z^{ij} Z_{jk} = \delta^i_k. \tag{6}$$

The metric tensors Z_{ij} and Z^{ij} are used to change the flavors of tensors. It is an operation also referred to as *juggling indices*. In particular, the contravariant basis $\mathbf{Z}^i$ is obtained from the covariant basis $\mathbf{Z}_j$ by *raising* the subscript

$$\mathbf{Z}^i = Z^{ij} \mathbf{Z}_j. \tag{7}$$

Let Z be the volume element defined as the determinant of the covariant metric tensor Z_{ij}. In order to avoid a false appearance of free indices, we denote this relationship informally by

$$Z = |Z_{..}|\,. \tag{8}$$

The Levi-Civita tensors $\varepsilon^{ijk}$and ε_{ijk} are defined as

$$\varepsilon^{ijk} = \frac{e^{ijk}}{\sqrt{Z}}, \qquad \varepsilon_{ijk} = \sqrt{Z} e_{ijk} \tag{9}$$

where e^{ijk} and e_{ijk} equal 1 when ijk is an even permutation, -1 when it is odd, and zero otherwise.

The Christoffel symbols Γ^i_{jk}, given by

$$\Gamma^i_{jk} = \mathbf{Z}^i \cdot \frac{\partial \mathbf{Z}_j}{\partial Z^k} \tag{10}$$

are used in the definition of the covariant derivative ∇_k of a tensor T^i_j with a representative collection of indices

$$\nabla_k T^i_j = \frac{\partial T^i_j}{\partial Z^k} + \Gamma^i_{kn} T^n_j - \Gamma^n_{kj} T^i_n. \tag{11}$$

We now turn to tensors on the embedded manifold S. Suppose that $\boldsymbol{S}$ is the position vector for the points on the manifold. Suppose $\boldsymbol{S}$ is expressed as a function of time t and surface coordinates S^α

$$\boldsymbol{S} = \boldsymbol{S}(t, S)\,. \tag{12}$$

We continue to follow the convention of suppressing tensor indices of function arguments.

Introduce the covariant basis $\boldsymbol{S}_\alpha$

$$\boldsymbol{S}_\alpha = \frac{\partial \boldsymbol{S}(t,S)}{\partial S^\alpha} \tag{13}$$

and the covariant metric tensor $S_{\alpha\beta}$

$$S_{\alpha\beta} = \boldsymbol{S}_\alpha \cdot \boldsymbol{S}_\beta. \tag{14}$$

The contravariant metric tensor $S^{\alpha\beta}$ is defined as the matrix inverse of $S_{\alpha\beta}$

$$S^{\alpha\beta} S_{\beta\gamma} = \delta^\alpha_\gamma. \tag{15}$$

The metric tensors $S_{\alpha\beta}$ and $S^{\alpha\beta}$ are used to change the flavors of surface indices. In particular, the contravariant basis $\boldsymbol{S}^\alpha$ is obtained from the covariant basis by raising the subscript

$$\boldsymbol{S}^\alpha = S^{\alpha\beta} \boldsymbol{S}_\beta. \tag{16}$$

Let S be the area element defined as the determinant of the covariant metric tensor $S_{\alpha\beta}$

$$S = |S_{\cdot\cdot}|. \tag{17}$$

The Levi-Civita tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ on the surface are defined as

$$\varepsilon^{\alpha\beta} = \frac{e^{\alpha\beta}}{\sqrt{S}}, \qquad \varepsilon_{\alpha\beta} = \sqrt{S} e_{\alpha\beta} \tag{18}$$

where $e^{\alpha\beta}$ and $e_{\alpha\beta}$ equal 1 if $\alpha\beta$ is an even permutation, -1 if it is odd, and zero otherwise.

The surface Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, given by

$$\Gamma^\alpha_{\beta\gamma} = \boldsymbol{S}^\alpha \cdot \frac{\partial \boldsymbol{S}_\beta}{\partial S^\gamma} \tag{19}$$

along with the space Christoffels Γ^i_{jk}, are used in the definition of the covariant derivative ∇_γ of a tensor $T^{i\alpha}_{j\beta}$ with a representative collection of indices

$$\nabla_\gamma T^{i\alpha}_{j\beta} = \frac{\partial T^{i\alpha}_{j\beta}}{\partial S^\gamma} + Z^k_\gamma \Gamma^i_{kn} T^{n\alpha}_{j\beta} - Z^k_\gamma \Gamma^n_{kj} T^{i\alpha}_{n\beta} + \Gamma^\alpha_{\gamma\eta} T^{i\eta}_{j\beta} - \Gamma^\eta_{\gamma\beta} T^{i\alpha}_{j\eta}. \tag{20}$$

