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EXACT DIVISORS OF POLYNOMIALS WITH PRIME VARIABLE

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In memory of Heini Halberstam

Abstract: In 1952 Paul Erdős obtained upper and lower bounds of the same order of magnitude for the number $N(x)$ of divisors of an irreducible polynomial $f(n)$ with integer coefficients for n up to x; an asymptotic formula for $N(x)$ when f has degree at least 3 has not yet been established. However progress has been made in the corresponding problem when the divisors of $f(n)$ are restricted in some way and f is not necessarily irreducible. In this paper we consider a polynomial f with integer coefficients that may not be irreducible or squarefree. Our aim is to obtain an asymptotic formula for the number of exact divisors up to y of $f(p)$ for p a prime less than x with y as large as possible in terms of x. We utilize the result that Vaughan established for his elementary proof of the Bombieri-Vinogradov Theorem.

Keywords: exact divisors, polynomials with prime variable, Siegel-Walfisz theorem, Bombieri-Vinogradov theorem.

1. Introduction

Let $f \in \mathbb{Z}[x]$ where f is not necessarily irreducible but the degree of each irreducible factor is at least 2. Many authors have investigated problems concerning the divisors of $f(n)$ for $n \leqslant x$. A key result is due to Paul Erdős [2] who proved using complicated elementary techniques that if f is irreducible then

$$x \log x \ll \sum_{n \leqslant x} \tau(f(n)) \ll x \log x$$

where $\tau(k)$ is the number of positive divisors of k. When f is an irreducible quadratic polynomial an asymptotic formula for this sum was established by Bellman and Shapiro (see [2]) and studied further in [5], [6], [7], [8], [12] but to the author's knowledge no corresponding asymptotic formula has been established for irreducible f of degree at least 3. However it is sometimes possible to derive an asymptotic formula for the number of divisors of $f(n)$ for $n \leqslant x$ satisfying an

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additional property. When $f = \prod_{i=1}^{l} f_i$ where the f_i are pairwise coprime and of degree at least 2 we proved in [13] that

$$\sum_{n \leqslant x} \#\{m \leqslant x : m | f(n)\} = Cx(\log x)^l \left(1 + O\left(\frac{1}{\log x}\right)\right) \tag{1.1}$$

where C is a constant, and we obtained an asymptotic formula for the corresponding sum when $P(m) := \max_{p|m} p \leqslant y$, valid for $y \geqslant \exp((\log\log x)^{\frac{5}{3}+\epsilon})$ with $\epsilon > 0$, so m is a smooth or friable divisor of $f(n)$. The proof of this latter result is related to and depends on ideas in [4] by Hanrot, Tenenbaum and Wu.

If $d|k$ and $(d, \frac{k}{d}) = 1$, d is called an exact divisor of k and we write $d \parallel k$. In this paper we consider divisors and exact divisors up to $y = y(x)$ of $f(p)$ for a general polynomial f over the integers and p a prime $\leqslant x$. For the simplest case considering the divisors of $f(p) = p - a$, Linnik used his dispersion method to establish that $\sum_{a<p\leqslant x} \tau(p-a) \sim Ex$ where the constant E depends on a. Using Bombieri's Theorem G. Rodriguez [11] and H. Halberstam [3] gave independently a much simpler proof of this result obtaining an error term $O(\frac{x \log\log x}{\log x})$. Our results below concern polynomials

$$f = \prod_{i=1}^{l} f_i^{r_i}$$

where each $f_i \in \mathbb{Z}[x]$, is irreducible and of degree $\geqslant 2$, the f_i are pairwise coprime, and $1 \leqslant r_1 \leqslant ... \leqslant r_l$. Our main aim is to investigate the exact divisors $\leqslant y$ of $f(p)$ for primes $p \leqslant x$, with y as large as possible in terms of x, but we also look at the analogous problem for the divisors of $f(p)$. As is usual in this type of problem we utilize the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ for } \alpha \geqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Our goal is to obtain an asymptotic formula for

$$\sum_{M \leqslant y} \sum_{\substack{n \leqslant x \\ M \parallel f(n)}} \Lambda(n)$$

for $M \in \mathbb{N}$, from which we deduce an asymptotic formula for

$$\sum_{M \leqslant y} \sum_{\substack{p \leqslant x \\ M \parallel f(p)}} 1.$$

Our result depends on applying the Siegel-Walfisz Theorem (see Lemma 4.3) and a result of Vaughan (see Lemma 4.4 and [15]) that he used to give an essentially elementary proof of the Bombieri-Vinogradov Theorem.

We denote a polynomial of degree k by $P_k(.)$ and assume that $\varepsilon > 0$ is arbitrarily small. If $r_1 \geqslant 2,\ l \geqslant 1$, let

$$\Delta = (1 - \frac{1}{lr_l} + \varepsilon), \qquad y^{\Delta} \leqslant x^{1/6}(\log x)^{-A-4}. \tag{1.3}$$

If $r_1 = 1,\ l \geqslant 1$, let

$$\Delta = (3 - \frac{1}{lr_l} + \varepsilon), \qquad y^{\Delta} \leqslant x^{1/2}(\log x)^{-A-4}. \tag{1.4}$$

In both cases we assume that $A > 0$ is the constant in Lemma 4.3 and that $y^{\frac{1}{lr_l}-\varepsilon} \gg (\log x)^{A+4}$.

Throughout we put

$$E(y) = \exp(-(\log y)^{\frac{3}{5}-\varepsilon}). \tag{1.5}$$

Theorem 1.1. *When y satisfies (1.3) or (1.4)*

$$\sum_{M \leqslant y} \sum_{\substack{p \leqslant x \\ M \| f(p)}} \log p = x(P_l(\log y) + O(E(y)) + O(x(\log x)^{-A})$$

where the leading coefficient of the polynomial P_l is given in (3.11).

Corollary 1.2.

$$\sum_{M \leqslant y} \sum_{\substack{p \leqslant x \\ M \| f(p)}} 1 = li(x)(P_l(\log y) + O(E(y)) + O(x(\log x)^{-A}).$$

The method used to prove Theorem 1.1 yields an analogous result when $M \parallel f(p)$ is replaced by $M \mid f(p)$. Let $R = \sum_{i=1}^{l} r_i$. Suppose that for $lr_l > 1$, y satisfies (1.3) and for $lr_l = 1,\ \Delta = 2 - \frac{1}{lr_l} + \varepsilon = 1 + \varepsilon$, assume that $y^{\Delta} \leqslant x^{1/2}(\log x)^{-A-4}$, and that in both cases $y^{\frac{1}{lr_l}-\varepsilon} \gg (\log x)^{A+4}$.

Theorem 1.3.

$$\sum_{M \leqslant y} \sum_{\substack{p \leqslant x \\ M \mid f(p)}} \log p = x(P_R(\log y) + O(E(y)) + O(x(\log x)^{-A})$$

where the polynomial P_R has degree R and leading coefficient given by (6.6).

As in the Corollary 1.2, to replace the summand $\log p$ by 1 just replace x by $li(x)$ in the main term.

The leading coefficients of P_l and P_R depend on the residue of certain Dedekind zeta functions at their pole $s = 1$.

We can ask the corresponding question when the prime variable p is replaced by $n \in \mathbb{N}$. The error term here is much easier to obtain, and in order to derive an asymptotic formula the maximum value of y can be much larger.

Theorem 1.4.

(i) *Suppose* $1 = r_1 = ... = r_j < r_{j+1} \leqslant ... \leqslant r_l$ *for some* j $(0 \leqslant j \leqslant l)$. *Then*

(a)

$$\sum_{M \leqslant y} \sum_{\substack{n \leqslant x \\ M \| f(n)}} 1 = x(P_l(\log y) + O(E(y)) + O(y(\log y)^{l+j-1});$$

(b)

$$\sum_{M \leqslant y} \sum_{\substack{n \leqslant x \\ M \| f(n)}} 1 = x(B_l(\log y)^l + O((\log y)^{l-1} \log\log y)) + O(y(\log y)^{l-1})$$

for $j > 0$ *where* B_l *is the leading coefficient of* $P_l(.)$.