Suppose that the evolution of the embedded surface is given by the parametric equations

$$Z^i = Z^i(t,S). \tag{21}$$

Then the shift tensor Z_α^i is defined as

$$Z_\alpha^i = \frac{\partial Z^i(t,S)}{\partial S^\alpha} \tag{22}$$

and it easy to show that the space basis and the surface basis are related by

$$\boldsymbol{S}_\alpha = \mathbf{Z}_i Z_\alpha^i. \tag{23}$$

Consequently the metric tensors are related as well by

$$S_{\alpha\beta} = Z_{ij} Z_\alpha^i Z_\beta^j \tag{24}$$

which can also be written as

$$S_{\alpha\beta} = Z_\alpha^i Z_{i\beta} \tag{25}$$

and even more concisely as

$$\delta_\alpha^\beta = Z_\alpha^i Z_i^\beta. \tag{26}$$

The components N^k of the surface normal $\mathbf{N}$ are given by

$$N^i = \frac{1}{2}\varepsilon^{ijk}\varepsilon_{\alpha\beta} Z_i^\alpha Z_j^\beta \tag{27}$$

and satisfy the following relationships

$$N_i N^i = 1, \qquad N^i Z_i^\alpha = 0, \qquad \delta_j^i - N^i N_j = Z_\alpha^i Z_j^\alpha. \tag{28}$$

The curvature tensor $B_{\alpha\beta}$ arises most naturally in the following way. It follows directly from the definition of the Christoffel symbols that

$$\boldsymbol{S}^\gamma \cdot \nabla_\alpha \boldsymbol{S}_\beta = 0. \tag{29}$$

Therefore, $\nabla_\alpha \boldsymbol{S}_\beta$ must point along the normal direction $\mathbf{N}$. Then let $B_{\alpha\beta}$ be the coefficients in the relationship

$$\nabla_\beta \boldsymbol{S}_\alpha = \mathbf{N} B_{\alpha\beta}. \tag{30}$$

Since $\nabla_\beta \boldsymbol{S}_\alpha$ can be expressed as the double derivative $\nabla_\beta \nabla_\alpha \boldsymbol{S}$, the curvature tensor $B_{\alpha\beta}$ is symmetric

$$B_{\alpha\beta} = B_{\beta\alpha}. \tag{31}$$

The trace B_α^α of the curvature tensor with one covariant and one contravariant index is called *mean curvature*. Its determinant $|B_\cdot^\cdot|$ is *Gaussian*, *intrinsic* or *total curvature* denoted by K. It can be computed on a surface by measuring distances and angles within the surface. It is a fact famously captured by Gauss's Theorema Egregium

$$B_{\alpha\nu} B_{\beta\mu} - B_{\beta\nu} B_{\alpha\mu} = 4K \varepsilon_{\alpha\beta}\varepsilon_{\nu\mu}. \tag{32}$$

3. Moving Surfaces and the $\delta/\delta t$-derivative

The original definition of the $\delta/\delta t$-derivative was given by Hadamard [17]. Many of the details pertaining to applications of the $\delta/\delta t$-derivative can be found in [18] and [19].

A moving surface is a one parameter family of submanifolds indexed by time t. We assume that each submanifold is sufficiently differentiable. Furthermore, given a family of parameterizations,

$$Z^i = Z^i\left(t, S\right) \tag{33}$$

we assume that $Z^i\left(t, S\right)$ is sufficiently differentiable with respect to t.

Define a quantity v^i according to

$$v^i = \frac{\partial Z^i\left(t, S\right)}{\partial t} \tag{34}$$

and its projection v^α onto the surface

$$v^\alpha = v^i Z_i^\alpha. \tag{35}$$

Then the $\delta/\delta t$-derivative for a tensor $T_{j\beta}^{i\alpha}$ with a typical collection of indices is defined by

$$\frac{\delta T_{j\beta}^{i\alpha}}{\delta t} = \frac{\partial T_{j\beta}^{i\alpha}}{\partial t} - v^\eta \nabla_\eta T_{j\beta}^{i\alpha} + v^m \Gamma_{mk}^i T_{j\beta}^{k\alpha} - v^m \Gamma_{mj}^k T_{k\beta}^{i\alpha} + \nabla_\eta v^\alpha T_{j\beta}^{i\eta} - \nabla_\beta v^\eta T_{j\eta}^{i\alpha}. \tag{36}$$

The $\delta/\delta t$-derivative commutes with contraction, satisfies the product rule for any collection of indices

$$\frac{\delta}{\delta t}\left(S_{::}^{::} T_{::}^{::}\right) = \frac{\delta S_{::}^{::}}{\delta t} T_{::}^{::} + S_{::}^{::} \frac{\delta T_{::}^{::}}{\delta t} \tag{37}$$

and obeys a chain rule for surface restrictions of spatial tensors

$$\frac{\delta F\left(t, S\right)}{\delta t} = \frac{\partial F\left(t, Z\right)}{\partial t} + C N^i \nabla_i F\left(t, Z\right). \tag{38}$$

In (38), $F(t, S)$ indicates the restriction of F onto the manifold S expressed with respect to surface coordinates S^α, and $F\left(t, Z\right)$ is the full tensor field F expressed with respect to space coordinates Z^i. Chain rule shows that the $\delta/\delta t$-derivative of spatial "metrics" vanishes

$$\frac{\delta Z_{ij}}{\delta t} = \frac{\delta Z^{ij}}{\delta t} = \frac{\delta Z_{ijk}}{\delta t} = \frac{\delta Z^{ijk}}{\delta t} = 0 \tag{39}$$

and it is also true that

$$\frac{\delta\delta^{\alpha}_{\beta}}{\delta t} = 0. \tag{40}$$

The $\delta/\delta t$-derivative of the key surface objects leads to highly concise and attractive formulas. When applied to the metric tensors, the curvature tensor appears

$$\begin{aligned} \frac{\delta S_{\alpha\beta}}{\delta t} &= -2CB_{\alpha\beta} \\ \frac{\delta S^{\alpha\beta}}{\delta t} &= 2CB^{\alpha\beta}. \end{aligned} \tag{41}$$