(ii) *With* $R = \sum_{i=1}^{l} r_i$ *as in Theorem 1.3*

$$\sum_{M \leqslant y} \sum_{\substack{n \leqslant x \\ M | f(n)}} 1 = x(P_R(\log y) + O(E(y)) + O(y(\log y)^{R-1}).$$

We obtain an asymptotic formula in (i)(a), with main term $xP_l(\log y)$, when $y(\log y)^{l+j-1} = o(x)$ and in (i)(b) when $\frac{y}{\log y} = o(x)$; however in (b) we only have one term of the polynomial $P_l(\log y)$. When f is irreducible, so $l = j = 1$, and $y = x$, we obtain an asymptotic formula from (i)(b) but not from (i)(a). For an asymptotic formula with main term $xP_R(\log y)$ in (ii) we need $y(\log y)^{R-1} = o(x)$. For a special case of (ii) with a weaker result see (1.1) above, and for a related result with an analogous proof see Lemma 3.9 in [14].

When f is an irreducible quadratic polynomial it is known that

$$\sum_{M} \sum_{\substack{n \leqslant x \\ M | f(n)}} 1 = Cx \log x + O(x)$$

for C a constant; see [5] for a stronger result in the special case $f(n) = n^2 + a$, and (1.1) and [13] for an arbitrary such f. For exact divisors of an irreducible quadratic polynomial f, in section 7 we deduce from Theorem 1.4(i)(b)

Corollary 1.5.

$$\sum_{M} \sum_{\substack{n \leqslant x \\ M \| f(n)}} 1 = 2B_1 x \log x + O(x \log\log x).$$

2. Notation and preliminary results

As in section 1 write $f = \prod_{i=1}^{l} f_i^{r_i} \in \mathbb{Z}[x]$ where the f_i are irreducible, have degree $\geqslant 2$, are pairwise coprime and $1 \leqslant r_1 \leqslant ... \leqslant r_l$. Let

$$f_0 = \prod_{i=1}^{l} f_i.$$

Throughout this paper p, q denote primes, and $\varepsilon > 0$ is arbitrary.

Lemma 2.1.

(i) *If* $q^\alpha | f_i^{r_i}(n)$ *then* $q^t | f_i(n)$ *with* $t = \lceil \frac{\alpha}{r_i} \rceil$, *so* $\alpha = (t-1)r_i + \beta_i$ *with* $t \geqslant 1$ *and* $1 \leqslant \beta_i \leqslant r_i$. *Hence if* $M_i | f_i^{r_i}(n)$ *and* $M_i = \prod_{q^\alpha \| M_i} q^\alpha$, *then* $m_i := \prod_{q^\alpha \| M_i} q^t | f_i(n)$ *where* $t = \lceil \frac{\alpha}{r_i} \rceil$.

(ii) *There exists* p_0 *such that if* $q \geqslant p_0$ *and* $q | f(n)$ *then* $q | f_i(n)$ *for exactly one value of* i. *Hence if* $(M, \prod_{q < p_0} q) = 1$ *and* $M | f(n)$ *then* $M = \prod_{i=1}^{l} M_i$ *where* $M_i | f_i^{r_i}(n)$ *for* $i = 1, ..., l$ *and* $(M_i, M_j) = 1$ *for* $i \neq j$.

(iii) *If* $M_i \parallel f_i^{r_i}(n)$ *then* $m_i = M_i^{1/r_i} \in \mathbb{N}$ *and* $m_i \parallel f_i(n)$. *Hence in (i) each* $\beta_i = r_i$ *and* $\alpha = r_i t$.

Proof. Part (ii) follows since if $i \neq j$ then f_i and f_j are coprime over $\mathbb{Q}$, and the other parts are routine. ∎

Notation. If $(M, \prod_{q < p_0} q) = 1$ and $M \parallel f(n)$ then $M = \prod_{i=1}^{l} M_i$ with $M_i \parallel f_i^{r_i}(n)$ and $m \parallel f_0(n)$ for $m = \prod_{i=1}^{l} m_i = \prod_{i=1}^{l} M_i^{1/r_i} \in \mathbb{N}$.

Let D_i denote the discriminant of f_i and choose p_0 large enough for $q \nmid \prod_{i=1}^{l} D_i$ when $q \geqslant p_0$.

Define

$$\rho_i(m_i) = \#\{n (\mathrm{mod}\, m_i) : m_i | f_i(n)\} \tag{2.1}$$

$$\rho(m) = \#\{n (\mathrm{mod}\, m) : m | f_0(n)\}. \tag{2.2}$$

Lemma 2.2.

(i) $\rho_i(m_i)$ *is multiplicative. If* $q \geqslant p_0$, *then* $\rho_i(q^\alpha) = \rho_i(q)$ *for all* $\alpha \geqslant 1$, *and if* $q < p_0$ *then* $\rho_i(q^\alpha) \ll 1$ *for all* $\alpha \geqslant 1$.

(ii) *If* $(m, \prod_{q<p_0} q) = 1$ *then* $\rho(m) = \prod_{i=1}^{l} \prod_{q|m_i} \rho_i(q) = \prod_{q|m} \rho(q)$.

Proof. See Theorems 53 and 54 in [9]. ■

We assume throughout that M, M_0 denote positive integers satisfying

$$(M, \prod_{q<p_0} q) = 1, \qquad (M_0, \prod_{q \geqslant p_0} q) = 1, \tag{2.3}$$

so $(M, M_0) = 1$. Then our aim in Theorem 1.1 is to consider when $MM_0 \parallel f(p)$. If $q < p_0$ and $l > 1$, both $q|f_i(n)$ and $q|f_j(n)$ may hold for the same n when $i \neq j$, so we need to treat M_0 separately from M.

Let

$$\left.\begin{aligned} \lambda(M) &= \#\{n(\operatorname{mod} M) : M|f(n)\} \\ \lambda(M_0) &= \#\{n(\operatorname{mod} M_0) : M_0|f(n)\} \\ \lambda^*(M) &= \#\{n(\operatorname{mod} M) : M \parallel f(n)\}. \end{aligned}\right\} \tag{2.4}$$

Lemma 2.3. *If* $2 \leqslant r_1 \leqslant \ldots \leqslant r_l$ *then*

$$\lambda^*(M) = \frac{M}{m}\rho(m)\prod_{q|m}(1-\frac{1}{q}) = \varphi(M)\frac{\rho(m)}{m}. \tag{2.5}$$

Proof. Let $M,\ m,\ M_i,\ m_i$ be as in the notaion above; then if $M_i > 1$ we see that $M_i > m_i$ since $r_i > 1$. Let $M_i = \prod_{q^\alpha \parallel M_i} q^\alpha$ with $r_i|\alpha$, so $m_i = \prod_{q^t \parallel m_i} q^t$ with $\alpha = r_i t$. It follows that if $q^\alpha \parallel f_i^{r_i}(n)$ then $q^t|f_i(n)$ but $q^{t+1} \nmid f_i(n)$. Since by Lemma 2.2(i) and (2.1)

$$\#\{n(\operatorname{mod} q^{t+1} : q^t \parallel f_i(n)\} = q\rho_i(q^t) - \rho_i(q^{t+1}) = (q-1)\rho_i(q)$$

we deduce that

$$\#\{n(\operatorname{mod} q^\alpha : q^\alpha \parallel f_i^{r_i}(n)\} = q^{\alpha-(t+1)}(q-1)\rho_i(q) = q^{\alpha-t}(1-\frac{1}{q})\rho_i(q).$$

Hence by multiplicity

$$\lambda^*(M) = \prod_{i=1}^{l}\frac{M_i}{m_i}\prod_{q|M_i}(1-\frac{1}{q})\rho_i(q) = \varphi(M)\frac{\rho(m)}{m}. \qquad ■$$

We treat the case $r_1 = 1$ when $M_1 = m_1$ differently from that of $r_1 > 1$. We observe that if $M \parallel f(n)$ then $M|f(n)$ but for all squarefree $k > 1$ with $k|M$ we have $Mk \nmid f(n)$.