Importantly, when applied to the curvature tensor B^{α}_{β}, it reappears in the result along with the metric tensors embedded in the co- and contravariant surface derivatives

$$\frac{\delta B^{\alpha}_{\beta}}{\delta t} = \nabla_{\beta}\nabla^{\alpha}C + CB^{\alpha}_{\gamma}B^{\gamma}_{\beta}. \tag{42}$$

The shift tensor and the normal produce one another

$$\frac{\delta Z^{i}_{\alpha}}{\delta t} = \nabla_{\alpha}\left(CN^{i}\right) \tag{43}$$

$$\frac{\delta N^{i}}{\delta t} = -Z^{i}_{\alpha}\nabla^{\alpha}C \tag{44}$$

Finally, for the Levi-Civita tensors, we have

$$\begin{aligned} \frac{\delta\varepsilon_{\alpha\beta}}{\delta t} &= -\varepsilon_{\alpha\beta}CB^{\gamma}_{\gamma} \\ \frac{\delta\varepsilon^{\alpha\beta}}{\delta t} &= \varepsilon^{\alpha\beta}CB^{\gamma}_{\gamma}. \end{aligned} \tag{45}$$

4. Time Differentiation of Surface Integrals

Central to the analysis of conserved quantities is the formula that governs time differentiation of surface integrals. Suppose that a scalar field F is defined on the evolving manifold

$$F \equiv F\left(t, S\right). \tag{46}$$

Then the evolution of the integral of F over a closed manifold satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S} F\mathrm{d}S = \int_{S}\frac{\delta F}{\delta t}\mathrm{d}S - \int_{S} CB^{\alpha}_{\alpha}F\mathrm{d}S. \tag{47}$$

The intuitive interpretation of this formula is this: the first term captures the rate of change in the tensor field F while the second captures the rate of change in the area.

If the manifold S is not closed and has a moving contour γ, then an additional term is needed to capture the change in the area due to γ. Suppose that c is the velocity of the contour γ with respect to S. Then the proper generalization of (47) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S F\mathrm{d}S = \int_S \frac{\delta F}{\delta t}\mathrm{d}S - \int_S CB_\alpha^\alpha F\mathrm{d}S + \int_\gamma cF\mathrm{d}\gamma. \tag{48}$$

5. Dynamic Equations of Motion

5.1. Full Equations of Motion

We model the fluid film as a two dimensional manifold with two dimensional density ρ. Let C be the normal component of the velocity field and V^α $(\alpha = 1, 2)$ be the tangential components. Suppose that the potential energy density per unit mass e is given as a function of ρ and that the total potential energy is given by the integral

$$V = \int_S \rho e(\rho)\,\mathrm{d}S. \tag{49}$$

The dynamic equations of motion, derived from the Least Action Principle, read $(e_\rho = e'(\rho))$

$$\begin{aligned}
\frac{\delta\rho}{\delta t} + \nabla_\alpha(\rho V^\alpha) &= \rho C B_\alpha^\alpha \\
\rho\left(\frac{\delta C}{\delta t} + 2V^\alpha\nabla_\alpha C + B_{\alpha\beta}V^\alpha V^\beta\right) &= -\rho^2 e_\rho B_\alpha^\alpha \\
\rho\left(\frac{\delta V^\alpha}{\delta t} + V^\beta\nabla_\beta V^\alpha - C\nabla^\alpha C - 2CV^\beta B_\beta^\alpha\right) &= -\nabla^\alpha\left(\rho^2 e_\rho\right).
\end{aligned} \tag{50}$$

The structure of these equations is similar to that of the classical hydrodynamic equations. The first equation in (50) is mass conservation. The remaining equations – the scalar one and the vector one – represent evolution of momentum. However, there are fundamental differences as well. The fluid film equations have a two dimensional deforming support, while the classical equations of fluid dynamics are typically solved in a Euclidean space with a stationary metric.

5.2. Conservation of Mass

Suppose that $P(t)$ is a material patch on the manifold S. In other words, $P(t)$ is comprised of the same set of material particles at different times t. The total mass $M(t)$ contained within $P(t)$ is given by the surface integral with a moving contour

$$M(t) = \int_{P(t)} \rho \mathrm{d}S. \tag{51}$$

Its rate of change $\mathrm{d}M(t)/\mathrm{d}t$ is analyzed by equation (48)

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = \int_{P(t)} \frac{\delta \rho}{\delta t} \mathrm{d}S - \int_{P(t)} C \rho B_\alpha^\alpha \mathrm{d}S + \int_{\gamma(t)} c \rho \mathrm{d}\gamma \tag{52}$$

where $\gamma(t)$ is the contour boundary of $P(t)$. Then, according to the first equation in (50) the first integral can be rewritten as

$$\int_{P(t)} \frac{\delta \rho}{\delta t} \mathrm{d}S = \int_{P(t)} \rho C B_\alpha^\alpha \mathrm{d}S - \int_{P(t)} \nabla_\alpha (\rho V^\alpha) \mathrm{d}S. \tag{53}$$