Lemma 2.4. *Let* $r_1 = 1$ *and* k *be as above. Then*

$$\lambda(Mk) = \lambda(M) = \frac{M}{m}\rho(m). \tag{2.6}$$

Proof. If $r_i|\alpha$ and $q^{\alpha+1}|f_i^{r_i}(n)$ then $q^{\frac{\alpha}{r_i}+1}|f_i(n)$. With the above notation, it follows that $Mk|f(n) \Leftrightarrow mk|f_0(n)$, and hence

$$\begin{aligned}\lambda(Mk) &= \#\{n(\operatorname{mod} Mk) : mk|f_0(n)\}\\ &= \frac{Mk}{mk}\#\{n(\operatorname{mod} mk) : mk|f_0(n)\} = \frac{M}{m}\rho(m)\end{aligned}$$

since $\rho(mk) = \prod_{q|mk}\rho(q) = \rho(m)$ as $k|m$. ■

Note: If $r_i \nmid \alpha$ and $q^{\alpha+1}|f_i^{r_i}(n)$ then $q^{\lceil \alpha/r_i \rceil}|f_i(n)$ so the above argument fails.

Lemma 2.5.

$$\lambda(M_0) \ll M_0^{1-\frac{1}{lr_l}} \prod_{q^\alpha \| M_0} e^{c\sqrt{\alpha}} \tag{2.7}$$

where $c = \pi\sqrt{\frac{2}{3}}$.

Proof. If $q|M_0$ and $l > 1$, we observed that both $q|f_i(n)$ and $q|f_j(n)$ may hold when $i \neq j$. However if $q^\alpha|f(n)$ then $q^{\alpha_i}|f_i^{r_i}(n)$ for $i = 1, ..., l$ with $\sum_{i=1}^{l}\alpha_i = \alpha$. Given $\alpha \geqslant 1$ the number of such sets $\{\alpha_1, ..., \alpha_l\}$ is at most $l!p(\alpha) \ll e^{c\sqrt{\alpha}}$, $c = \pi\sqrt{\frac{2}{3}}$, where $p(\alpha)$ denotes the number of partitions of α into positive integers; see Theorem 10.12 in [10]. If $q < p_0$ and $q^\alpha|f(n)$ with $\alpha \geqslant 1$ then for at least one partition of α of the above type

$$\begin{aligned}0 &< \#\{n(\operatorname{mod} q^\alpha) : q^{\alpha_i}|f_i^{r_i}(n),\ i = 1, ..., l.\}\\ &\leqslant \min_{\alpha_i \geqslant 1, 1\leqslant i\leqslant l} \#\{n(\operatorname{mod} q^\alpha) : q^{\alpha_i}|f_i^{r_i}(n)\}\\ &\ll \min_{\alpha_i \geqslant 1, 1\leqslant i\leqslant l} q^{\alpha-\lceil \alpha_i/r_i \rceil}\rho_i(q^{\lceil \alpha_i/r_i \rceil}) \ll q^{\alpha(1-\frac{1}{lr_l})}\end{aligned}$$

since $\lceil \alpha_i/r_i \rceil \geqslant \alpha_i/r_i$, $\max_{1\leqslant i\leqslant l}\alpha_i \geqslant \alpha/l$ and $r_i \leqslant r_l$. Hence $\lambda(q^\alpha) \ll q^{\alpha(1-\frac{1}{lr_l})}e^{c\sqrt{\alpha}}$, and the result of Lemma 2.5 follows for $l > 1$. When $l = 1$, (2.7) holds since $\lambda(M_0) \ll M_0^{1-\frac{1}{r_1}}$. ■

Corollary 2.6. *Let* $M,\ k,\ M_0$ *be as above and* k_0 *be a squarefree divisor of* M_0. *Then*

$$\lambda(MkM_0k_0) = \lambda(M)\lambda(M_0k_0) \tag{2.8}$$

where $\lambda(M_0k_0) \ll M_0^{1-\frac{1}{lr_l}+\varepsilon}$.

3. Generating functions

Let $K_i = \mathbb{Q}(\theta_i)$ where $f_i(\theta_i) = 0$, so $\theta_i \notin \mathbb{Q}$. For $\sigma = \operatorname{Re} s > 1$ the Dedekind zeta function $\zeta_i(s)$ associated with K_i is defined by

$$\zeta_i(s) = \sum_{\mathfrak{a}} (N(\mathfrak{a}))^{-s} = \prod_{\mathfrak{p}} (1 - (N(\mathfrak{p}))^{-s})^{-1} \tag{3.1}$$

where $\mathfrak{a}$ denotes an ideal and $\mathfrak{p}$ a prime ideal of K_i and $N(.)$ denotes the norm. $\zeta_i(s)$ has a simple pole at $s = 1$ with residue that we denote by λ_i.

Lemma 3.1. *For a suitable choice of* p_0 *and for* $\sigma > 1$

$$\zeta_i(s) = \prod_{q \geqslant p_0} (1 - q^{-s})^{-\rho_i(q)} h_i(s) \tag{3.2}$$

where $h_i(s)$ *is analytic in* $\sigma > \frac{1}{2}$ *and* $h_i(1) > 0$.

This is well known; for example, see Lemma 2 and equation (2.19) of [13].

Suppose that $F(s)$ is a product of j Dedekind zeta functions (not necessarily distinct), each having a simple pole at $s = 1$, together with a function $H(s)$ analytic in $\sigma > 1 - \delta$ for some δ $(0 < \delta < 1)$. We assume that the Dirichlet series for $F(s)$ in $\sigma > 1$ is of the form

$$F(s) = \sum_{n=1}^{\infty} n a(n) n^{-s} \tag{3.3}$$

where $a(n)$ is multiplicative and non-negative. We consider several such functions below and we require estimates for

$$S(y) := \sum_{n \leqslant y} n a(n), \qquad T(y) := \sum_{n \leqslant y} a(n). \tag{3.4}$$

Lemma 3.2.

(i)
$$S(y) = y P_{j-1}(\log y) + O(yE(y)) \tag{3.5}$$

where P_{j-1} *is a polynomial of degree* $j-1$ *and* $E(y)$ *is given by (1.5).*

(ii)
$$T(y) = P_j(\log y) + O(E(y)) \tag{3.6}$$

where P_j *has degree* j *and leading coefficient*

$$\frac{1}{j!} \lim_{s \to 1} \left((s-1)^j F(s) \right). \tag{3.7}$$

Proof. This follows by standard analytical methods, starting from

$$\frac{1}{y} \int_{1-}^{y} S(u) du = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} F(s) \frac{y^s}{s(s+1)} ds = y P_{j-1}(\log y) + O(yE(y))$$

with $\kappa = 1 + \frac{1}{\log y}$, and then using that $S(u)$ in (3.4) is non-decreasing to obtain (3.5) and partial summation to deduce (3.6). ∎

Let $M,\ M_0$ be as in (2.3), m be as in the notation and $k_0|M_0$ be squarefree.

Lemma 3.3.

$$\sum_{MM_0\leqslant y}\frac{\rho(m)}{m}\frac{1}{\varphi(M_0)}\sum_{k_0|M_0}\frac{\mu(k_0)}{k_0}\lambda(M_0k_0)=P_l(\log y)+O(E(y)) \tag{3.8}$$

where the polynomial P_l of degree l has leading coefficient given in (3.11).

Proof. Let

$$\begin{aligned}F_1(s)&:=\sum_M\frac{M}{m}\rho(m)M^{-s}=\prod_{i=1}^{l}\prod_{q\geqslant p_0}\left(1+\rho_i(q)\sum_{t=1}^{\infty}q^{t(r_i-1)-tr_is}\right)\\&=\prod_{i=1}^{l}\zeta_i(r_i(s-1)+1)H_1(s)\end{aligned} \tag{3.9}$$

on using (3.2), where $H_1(s)$ is analytic in $\sigma>1-\frac{1}{2r_l}$. Let

$$G_1(s)=\sum_{M_0}\frac{M_0}{\varphi(M_0)}\sum_{k_0|M_0}\frac{\mu(k_0)}{k_0}\lambda(M_0k_0)M_0^{-s}. \tag{3.10}$$

The coefficient of M_0^{-s} is $\ll M_0^{1-\frac{1}{lr_l}+\varepsilon}$ by Lemma 2.5 and since the sum over k_0 has a bounded number of terms so $M_0/\varphi(M_0)\ll 1$. Hence $G_1(s)$ is analytic in $\sigma>1-\frac{1}{lr_l}+\varepsilon$.

We now apply Lemma 3.2(ii) to obtain (3.8) with the function in (3.3) given by $F(s)=F_1(s)G_1(s)$; by (3.7) the leading coefficient of P_l is

$$\begin{aligned}&\frac{1}{l!}\lim_{s\to 1}\prod_{i=1}^{l}\left((s-1)\zeta_i(r_i(s-1)+1)\right)H_1(s)G_1(s)\\&\qquad=\frac{1}{l!}\prod_{i=1}^{l}\frac{\lambda_i}{r_i}H_1(1)G_1(1).\end{aligned} \tag{3.11}$$

■

By a similar argument we have:

Lemma 3.4.

$$\sum_{MM_0\leqslant y}\frac{\rho(m)}{m}\prod_{q|m}\frac{1+\frac{1}{q}}{1-\frac{1}{q}}\frac{1}{\varphi(M_0)}M_0^{1-\frac{1}{lr_l}+\varepsilon}\ll(\log y)^l. \tag{3.12}$$

Lemma 3.5. *Let $\omega(m)$ denote the number of distinct primes dividing m.*

$$\sum_{MM_0\leqslant y}\frac{M}{m}\rho(m)2^{\omega(m)}\lambda(M_0)\ll y(\log y)^{2l-1}. \tag{3.13}$$

Proof. In this case the generating function $F_1(s)$ in (3.9) is replaced by

$$F_2(s) = \prod_{i=1}^{l} \prod_{q \geqslant p_0} \left(1 + 2\rho_i(q) \sum_{t=1}^{\infty} q^{t(r_i-1)-tr_is}\right) = \prod_{i=1}^{l} \zeta_i^2(r_i(s-1)+1) H_2(s)$$

with $H_2(s)$ analytic in $\sigma > 1 - \frac{1}{2r_l}$, and instead of $G_1(s)$ in (3.10) we use

$$G_2(s) = \sum_{M_0} \lambda(M_0) M_0^{-s}.$$

Now apply Lemma 3.2(ii) again to obtain (3.13). ∎

Lemma 3.6.

$$M\frac{\rho(m)}{m} 2^{\omega(m)} \ll M^{1-\frac{1}{r_l}+\varepsilon}. \tag{3.14}$$

Proof. Let $d = \deg f_0$. Then since $\rho(q) \leqslant d$ for each prime $q \geqslant p_0$, $\rho(m) \leqslant d^{\omega(m)}$. We now use that

$$\omega(m) \leqslant \frac{\log m}{\log_2 m}(1 + O(\frac{1}{\log_2 m})), \qquad M^{1/r_l} \leqslant m = \prod_{i=1}^{l} M_i^{1/r_i} \leqslant M^{1/r_1}$$

to deduce that

$$\begin{aligned} \frac{M}{m}\rho(m)2^{\omega(m)} &\leqslant M^{1-\frac{1}{r_l}} \exp\left(\log(2d)\frac{\log m}{\log_2 m}(1 + O(\frac{1}{\log_2 m}))\right) \\ &\leqslant M^{1-\frac{1}{r_l}} \exp\left(\frac{\log(2d)}{r_1}\frac{\log M}{\log_2 M}(1 + O(\frac{1}{\log_2 M}))\right) \end{aligned}$$

and now (3.14) follows. ∎

4. Properties of the Ψ function

The von Mangoldt function $\Lambda(n)$ is defined in (1.2). Let χ denote a character and χ_0 the principal character $(\bmod K)$ and suppose $(b, K) = 1$. Define

$$\Psi(x; b, K) = \sum_{\substack{n \leqslant x \\ n \equiv b (\bmod K)}} \Lambda(n), \tag{4.1}$$

$$\Psi(x; \chi) = \sum_{n \leqslant x} \Lambda(n)\chi(n). \tag{4.2}$$

Lemma 4.1. *Let $\chi (\bmod K)$ be induced by the primitive character χ'. Then*

$$\Psi(x; \chi) = \Psi(x; \chi') + O((\log(xK))^2). \tag{4.3}$$

Lemma 4.2.

$$\Psi(x; b, K) = \frac{x}{\varphi(K)} + O\left(\frac{1}{\varphi(K)}\left(\sum_{\chi \neq \chi_0} |\Psi(x;\chi)| + xE(x)\right)\right) \tag{4.4}$$

where $E(x)$ is given by (1.5).

For these two lemmas see, for example, section 28 of [1] where the argument after equation (2) includes an extra term $\Psi(x;\chi_0) - x$ in the sum on the right of (4.4) above and this is $\ll xE(x)$ by the prime number theorem.

Lemma 4.3 (Siegel-Walfisz Theorem). *Let χ be a non-principal character $(\operatorname{mod} K)$ and suppose $K \leqslant (\log x)^A$ for some $A > 0$. Then there exists an ineffective constant $C(A)$ such that*

$$\Psi(x;\chi) \ll_A x \exp(-C(A)(\log x)^{1/2}). \tag{4.5}$$

See, for example, equation (3) of section 22 in [1].

Lemma 4.4. *Let $\sum_{\chi}^{*}$ denote the sum over all non-principal primitive characters $\chi(\operatorname{mod} K)$. For $Y \geqslant 1$, $x \geqslant 2$*

$$\sum_{K \leqslant Y} \frac{K}{\varphi(K)} \sum_{\chi}^{*} \max_{X \leqslant x} |\Psi(X;\chi)| \ll (x + x^{5/6}Y + x^{1/2}Y^2)(\log(xY)^4. \tag{4.6}$$

See, for example, the main Theorem in [15], or equation (2) in section 28 of [1].
Our aim is to investigate the sum

$$\sum_{MM_0 \leqslant y} \sum_{\substack{p^\alpha \leqslant x \\ MM_0 \| f(p)}} \Lambda(p^\alpha).$$

We assume M, M_0 satisfy (2.3) and consider separately the cases $r_1 > 1$, $r_1 = 1$.
Let $\delta = \frac{1}{lr_l} - \varepsilon > 0$, so $0 < \delta < \frac{1}{2}$ if $lr_l \geqslant 2$ and $\delta = 1 - \varepsilon$ when $r_l = l = 1$. Assume in the next two lemmas that $Y^\delta > (\log x)^{A+4}$ with A as in Lemma 4.3.

Lemma 4.5. *For $0 < \delta < \frac{1}{2}$ suppose $Y^{1-\delta} \leqslant x^{1/6}(\log x)^{-A-4}$ with $A > 0$. Then*

$$\sum_{K \leqslant Y} \frac{K^{1-\delta}}{\varphi(K)} \sum_{\chi}^{*} |\Psi(x,\chi)| \ll x(\log x)^{-A}. \tag{4.7}$$

When $\delta = 1 - \varepsilon$ with ε positive and small, then (4.7) holds for $Y^{1+\varepsilon} \leqslant x^{1/2}(\log x)^{-A-4}$.