Note that the integrals containing $\rho C B_\alpha^\alpha$ cancel, leaving

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = -\int_{P(t)} \nabla_\alpha (\rho V^\alpha) \mathrm{d}S + \int_{\gamma(t)} c \rho \mathrm{d}\gamma. \tag{54}$$

By Gauss's theorem, the surface integral is converted into the contour integral

$$-\int_{P(t)} \nabla_\alpha (\rho V^\alpha) \mathrm{d}S = -\int_{\gamma(t)} n_\alpha \rho V^\alpha \mathrm{d}\gamma \tag{55}$$

where n_α is the unit normal to contour $\gamma(t)$ that lies within the ambient manifold S. Finally, observe that the velocity c of the contour $\gamma(t)$ with respect to the ambient manifold S equals the normal component of the velocity field V^α

$$c = n_\alpha V^\alpha. \tag{56}$$

As a result, the two remaining integrals cancel and we have

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = 0 \tag{57}$$

for any material patch $P(t)$, Q.E.D.

5.3. Conservation of Energy

Let q be the absolute value of the total velocity field

$$q^2 = C^2 + V_\alpha V^\alpha. \tag{58}$$

The total energy E is given by the expression

$$E = \int_S \rho \left(\frac{1}{2} q^2 + e \right) \mathrm{d}S. \tag{59}$$

Its evolution $\mathrm{d}E/\mathrm{d}t$ is obtained by differentiating the integral according to equation (47)

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_S \rho \left(\frac{1}{2} q^2 + e \right) \mathrm{d}S \\ &= \int_S \left(\frac{\delta \rho}{\delta t} - \rho C B_\alpha^\alpha \right) \left(\frac{1}{2} q^2 + e \right) + \rho \left(q \frac{\delta q}{\delta t} + e_\rho \frac{\delta \rho}{\delta t} \right) \mathrm{d}S. \end{aligned} \tag{60}$$

The quantity $\delta \rho / \delta t$ is available from the first dynamic equation (50). We therefore focus on $q \frac{\delta q}{\delta t}$ which can be obtained from the rest of the dynamic equations. Apply $\delta / \delta t$ to equation (58). By the product rule, we have

$$q \frac{\delta q}{\delta t} = C \frac{\delta C}{\delta t} + \frac{1}{2} \left(\frac{\delta V^\alpha}{\delta t} V_\alpha + V^\alpha \frac{\delta V_\alpha}{\delta t} \right). \tag{61}$$

Therefore, we must calculate $\frac{\delta C}{\delta t}$, $\frac{\delta V^\alpha}{\delta t}$ and $\frac{\delta V_\alpha}{\delta t}$. The second equation in (50) gives $\frac{\delta C}{\delta t}$

$$\frac{\delta C}{\delta t} = -\rho e_\rho B_\alpha^\alpha - 2 V^\alpha \nabla_\alpha C - B_{\alpha\beta} V^\alpha V^\beta \tag{62}$$

while the third equation in (50) gives us $\frac{\delta V^\alpha}{\delta t}$

$$\frac{\delta V^\alpha}{\delta t} = -\frac{1}{\rho} \nabla^\alpha \left(\rho^2 e_\rho \right) - V^\beta \nabla_\beta V^\alpha + C \nabla^\alpha C + 2 C V^\beta B_\beta^\alpha \tag{63}$$

from where it follows (by lowering α) that

$$\frac{\delta V_\alpha}{\delta t} = -\frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho \right) - V^\beta \nabla_\beta V_\alpha + C \nabla_\alpha C. \tag{64}$$

Combining equations (62)-(63) gives $\rho \delta q / \delta t$

$$q \frac{\delta q}{\delta t} = -\frac{1}{\rho} V_\alpha \nabla^\alpha \left(\rho^2 e_\rho \right) - \rho e_\rho C B_\alpha^\alpha - \frac{1}{2} V^\beta \nabla_\beta q^2. \tag{65}$$

Substituting this expression in equation (60) we end with

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_S \left(\nabla_\alpha \left(\frac{1}{2} q^2 \rho V^\alpha\right) + \nabla_\alpha \left(\rho V^\alpha\right) e + V_\alpha \nabla^\alpha \left(\rho^2 e_\rho\right) + \rho e_\rho \nabla_\alpha \left(\rho V^\alpha\right)\right) \mathrm{d}S \tag{66}$$

The rest of the analysis proceeds by a repeated application of Gauss's theorem. The first term in the integrand yields

$$\int_S \nabla_\alpha \left(\frac{1}{2} q^2 \rho V^\alpha\right) \mathrm{d}S = \int_\gamma \frac{1}{2} q^2 \rho V^\alpha n_\alpha \mathrm{d}\gamma \tag{67}$$

where γ is the stationary contour of the fluid film and n_α is the normal to the contour that lies in the tangent plane to S. Since V^α is orthogonal to the normal at the boundary ($V^\alpha n_\alpha = 0$), this term integrates to zero.

Also by Gauss's theorem, the second term can be converted to

$$\begin{aligned} \int_S \nabla_\alpha \left(\rho V^\alpha\right) e\left(\rho\right) \mathrm{d}S &= -\int_S \rho V^\alpha \nabla_\alpha e\left(\rho\right) \mathrm{d}S \\ &= -\int_S \rho V^\alpha e_\rho \nabla_\alpha \rho \mathrm{d}S. \end{aligned} \tag{68}$$

Elementary calculus shows that the remaining terms also sum up to a single divergence expression

$$-\rho V^\alpha e_\rho \nabla_\alpha \rho + V_\alpha \nabla^\alpha \left(\rho^2 e_\rho\right) + \rho e_\rho \nabla_\alpha \left(\rho V^\alpha\right) = \nabla_\alpha \left(\rho^2 e_\rho V^\alpha\right) \tag{69}$$

whose integral vanishes due to Gauss's theorem. We have therefore shown that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 \tag{70}$$

Q.E.D.

5.4. Conservation of Vorticity

Define two dimensional vorticity ω as

$$\omega = \varepsilon^{\alpha\beta} \nabla_\alpha V_\beta. \tag{71}$$

We next show that the quantity ω/ρ is pointwise conserved. To track a specific material point, we evaluate the material derivative D/Dt

$$\frac{D}{Dt} = \frac{\delta}{\delta t} + V^\gamma \nabla_\gamma. \tag{72}$$

Our goal is to prove that

$$\frac{D\left(\omega/\rho\right)}{Dt}=0. \tag{73}$$

We have the following chain of identities

$$\begin{aligned}
\rho\frac{D\left(\omega/\rho\right)}{Dt} &= \rho\frac{\delta\left(\omega/\rho\right)}{\delta t}+\rho V^{\gamma}\nabla_{\gamma}\frac{\omega}{\rho}\\
&= \frac{\delta\omega}{\delta t}-\frac{\omega}{\rho}\frac{\delta\rho}{\delta t}+\rho V^{\gamma}\nabla_{\gamma}\frac{\omega}{\rho}\\
&= \frac{\delta\omega}{\delta t}-\omega CB_{\alpha}^{\alpha}+\frac{\omega}{\rho}\nabla_{\alpha}\left(\rho V^{\alpha}\right)+\rho V^{\gamma}\nabla_{\gamma}\frac{\omega}{\rho}\\
&= \frac{\delta\omega}{\delta t}-\omega CB_{\alpha}^{\alpha}+\nabla_{\alpha}\left(\omega V^{\alpha}\right).
\end{aligned} \tag{74}$$

Continue with $\delta\omega/\delta t$

$$\begin{aligned}
\frac{\delta\omega}{\delta t} &= \frac{\delta\left(\varepsilon^{\alpha\beta}\nabla_{\alpha}V_{\beta}\right)}{\delta t}\\
&= \frac{\delta\varepsilon^{\alpha\beta}}{\delta t}\nabla_{\alpha}V_{\beta}+\varepsilon^{\alpha\beta}\frac{\delta\nabla_{\alpha}V_{\beta}}{\delta t}\\
&= \varepsilon^{\alpha\beta}CB_{\gamma}^{\gamma}\nabla_{\alpha}V_{\beta}+\varepsilon^{\alpha\beta}\nabla_{\alpha}\frac{\delta V_{\beta}}{\delta t}\\
&= \omega CB_{\gamma}^{\gamma}+\varepsilon^{\alpha\beta}\nabla_{\alpha}\frac{\delta V_{\beta}}{\delta t}.
\end{aligned} \tag{75}$$

Importantly, in the above chain, we commuted the operators $\delta/\delta t$ and ∇_{α} as applied to V_{β}. This operation is not valid in general. However, since

$$\frac{\delta}{\delta t}\nabla_{\alpha}V_{\beta}-\nabla_{\alpha}\frac{\delta V_{\beta}}{\delta t}=\left(\nabla_{\beta}CB_{\alpha}^{\eta}+\nabla_{\alpha}CB_{\beta}^{\eta}+C\nabla_{\alpha}B_{\beta}^{\eta}-B_{\alpha\beta}\nabla^{\eta}C\right)V_{\eta} \tag{76}$$

we observe that the commutator on the left is symmetric with respect to α and β due to Codazzi equations [14].

Substitute equation (75) in the original chain of identities (74)

$$\rho\frac{D\left(\omega/\rho\right)}{Dt}=\varepsilon^{\alpha\beta}\nabla_{\alpha}\frac{\delta V_{\beta}}{\delta t}+\nabla_{\alpha}\left(\omega V^{\alpha}\right). \tag{77}$$

At this point, the dynamic equations of motion enter the analysis. Rename indices $\alpha\rightarrow\beta$ and $\beta\rightarrow\gamma$ in equation (64)

$$\frac{\delta V_{\beta}}{\delta t}=-\frac{1}{\rho}\nabla_{\beta}\left(\rho^{2}e_{\rho}\right)-V^{\gamma}\nabla_{\gamma}V_{\beta}+C\nabla_{\beta}C. \tag{78}$$