Proof. Let $W = 2^k$ for k in the range $(\log x)^{\frac{A+4}{\delta}} < 2^k \leqslant Y$. Then by (4.6)

$$\begin{aligned} \sum_{W < K \leqslant 2W} \frac{K^{1-\delta}}{\varphi(K)} \sum_{\chi}^{*} |\Psi(x,\chi)| &\ll W^{-\delta} \sum_{W < K \leqslant 2W} \frac{K}{\varphi(K)} \sum_{\chi}^{*} |\Psi(x,\chi)| \\ &\ll W^{-\delta}(x + x^{5/6}W + x^{1/2}W^2)(\log(xW))^4. \end{aligned} \tag{4.8}$$

Summing (4.8) over k in the range given above we find that

$$\sum_{(\log x)^{(A+4)/\delta}<K\leqslant Y} \frac{K^{1-\delta}}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| \ll \left(\frac{x}{(\log x)^{A+4}} + x^{5/6}Y^{1-\delta} + x^{1/2}Y^{2-\delta}\right)(\log(xY))^4. \quad (4.9)$$

For $0<\delta<\frac{1}{2}$, $\min(x^{1/6(1-\delta)}, x^{1/2(2-\delta)}) = x^{1/6(1-\delta)}$ and for $\delta = 1-\varepsilon$, $\min(x^{1/6(1-\delta)}, x^{1/2(2-\delta)}) = x^{1/2(1+\varepsilon)}$, so the right side of (4.9) is $\ll x(\log x)^{-A}$ when Y satisfies the conditions in the lemma.

By (4.5) with A replaced by $(A+4)/\delta$, we deduce, since $\sum_{\chi}^{*}$ has at most $\varphi(K)$ terms, that

$$\begin{aligned}\sum_{K\leqslant(\log x)^{(A+4)/\delta}} \frac{K^{1-\delta}}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| &\ll (\log x)^{(A+4)(2-\delta)/\delta} x \exp(-C((A+4)/\delta)(\log x)^{1/2}) \\ &\ll x(\log x)^{-A} \quad (4.10)\end{aligned}$$

for x sufficiently large. The result of the Lemma 4.5 now follows from (4.9) and (4.10). ■

Lemma 4.6. *For* $0<\delta<1$ *and* $Y^{3-\delta} \leqslant x^{1/2}(\log x)^{-A-4}$

$$Y^{1-\delta} \sum_{Y<K\leqslant Y^2} \frac{1}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| \ll x(\log x)^{-A}. \quad (4.11)$$

Proof. Let $W = 2^k$ for k in the range $Y < 2^k \leqslant Y^2$. Then by (4.6)

$$\sum_{W<K\leqslant 2W} \frac{1}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| \ll \left(\frac{x}{W} + x^{5/6} + x^{1/2}W\right)(\log(xW))^4. \quad (4.12)$$

Summing (4.12) over k in the range given above we have

$$Y^{1-\delta} \sum_{Y<K\leqslant Y^2} \frac{1}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| \ll Y^{1-\delta}\left(\frac{x}{Y} + x^{5/6}\log Y + x^{1/2}Y^2\right)(\log(xY))^4. \quad (4.13)$$

Since $\min(x^{1/6(1-\delta)}, x^{1/2(3-\delta)}) = x^{1/2(3-\delta)}$ for all δ with $0<\delta<1$, (4.11) follows from (4.13). ■

Corollary 4.7. *For* $0<\delta<1$ *and* $Y^{3-\delta} \leqslant x^{1/2}(\log x)^{-A-4}$

$$\sum_{K\leqslant Y^2} \frac{\min(K^{1-\delta}, Y^{1-\delta})}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x,\chi)| \ll x(\log x)^{-A}. \quad (4.14)$$

This follows from Lemmas 4.5 and 4.6, and is required when $r_1 = 1$. When $r_1 > 1$, Lemma 4.5 suffices.

When $2 \leqslant r_1 \leqslant \dots \leqslant r_l$ we see by Lemma 2.3 that $M \parallel f(n)$ for n lying in $\lambda^*(M)$ residue classes $(\operatorname{mod} M)$. Let $k_0|M_0$ be squarefree, and let $b_j,\ \ 1 \leqslant j \leqslant \lambda^*(M)\lambda(M_0k_0)$, denote the residue classes $(\operatorname{mod} MM_0k_0)$ for which

$$M \parallel f(n), \qquad M_0k_0|f(n).$$

Lemma 4.8. *When* $2 \leqslant r_1 \leqslant \dots \leqslant r_l$

$$\sum_{\substack{n\leqslant x\\ MM_0\|f(n)}} \Lambda(n) = \sum_{k_0|M_0} \mu(k_0) \sum_{j=1}^{\lambda^*(M)\lambda(M_0k_0)} \sum_{\substack{n\leqslant x\\ n\equiv b_j\,(\operatorname{mod} MM_0k_0)}} \Lambda(n). \tag{4.15}$$

Now suppose $r_1 = 1$ and $k|M,\ \ k_0|M_0$ are both squarefree. As in the remark before Lemma 2.4 $MM_0 \parallel f(p) \iff MM_0|f(p)$ but $MkM_0k_0 \nmid f(p)$ for all squarefree $kk_0 > 1$. By (2.8), (2.6) and (2.7)

$$\lambda(MkM_0k_0) = \lambda(M)\lambda(M_0k_0) = \frac{M}{m}\rho(m)\lambda(M_0k_0)$$

where $\lambda(M_0k_0) \ll M_0^{1-\frac{1}{lr_l}+\varepsilon}$ since $k_0 \leqslant \prod_{q<p_0} q \ll 1$. Let $b_j,\ \ 1 \leqslant j \leqslant \lambda(MkM_0k_0)$, denote the residue classes $(\operatorname{mod} MkM_0k_0)$ for which $MkM_0k_0|f(n)$, so b_j depends on k, k_0 as well as M, M_0.

Lemma 4.9. *When* $r_1 = 1$

$$\sum_{\substack{n\leqslant x\\ MM_0\|f(n)}} \Lambda(n) = \sum_{\substack{k|M\\ k_0|M_0}} \mu(k)\mu(k_0) \sum_{j=1}^{\lambda(Mk)\lambda(M_0k_0)} \sum_{\substack{n\leqslant x\\ n\equiv b_j\,(\operatorname{mod} MkM_0k_0)}} \Lambda(n). \tag{4.16}$$

Lemmas 4.8 and 4.9 follow from above and the inclusion-exclusion principle.

We note that

$$\sum_{\substack{n\leqslant x\\ MM_0\|f(n)}} \Lambda(n) - \sum_{\substack{p^\alpha\leqslant x\\ MM_0\|f(p)}} \Lambda(p^\alpha) \ll \sum_{\substack{p^\alpha\leqslant x\\ \alpha\geqslant 2}} \Lambda(p^\alpha) \ll \sqrt{x}. \tag{4.17}$$