Then for $\varepsilon^{\alpha\beta}\nabla_\alpha\left(\delta V_\beta/\delta t\right)$ we have

$$\begin{aligned}
\varepsilon^{\alpha\beta}\nabla_\alpha\frac{\delta V_\beta}{\delta t} &= \varepsilon^{\alpha\beta}\nabla_\alpha\left(-\frac{1}{\rho}\nabla_\beta\left(\rho^2 e_\rho\right)-V^\gamma\nabla_\gamma V_\beta+C\nabla_\beta C\right)\\
&= \varepsilon^{\alpha\beta}\left(\begin{array}{c}\nabla_\alpha\left(-\frac{1}{\rho}\nabla_\beta\left(\rho^2 e_\rho\right)\right)\\ -\nabla_\alpha V^\gamma\nabla_\gamma V_\beta-V^\gamma\nabla_\gamma\nabla_\alpha V_\beta\\ +\nabla_\alpha C\nabla_\beta C+C\nabla_\alpha\nabla_\beta C\end{array}\right).
\end{aligned} \tag{79}$$

Upon expanding the expression $\nabla_\alpha\left(-\rho^{-1}\nabla_\beta\left(\rho^2 e_\rho\right)\right)$, one notices that it is symmetric with respect to α and β. The same is easily seen for the last two terms, as well. We are, therefore, left with

$$\varepsilon^{\alpha\beta}\nabla_\alpha\frac{\delta V_\beta}{\delta t}=-\varepsilon^{\alpha\beta}\nabla_\alpha V^\gamma\nabla_\gamma V_\beta-\varepsilon^{\alpha\beta}V^\gamma\nabla_\gamma\nabla_\alpha V_\beta. \tag{80}$$

Combine all intermediate relationships

$$\begin{aligned}
\rho\frac{D\left(\omega/\rho\right)}{Dt} &= -\varepsilon^{\alpha\beta}\nabla_\alpha V^\gamma\nabla_\gamma V_\beta-\varepsilon^{\alpha\beta}V^\gamma\nabla_\gamma\nabla_\alpha V_\beta+\nabla_\gamma\left(\omega V^\gamma\right)\\
&= -\varepsilon^{\alpha\beta}\nabla_\alpha V^\gamma\nabla_\gamma V_\beta-\varepsilon^{\alpha\beta}V^\gamma\nabla_\gamma\nabla_\alpha V_\beta\\
&\quad+\nabla_\gamma\left(\varepsilon^{\alpha\beta}\nabla_\alpha V_\beta V^\gamma\right)\\
&= -\varepsilon^{\alpha\beta}\nabla_\alpha V^\gamma\nabla_\gamma V_\beta-\varepsilon^{\alpha\beta}V^\gamma\nabla_\gamma\nabla_\alpha V_\beta\\
&\quad+\varepsilon^{\alpha\beta}\nabla_\alpha\nabla_\gamma V_\beta V^\gamma+\varepsilon^{\alpha\beta}\nabla_\alpha V_\beta\nabla_\gamma V^\gamma\\
&= -\varepsilon^{\alpha\beta}\nabla_\alpha V^\gamma\nabla_\gamma V_\beta+\varepsilon^{\alpha\beta}\nabla_\alpha V_\beta\nabla_\gamma V^\gamma.
\end{aligned} \tag{81}$$

There are several ways to show that the resulting expression vanishes. The most straightforward way is to introduce a geodesic coordinate system [14] (which allows arbitrary juggling of indices), expand each term in the summations and notice that all terms cancel, Q.E.D.

6. Kelvin's Circulation Theorem

Finally, we turn to Kelvin's circulation theorem. Suppose that $\gamma\left(t\right)$ is a closed contour comprised of the same material particles over time. Suppose that γ is given parametrically by

$$S^\alpha\equiv S^\alpha\left(\theta\right) \tag{82}$$

where $\theta\in\left[\Theta_1,\Theta_2\right]$ is a material coordinate, thus no t dependence in equation (82), and

$$S^\alpha\left(\Theta_1\right)=S^\alpha\left(\Theta_2\right). \tag{83}$$

Define circulation Γ around the closed loop γ as

$$\Gamma = \int_\gamma \mathbf{V} \cdot \mathrm{d}\gamma \tag{84}$$

where $\mathrm{d}\gamma$ is an element of length pointing in the tangential direction.

Suppose that $\mathbf{R}(t, \theta)$ is the position vector that traces the material points on the contour γ. Then the velocity $\mathbf{V}$ is given by

$$\mathbf{V} = \frac{\partial \mathbf{R}(t, \theta)}{\partial t} \tag{85}$$

and the element of length $\mathrm{d}\gamma$ by

$$\mathrm{d}\gamma = \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} \mathrm{d}\theta. \tag{86}$$

The evolution $\mathrm{d}\Gamma/\mathrm{d}t$ is analyzed by the following chain of identities

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Theta_1}^{\Theta_2} \frac{\partial \mathbf{R}(t, \theta)}{\partial t} \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} \mathrm{d}\theta \\
&= \int_{\Theta_1}^{\Theta_2} \left(\frac{\partial^2 \mathbf{R}(t, \theta)}{\partial t^2} \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} + \frac{\partial \mathbf{R}(t, \theta)}{\partial t} \cdot \frac{\partial^2 \mathbf{R}(t, \theta)}{\partial \theta \partial t} \right) \mathrm{d}\theta \\
&= \int_{\Theta_1}^{\Theta_2} \left(\frac{\partial^2 \mathbf{R}(t, \theta)}{\partial t^2} \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} + \frac{1}{2} \frac{\partial}{\partial \theta} \left(\frac{\partial \mathbf{R}(t, \theta)}{\partial t} \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial t} \right) \right) \mathrm{d}\theta \\
&= \int_{\Theta_1}^{\Theta_2} \frac{\partial^2 \mathbf{R}(t, \theta)}{\partial t^2} \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} \mathrm{d}\theta.
\end{aligned} \tag{87}$$

The element of length $\mathrm{d}\gamma = (\partial \mathbf{R}/\partial \theta)\, \mathrm{d}\theta$ points in the tangential direction. Therefore only the tangential component of acceleration $\partial^2 \mathbf{R}/\partial t^2$ is relevant. That component is given by (ρ^{-1} times) the right hand side of the third equation in (50). Continuing with our calculation, we have

$$\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= -\int_{\Theta_1}^{\Theta_2} \frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \boldsymbol{S}^\alpha \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta} \mathrm{d}\theta \\
&= -\int_{\Theta_1}^{\Theta_2} \frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \gamma^\alpha \mathrm{d}\theta
\end{aligned} \tag{88}$$

where $\gamma^\alpha \mathrm{d}\theta$ are the coordinates of the length element $\mathrm{d}\gamma$ with respect to the surface basis

$$\gamma^\alpha = \boldsymbol{S}^\alpha \cdot \frac{\partial \mathbf{R}(t, \theta)}{\partial \theta}. \tag{89}$$

The tensor γ^α is also given by

$$\gamma^\alpha = \varepsilon^{\alpha\beta} n_\beta \tag{90}$$

where $\varepsilon^{\alpha\beta}$ is the Levi-Civita tensor defined in equation (18) and n_β is the covariant component of the unit normal $\mathbf{n}$ to the contour γ that lies in the tangential plane to manifold S. Therefore, $\mathrm{d}\Gamma/\mathrm{d}t$ is expressed by the integral

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -\int_{\Theta_1}^{\Theta_2} \frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \varepsilon^{\alpha\beta} n_\beta \mathrm{d}\theta.$$

Once again, apply Gauss's theorem

$$\begin{aligned} \frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= -\int_{\Theta_1}^{\Theta_2} \frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \varepsilon^{\alpha\beta} n_\beta \mathrm{d}\theta \\ &= -\int_{P(t)} \varepsilon^{\alpha\beta} \nabla_\beta \left(\frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \right) \mathrm{d}S \end{aligned} \tag{91}$$

where $P(t)$ is the material patch enclosed by $\gamma(t)$ and the Levi-Civita tensor is outside of the covariant derivative operator since its covariant derivative vanishes [14]. Also, note that $\nabla_\beta \left(\frac{1}{\rho}\nabla_\alpha \left(\rho^2 e_\rho\right)\right)$ is symmetric in α and β

$$\begin{aligned} \nabla_\beta \left(\frac{1}{\rho} \nabla_\alpha \left(\rho^2 e_\rho\right) \right) &= \nabla_\beta \left(\frac{1}{\rho} \left(\rho^2 e_\rho\right)_\rho \nabla_\alpha \rho \right) \\ &= \left(\frac{1}{\rho} \left(\rho^2 e_\rho\right)_\rho \right)_\rho \nabla_\beta \rho \nabla_\alpha \rho + \frac{1}{\rho} \left(\rho^2 e_\rho\right)_\rho \nabla_\beta \nabla_\alpha \rho. \end{aligned} \tag{92}$$

Therefore, the integrand of (92) vanishes and we have shown that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \tag{93}$$

Q.E.D.

7. Equations for Normal Deformations

Fluid films can undergo enormous deformations and display variations in thickness from nanometers to millimeters. To adequately capture these effects, analysis must be fully nonlinear and assume small neither deformations nor constant thickness. However, other types of simplifications do not detract from the nonlinear nature of fluid films. One of the most appealing simplifying assumptions commonly employed in contemporary literature is to neglect the tangential components of the velocity field. This simplified model is still capable of displaying a wide range of deeply nonlinear effects. In this paper, we derive exact equations for the dynamics of fluid films under this assumption.

This assumption cannot be implemented in a consistent manner simply by neglecting the terms containing V^α is in the full dynamic system (50). If one were to simply eliminate these terms, the first two equations would become

$$\begin{aligned} \frac{\delta \rho}{\delta t} &= \rho C B_\alpha^\alpha \\ \rho \frac{\delta C}{\delta t} &= -\rho^2 e_\rho B_\alpha^\alpha . \end{aligned} \tag{94}$$

This is a system that certainly deserves attention. It is nonlinear and conserves mass and energy. However, it most likely cannot be obtained from a proper Least Action Principle. Further, this approach based on neglecting terms containing V^α leaves the third equation in a nonsensical form

$$\rho C \nabla^\alpha C = -\nabla^\alpha \left(\rho^2 e_\rho \right) .$$

An alternative way to see the problem is to notice that the third equation in (50) shows that finite C will necessarily cause finite V^α.