5. Proof of Theorem 1.1

Case 1: $2 \leqslant r_1 \leqslant \ldots \leqslant r_l$

By (4.4), (4.17) and Lemma 4.8

$$\begin{aligned}\sum_{\substack{p \leqslant x \\ MM_0 \| f(p)}} \log p &= \sum_{\substack{n \leqslant x \\ MM_0 \| f(n)}} \Lambda(n) + O(\sqrt{x}) \\ &= \sum_{k_0 | M_0} \mu(k_0) \sum_{j=1}^{\lambda^*(M)\lambda(M_0k_0)} \Psi(x; b_j, MM_0k_0) + O(\sqrt{x}) \\ &= x\frac{\lambda^*(M)}{\varphi(M)} \sum_{k_0|M_0} \mu(k_0)\frac{\lambda(M_0k_0)}{\varphi(M_0k_0)} + \mathcal{E}(x; M, M_0) \qquad (5.1)\end{aligned}$$

where by Lemmas 4.1 and 4.2

$$\begin{aligned}&\mathcal{E}(x; M, M_0) \\ &\quad \ll \lambda^*(M) \sum_{k_0|M_0} |\mu(k_0)|\, \lambda(M_0k_0) \\ &\quad \times \left\{ \frac{1}{\varphi(MM_0k_0)} \left({\sum_\chi}^* |\Psi(x;\chi)| + xE(x) \right) + (\log(xMM_0k_0))^2 \right\} \\ &\quad + \sqrt{x} \qquad (5.2)\end{aligned}$$

with ${\sum_\chi}^*$ defined in Lemma 4.4 and χ denoting a character $(\bmod\, MM_0k_0)$. The sum over k_0 in (5.1) and (5.2) has a bounded number of terms since $q|k_0 \implies q < p_0$, and by (2.7) $\lambda(M_0k_0) \ll M_0^{1-\frac{1}{lr_l}+\varepsilon}$ on using that $k_0 \ll 1$ for squarefree $k_0|M_0$. Hence

$$C(M_0) := \frac{1}{\varphi(M_0)} \sum_{k_0|M_0} \frac{\mu(k_0)}{k_0} \lambda(M_0k_0) \ll M_0^{-\frac{1}{lr_l}+\varepsilon}. \qquad (5.3)$$

By (2.5) the coefficient of x in the main term of (5.1) is $\frac{\rho(m)}{m}C(M_0)$. By (3.8)

$$\sum_{MM_0 \leqslant y} \frac{\rho(m)}{m} C(M_0) = P_l(\log y) + O(E(y)) \qquad (5.4)$$

where the leading coefficient of the polynomial P_l is given in (3.11).

It remains to estimate $\sum_{MM_0 \leqslant y} |\mathcal{E}(x; M, M_0)|$ which we split into several parts.

(i) For squarefree $k_0|M_0$, $k_0 \leqslant \prod_{q|M_0} q = c_0 \ll 1$, so $MM_0 \leqslant y \Rightarrow MM_0k_0 \leqslant c_0y$. By (2.5), (2.7) and (3.14) $\lambda^*(M)\lambda(M_0k_0) \ll (MM_0)^{1-\delta}$ where $\delta =$

$\frac{1}{lr_l} - \varepsilon$. Hence with $K = MM_0k_0$ and $Y = c_0 y$, we have by (4.7) provided $Y^{1-\delta} \leqslant x^{1/6}(\log x)^{-A-4}$ that

$$\sum_{MM_0 \leqslant y} \lambda^*(M) \sum_{k_0 | M_0} |\mu(k_0)|\, \lambda(M_0k_0) \frac{1}{\varphi(MM_0k_0)} \sum\nolimits_{\chi}^{*} |\Psi(x;\chi)| \\ \ll \sum_{K \leqslant Y} \frac{K^{1-\delta}}{\varphi(K)} \sum\nolimits_{\chi}^{*} |\Psi(x;\chi)| \ll x(\log x)^{-A} \quad (5.5)$$

where χ denotes a primitive character $(\operatorname{mod} K)$.

(ii) By (2.5) and (3.12)

$$xE(x) \sum_{MM_0 \leqslant y} \frac{\lambda^*(M)}{\varphi(M)} \sum_{k_0 | M_0} |\mu(k_0)|\, \lambda(M_0k_0) \frac{1}{\varphi(M_0k_0)} \\ \ll xE(x) \sum_{MM_0 \leqslant y} \frac{\rho(m)}{m} \frac{\lambda(M_0)}{\varphi(M_0)} \\ \ll xE(x)(\log y)^l. \quad (5.6)$$

(iii)

$$\sum_{MM_0 \leqslant y} \lambda^*(M) \sum_{k_0 | M_0} |\mu(k_0)| \lambda(M_0k_0)(\log(xMM_0k_0))^2 \\ \ll (\log(xy))^2 \sum_{MM_0 \leqslant y} \varphi(M) \frac{\rho(m)}{m} \lambda(M_0) \\ \ll (\log(xy))^2 y(\log y)^{l-1} \quad (5.7)$$

on adapting the proof of (3.13) by removing the factor $2^{\omega(m)}$.

On combining (5.2), (5.5), (5.6) and (5.7) and assuming (1.3) we find that

$$\sum_{MM_0 \leqslant y} |\mathcal{E}(x; M, M_0)| \\ \ll x(\log x)^{-A} + xE(x)(\log y)^l + (\log(xy))^2 y(\log y)^{l-1} + \sqrt{x}y \\ \ll x(\log x)^{-A}. \quad (5.8)$$

From (5.1), (5.4) and (5.8) we obtain under the assumption (1.3) that when $r_1 > 1$

$$\sum_{MM_0 \leqslant y} \sum_{\substack{p \leqslant x \\ MM_0 \| f(p)}} \log p = x(P_l(\log y) + O(E(y)) + O(x(\log x)^{-A}) \quad (5.9)$$

which is Theorem 1.1 for this case. ∎

Case 2: $1 = r_1 \leqslant r_2 \leqslant \ldots \leqslant r_l$

In this case (2.5) is not relevant so we proceed in a different way using the remark after the proof of Lemma 2.3. By (4.16) and (4.17)

$$\begin{aligned}
\sum_{\substack{p\leqslant x\\ MM_0\|f(p)}} \log p &= \sum_{\substack{k|M\\ k_0|M_0}} \mu(k)\mu(k_0) \sum_{j=1}^{\lambda(Mk)\lambda(M_0k_0)} \Psi(x; b_j, MkM_0k_0) + O(\sqrt{x}) \\
&= x \sum_{\substack{k|M\\ k_0|M_0}} \mu(k)\mu(k_0) \frac{\lambda(Mk)}{\varphi(Mk)} \frac{\lambda(M_0k_0)}{\varphi(M_0k_o)} + \mathcal{E}(x; M, M_0) \\
&= x\frac{\rho(m)}{m} C(M_0) + \mathcal{E}(x; M, M_0) \qquad (5.10)
\end{aligned}$$

on using (2.6), (4.4), (5.3) and noting that $\sum\limits_{k|M} \frac{\mu(k)}{\varphi(Mk)} = \frac{1}{M}$.

With χ a character $(\operatorname{mod} MkM_0k_0)$, we have using (4.3) and (4.4) that

$$\begin{aligned}
&\mathcal{E}(x; M, M_0) \\
&\ll \sum_{\substack{k|M\\ k_0|M_0}} |\mu(k)\mu(k_0)|\lambda(Mk)\lambda(M_0k_0) \\
&\quad \times \left\{ \frac{1}{\varphi(MkM_0k_0)} \left({\sum_\chi}^* |\Psi(x;\chi)| + xE(x) \right) + (\log(xMkM_0k_0))^2 \right\} \\
&\quad + \sqrt{x}. \qquad (5.11)
\end{aligned}$$

We now want to sum (5.10) and (5.11) over $MM_0 \leqslant y$, and we have by (5.4) that the main term is $x(P_l(\log y) + O(E(y))$ as required. We split the sum of the error term into three parts as in case 1. Note that $k \leqslant \prod\limits_{q|m} q \leqslant m \leqslant M$ for $r_1 = 1$ and that the sum over k has $2^{\omega(m)}$ terms. When $MM_0 \leqslant y$ we can certainly say that $MkM_0k_0 \leqslant y^2$ for each squarefree $k|M$, $k_0|M_0$. Moreover given $K \leqslant y^2$, $K = MkM_0k_0$ in at most $2^{\omega(mm_0)}$ ways.

(i) By (2.6), (2.7) and (3.14) $\lambda(Mk)\lambda(M_0k_0)2^{\omega(mm_0)} \ll (MM_0)^{1-\delta}$ where $\delta = \frac{1}{lr_l} - \varepsilon$. Hence with χ a character $(\operatorname{mod} K)$ and $Y = y$, $Y^{3-\delta} \leqslant x^{1/2}(\log x)^{-A-4}$ we deduce from (4.14) that

$$\begin{aligned}
&\sum_{MM_0\leqslant y} \sum_{\substack{k|M\\ k_0|M_0}} |\mu(k)\mu(k_0)|\lambda(Mk)\lambda(M_0k_0) \frac{1}{\varphi(MkM_0k_0)} {\sum_\chi}^* |\Psi(x;\chi)| \\
&\qquad \ll \sum_{K\leqslant Y^2} \frac{\min(K^{1-\delta}, Y^{1-\delta})}{\varphi(K)} {\sum_\chi}^* |\Psi(x;\chi)| \ll x(\log x)^{-A}. \qquad (5.12)
\end{aligned}$$

This holds even when $lr_l = 1$.

(ii)

$$\sum_{MM_0\leqslant y}\sum_{\substack{k|M\\ k_0|M_0}}|\mu(k)\mu(k_0)|\frac{\lambda(Mk)}{\varphi(Mk)}\frac{\lambda(M_0k_0)}{\varphi(M_0k_o)}$$

$$\ll \sum_{MM_0\leqslant y}\frac{\rho(m)}{m}\prod_{q|m}\frac{1+\frac{1}{q}}{1-\frac{1}{q}}M_0^{-\frac{1}{lr_l}+\varepsilon}$$

$$\ll (\log y)^l \tag{5.13}$$

on using (2.6), (2.7) and (3.12).

(iii)

$$\sum_{MM_0\leqslant y}\sum_{\substack{k|M\\ k_0|M_0}}|\mu(k)\mu(k_0)|\lambda(Mk)\lambda(M_0k_0)(\log(xMkM_0k_0))^2$$

$$\ll (\log(xy))^2\sum_{MM_0\leqslant y}M\frac{\rho(m)}{m}2^{\omega(m)}M_0^{1-\frac{1}{lr_l}+\varepsilon}$$

$$\ll y(\log y)^{2l-1}(\log(xy))^2 \tag{5.14}$$

by (3.13).

From equations (5.11) to (5.14) we deduce

$$\sum_{MM_0\leqslant y}|\mathcal{E}(x;M,M_0)|$$

$$\ll x(\log x)^{-A}+xE(x)(\log y)^l+y(\log y)^{2l-1}(\log(xy))^2+y\sqrt{x}$$

$$\ll x(\log x)^{-A} \tag{5.15}$$

provided (1.4) holds. Then it follows from (5.10), (5.15) and (5.4) that (5.9) holds in this case. This completes the proof of Theorem 1.1. ■

Proof of Corollary 1.2. This follows from the formula

$$\pi(x;b,K)=\frac{1}{\varphi(K)}li(x)+O(\max_{n\leqslant x,\ (b,K)=1}\left|\Psi(n;b,K)-\frac{n}{\varphi(K)}\right|+\sqrt{x}). \tag{5.16}$$

To establish this, substitute $\Lambda(n)=\Psi(n;b,K)-\Psi(n-1;b,K)$ in $\pi(x;b,K)=\sum_{\substack{2\leqslant n\leqslant x\\ n\equiv b(\mathrm{mod}\,K)}}\frac{\Lambda(n)}{\log n}+O(\sqrt{x})$ and then use partial summation, subtract $\frac{n}{\varphi(K)}$ from $\Psi(n;b,K)$ and compensate; then use partial summation again to obtain the main term. We can now use (5.16) in an argument analogous to the two cases of the proof of Theorem 1.1 and we obtain the same polynomial $P_l(\log y)$ and an error term involving the same estimate of $\sum_{MM_0\leqslant y}|\mathcal{E}(x;M,M_0)|$ in the result. ■

6. Sketch proof of Theorem 1.3

As in Lemma 2(i) write $\alpha = (t_i - 1)r_i + \beta_i$ with $1 \leqslant \beta_i \leqslant r_i$ and assume $q|M_i \Rightarrow q > p_0$. Then

$$M_i := \prod_{q^\alpha \| M_i} q^\alpha \mid f_i^{r_i}(n) \Longleftrightarrow m_i := \prod_{\substack{q^\alpha \| M_i \\ t_i = \lceil \frac{\alpha}{r_i} \rceil}} q^{t_i} | \ f_i(n).$$

Note that $m_i \geqslant M_i^{1/r_i} \geqslant M_i^{1/r_l}$. With $M = \prod_{i=1}^{l} M_i, \quad m = \prod_{i=1}^{l} m_i$ we have

$$M \mid f(n) \Longleftrightarrow m \mid f_0(n), \qquad \frac{M}{m} \leqslant M^{1-\frac{1}{r_l}}.$$

By (2.2), (2.4), (2.6) and (3.14)

$$\lambda(M) = \frac{M}{m}\rho(m) < M^{1-\frac{1}{r_l}+\varepsilon} \leqslant M^{1-\frac{1}{lr_l}+\varepsilon}. \tag{6.1}$$

By (2.3), (2.4) and (2.7) $\lambda(M_0) \ll M_0^{1-\frac{1}{lr_l}+\varepsilon}$.

Denote the residue classes $(\operatorname{mod} MM_0)$ for which $MM_0 \mid f(n)$ by a_j, $j = 1, ..., \lambda(M)\lambda(M_0)$. Then

$$\begin{aligned} \sum_{MM_0 \leqslant y} \sum_{\substack{n=p^\alpha \leqslant x \\ MM_0 \mid f(p)}} \Lambda(n) &= \sum_{MM_0 \leqslant y} \left\{ \sum_{j=1}^{\lambda(M)\lambda(M_0)} \Psi(x; a_j, MM_0) + O(\sqrt{x}) \right\} \\ &= x \sum_{MM_0 \leqslant y} \frac{\lambda(M)\lambda(M_0)}{\varphi(M)\varphi(M_0)} + O\left(\sum_{MM_0 \leqslant y} |\mathcal{E}(x; M, M_0)| \right) \end{aligned} \tag{6.2}$$

where for χ a character $(\operatorname{mod} MM_0)$ we have by (4.2), (4.3) and (4.4)

$$\begin{aligned} &\mathcal{E}(x; M, M_0) \\ &\quad \ll \frac{\lambda(M)\lambda(M_0)}{\varphi(M)\varphi(M_0)} \left\{ \sum\nolimits_{\chi}^{*} |\Psi(x;\chi)| + xE(x) \right\} + (\log(xMM_0))^2 \lambda(M)\lambda(M_0) \\ &\quad + \sqrt{x}. \end{aligned} \tag{6.3}$$

The main term in (6.2) is obtained from Lemma 3.2 on using the generating functions

$$\begin{aligned} F(s) = \sum_M M \frac{\lambda(M)}{\varphi(M)} M^{-s} &= \prod_{i=1}^{l} \prod_{q \geqslant p_0} \left\{ 1 + \frac{q\rho(q)}{q-1} \sum_{\beta_i=1}^{r_i} \sum_{t_i=1}^{\infty} q^{-t_i - ((t_i-1)r_i + \beta_i)(s-1)} \right\} \\ &= \prod_{i=1}^{l} \prod_{\beta_i=1}^{r_i} \zeta_i(\beta_i(s-1)+1) H(s) \end{aligned} \tag{6.4}$$

where $H(s)$ is analytic in $\sigma > 1 - \frac{1}{2r_l}$, and $G(s) = \sum_{M_0} M_0 \frac{\lambda(M_0)}{\varphi(M_0)} M_0^{-s}$ which is analytic in $\sigma > 1 - \frac{1}{lr_l} + \varepsilon$. Since $F(s)$ has a pole at $s = 1$ of order $\sum_{i=1}^{l} r_i = R$, we deduce from (3.6) and (3.7) that

$$\sum_{MM_0 \leqslant y} \frac{\lambda(M)\lambda(M_0)}{\varphi(M)\varphi(M_0)} = P_R(\log y) + O(E(y)) \tag{6.5}$$

where the leading coefficient of P_R is

$$\frac{1}{R!} \prod_{i=1}^{l} \frac{\lambda_i^{r_i}}{r_i!} H(1)G(1). \tag{6.6}$$

By (2.7), (6.1), (6.3), Lemma 4.5 and an argument similar to that in section 5 it follows that

$$\sum_{MM_0 \leqslant y} |\mathcal{E}(x; M, M_0)| \ll x(\log x)^{-A} \tag{6.7}$$

provided Δ and y satisfy the conditions given in Theorem 1.3.