We must therefore incorporate the constraint

$$V^\alpha = 0 \tag{95}$$

at an earlier stage in our modeling. We propose to modify the action by incorporating the constraint (95) in the following way

$$A = \int_{t_1}^{t_2} \left(\int_S \rho \left(\frac{1}{2} C^2 - e - \Lambda^\alpha V_\alpha \right) \mathrm{d}S \right) \mathrm{d}t \tag{96}$$

where Λ^α is a pointwise time-dependent field of Lagrange multipliers. This action leads to the following simplified system

$$\begin{aligned} \frac{\delta \rho}{\delta t} &= \rho C B_\alpha^\alpha \\ \rho \frac{\delta C}{\delta t} - 2\rho \Lambda^\alpha \nabla_\alpha C - C \nabla_\alpha \left(\rho \Lambda^\alpha \right) &= -\rho^2 e_\rho B_\alpha^\alpha \\ \rho \left(\frac{\delta \Lambda^\alpha}{\delta t} - 2 C B_\beta^\alpha \Lambda^\beta + C \nabla^\alpha C \right) &= -\nabla^\alpha \left(\rho^2 e_\rho \right) . \end{aligned} \tag{97}$$

This system is certainly more complicated than the simplified system (95) obtained by intuitive reasoning, but offers a number of advantages over the full system (50). In particular, contrary to the equations (50), the new system (98) has properties similar to those of the Lagrange equations of the second kind in classical dynamics [20]. It can therefore be analyzed by the universal methods of classical dynamics and control theory [21].

We show that the simplified system conserves mass and total energy. Mass conservation is particularly straightforward. Its proof follows that of Section 5.2, except it is simpler for lack of tangential components. Once again, consider a material patch $P(t)$ and its mass M. Its evolution $\mathrm{d}M/\mathrm{d}t$ is given by the integral

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{P(t)} \rho \mathrm{d}S. \tag{98}$$

The surface integral is differentiated according to rule (47)

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{P(t)} \frac{\delta\rho}{\delta t}\mathrm{d}S - \int_{P(t)} CB^\alpha_\alpha \rho \mathrm{d}S \tag{99}$$

and this time there is no contour term because the velocity c of the contour ($c = V^\alpha n_\alpha$) vanishes. Combining the two terms and applying the second equation in (98), yields

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{P(t)} \left(\frac{\delta\rho}{\delta t} - \rho CB^\alpha_\alpha\right)\mathrm{d}S = 0 \tag{100}$$

Q.E.D.

We now turn to energy conservation. The total energy E is given by

$$E = \int_S \rho\left(\frac{1}{2}C^2 + e\right)\mathrm{d}S. \tag{101}$$

The analysis of its evolution starts with an application of equation (47)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S \frac{\delta\left(\rho\left(\frac{1}{2}C^2 + e\right)\right)}{\delta t}\mathrm{d}S - \int_S \rho CB^\alpha_\alpha\left(\frac{1}{2}C^2 + e\right)\mathrm{d}S. \tag{102}$$

The first integral on the right hand side is expanded by the product rule

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_S \left(\frac{\delta\rho}{\delta t}\left(\frac{1}{2}C^2 + e\right) + \rho\left(C\frac{\delta C}{\delta t} + e_\rho\frac{\delta\rho}{\delta t}\right)\right)\mathrm{d}S \\ &\quad - \int_S \rho CB^\alpha_\alpha\left(\frac{1}{2}C^2 + e\right)\mathrm{d}S.\end{aligned} \tag{103}$$

According to the first equation in (98), substitute $\rho C B_\alpha^\alpha$ for $\delta\rho/\delta t$ and cancel the two equal and opposite terms

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S \rho \left(C\frac{\delta C}{\delta t} + e_\rho \rho C B_\alpha^\alpha \right) \mathrm{d}S. \tag{104}$$

Next, solve the first equation in (98) for $\delta C/\delta t$, substitute in equation (104), and once again cancel two equal and opposite terms

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S \rho C \left(2\Lambda^\alpha \nabla_\alpha C + C\rho^{-1}\nabla_\alpha \left(\rho\Lambda^\alpha\right)\right) \mathrm{d}S. \tag{105}$$

The product combines the two terms into a single divergence term

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S \nabla_\alpha \left(\rho\Lambda^\alpha C^2\right) \mathrm{d}S. \tag{106}$$

Finally, use Gauss's theorem to convert this surface integral to a contour integral

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_\gamma \rho n_\alpha \Lambda^\alpha C^2 \mathrm{d}\gamma \tag{107}$$

which vanishes because $C = 0$ at the boundary γ, Q.E.D.

8. Conclusion

In this paper, we offered proofs for four fundamental properties of the full governing equations for the dynamics of fluid films: conservation of mass, conservation of energy, pointwise conservation of momentum, and conservation of circulation around a closed material loop. These properties are retained from Euler's classical equations of hydrodynamics.

We have also analyzed a feature unique to fluid films. It is commonly assumed in existing literature, albeit in an infinitesimal linear framework, that the material particles move in the instantaneously normal direction to the fluid film. In this paper, we put this assumption on a rigorous variation framework. We have formulated a least action principle in which the constraint is enforced by a field of Lagrange multipliers. The resulting equations are simpler than the full unconstrained system (50) but still maintain their full nonlinear nature. Furthermore, these simplified equations also satisfy mass conservation and total energy conservation.

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