Theorem 1.3 now follows from (6.2), (6.5) and (6.7) with (6.6) giving the leading coefficient of P_R. ■

7. Outline proof of Theorem 1.4

(i)(a) Suppose $r_1 = \dots = r_j < r_{j+1} \leqslant \dots \leqslant r_l$ for some j $(0 \leqslant j \leqslant l)$, and let $g = \prod_{i=1}^{j} f_i$, $h = \prod_{i=j+1}^{l} f_i^{\,r_i}$. If $M|f(n)$ then $M = M_g M_h$ with $M_g|g(n)$, $M_h|h(n)$, $(M_g, M_h) = 1$, so $M_g = \prod_{i=1}^{j} M_i = m_g$, $M_h = \prod_{i=j+1}^{l} M_i$, $m_h = \prod_{i=j+1}^{l} m_i$, $m = m_g m_h$ on using our previous notation. Define $\lambda_h^*(M_h)$, $\lambda_h(M_h)$, $\lambda_g(M_g)$, $\lambda(M_0)$ in an analogous way to (2.4), and then by (2.5), (2.6), (2.7)

$$\begin{aligned} \lambda_h^*(M_h) &= \varphi(M_h)\frac{\rho_h(m_h)}{m_h}, & \lambda_h(M_h) &= \frac{M_h}{m_h}\rho_h(m_h), \\ \lambda_g(M_g) &= \lambda_g(m_g) = \rho_g(m_g), & \lambda(M_0) &\ll M_0^{1-\frac{1}{lr_l}+\varepsilon} \end{aligned} \tag{7.1}$$

and $\rho_f(m) = \rho_g(m_g)\rho_h(m_h)$.

Suppose $k_g|m_g$, $k_0|M_0$ are squarefree. In order to consider when $MM_0 \parallel f(n)$ we use ideas from case 2 in the proof of Theorem 1.1. We see that

$$\begin{aligned} \#\{n(\operatorname{mod} M_h m_g k_g M_0 k_0) : M_h \parallel h(n),\ m_g k_g|g(n),\ M_0 k_0|f(n)\} \\ = \lambda_h^*(M_h)\lambda_g(m_g)\lambda(M_0 k_0) \end{aligned} \tag{7.2}$$

since $\lambda_g(m_g k_g) = \lambda_g(m_g) = \rho_g(m_g)$. Then by the inclusion-exclusion principle we have on using (7.1) and (7.2) that

$$\begin{aligned}
&\#\{n \leqslant x : M_h \parallel h(n), m_g \parallel g(n), M_0 \parallel f(n)\} \\
&= \lambda_h^*(M_h) \sum_{k_g | m_g} \mu(k_g)\lambda_g(m_g k_g) \sum_{k_0 | M_0} \mu(k_0)\lambda(M_0 k_0) \left(\frac{x}{M_h m_g k_g M_0 k_0} + O(1) \right) \\
&= x \frac{\rho_f(m)}{m} \prod_{q|m} \left(1 - \frac{1}{q}\right) \frac{\varphi(M_0)}{M_0} C(M_0) + O\left(\rho_f(m) \frac{\varphi(M_h)}{m_h} 2^{\omega(m_g)} M_0^{1-\frac{1}{lr_l}+\varepsilon} \right)
\end{aligned} \tag{7.3}$$

where $C(M_0)$ is defined in (5.3). On summing (7.3) over $MM_0 \leqslant y$ and using arguments similar to those in (3.8) and (3.13) we establish Theorem 1.4(i)(a) that

$$\sum_{MM_0 \leqslant y} \#\{n \leqslant x : MM_0 \parallel f(n)\} = x(P_l(\log y) + O(E(y))) + O(y(\log y)^{l+j-1}).$$

The leading coefficient of P_l is similar to (3.11) except that the values of $H_1(1)$, $G_1(1)$ are marginally different. This establishes (i)(a).

To derive (i)(b) let $Y = y(\log y)^{-j}$. Then by (i)(a)

$$\begin{aligned}
\sum_{MM_0 \leqslant Y} \#\{n \leqslant x : MM_0 \parallel f(n)\} &= x(P_l(\log Y) + O(E(Y)) + O(Y(\log Y)^{l+j-1}) \\
&= x(B_l(\log y)^l + O((\log y)^{l-1} \log\log y)).
\end{aligned}$$

With $M = \prod_{i=1}^{l} M_i,\ M_i^{1/r_i} \in \mathbb{N}$ as in section 6 and using an argument similar to that used to prove Theorem 1.4(ii) we have

$$\begin{aligned}
&\sum_{Y < MM_0 \leqslant y} \#\{n \leqslant x : MM_0 \parallel f(n)\} \\
&\qquad \leqslant \sum_{Y < MM_0 \leqslant y} \#\{n \leqslant x : MM_0 | f(n)\} \\
&\qquad = x(Q_l(\log y) - Q_l(\log Y) + O(E(y)) + O(y(\log y)^{l-1}) \\
&\qquad \ll x(\log y)^{l-1} \log\log y
\end{aligned}$$

where $Q_l(.)$ is a polynomial of degree l. The result of (i)(b) follows.

(ii) Defining $M,\ m$ as in section 6, by (6.1) and an argument similar to (6.5) and (6.6) we deduce Theorem 1.4(ii) that

$$\begin{aligned}
&\sum_{MM_0 \leqslant y} \#\{n \leqslant x : MM_0 | f(n)\} \\
&\qquad = x \sum_{MM_0 \leqslant y} \frac{\rho(m)}{m} \frac{\lambda(M_0)}{M_0} + O\left(\sum_{MM_0 \leqslant y} \frac{M}{m} \rho(m) \lambda(M_0) \right) \\
&\qquad = x\left(P_R(\log y) + O(E(y))\right) + O(y(\log y)^{R-1})
\end{aligned}$$

where the leading coefficient of P_R is similar to that in (6.6) but with slightly different values for $H(1),\ G(1)$. ■

To deduce the Corollary 1.5 from (i)(b) for f an irreducible quadratic, let $X = \left(\sup_{n \leqslant x} |f(n)|\right)^{1/2} \times x$. To each exact divisor $M > X$ of $f(n)$, $\frac{f(n)}{M}$ is an exact divisor $< X$. Hence

$$\sum_{M} \sum_{\substack{n \leqslant x \\ M \| f(n)}} 1 = 2 \sum_{M \leqslant X} \sum_{\substack{n \leqslant x \\ M \| f(n)}} 1 - \sum_{M \leqslant X} \sum_{\substack{n \leqslant x \\ M \| f(n) \\ \frac{|f(n)|}{M} \leqslant X}} 1. \tag{7.4}$$

The first double sum on the right of (7.4) equals $2B_1 x \log x + O(x \log\log x)$ by (i)(b). Since $|f(n)| \geqslant cn^2$ for some $c > 0$, the second double sum on the right of (7.4) is

$$\leqslant \sum_{M \leqslant X} \sum_{\substack{n \leqslant \sqrt{\frac{XM}{c}} \\ M | f(n)}} 1 \leqslant \sum_{M \leqslant X} \rho(M) \left(\sqrt{\frac{X}{cM}} + O(1)\right) \ll x.$$

Corollary 1.5 now follows from (7.4). ■

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