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APPROXIMATE FUNCTIONAL EQUATION AND MEAN VALUE FORMULA FOR THE DERIVATIVES OF L-FUNCTIONS ATTACHED TO CUSP FORMS

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Abstract: Let f be a holomorphic cusp form of weight k with respect to the full modular group $SL_2(\mathbb{Z})$. We suppose that f is a normalized Hecke eigenform. Let $L_f(s)$ be the L-function attached to the form f. Good gave the approximate functional equation and mean square formula of $L_f(s)$. In this paper, we shall generalize these formulas for the derivatives of $L_f(s)$.

Keywords: cusp forms, L-functions, derivative, approximate functional equation, mean value formula.

1. Introduction

Let S_k be the space of cusp forms of even weight $k \in \mathbb{Z}_{\geqslant 12}$ with respect to the full modular group $SL_2(\mathbb{Z})$. Let $f \in S_k$ be a normalized Hecke eigenform, and $a_f(n)$ the n-th Fourier coefficient of f. Set $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. The L-function attached to f is defined by

$$L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p\colon \text{prime}} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \quad (\text{Re } s > 1), \tag{1.1}$$

where $\alpha_f(p)$ and $\beta_f(p)$ satisfy $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$. Then it is well-known that the function $L_f(s)$ is analytically continued to the whole s-plane by

$$(2\pi)^{-s-\frac{k-1}{2}}\Gamma(s+\tfrac{k-1}{2})L_f(s) = \int_0^{\infty} f(iy)y^{s+\frac{k-1}{2}-1}dy, \tag{1.2}$$

and has a functional equation

$$L_f(s) = \chi_f(s)L_f(1-s)$$

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where $\chi_f(s)$ is given by

$$\chi_f(s) = (-1)^{\frac{k}{2}}(2\pi)^{2s-1}\frac{\Gamma(1-s+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \tag{1.3}$$

$$= (-1)^{\frac{k}{2}}(2\pi)^{2\sigma-1}|t|^{1-2\sigma}e^{i\left(\frac{\pi}{2}(1-k)\mathrm{sgn}(t)-2t\log\frac{|t|}{2\pi e}\right)}(1+O(|t|^{-1})) \tag{1.4}$$

where $\mathrm{sgn}(t)$ is defined by $\mathrm{sgn}(t) = 1$ for $t \in \mathbb{R}_{>0}$ and $\mathrm{sgn}(t) = -1$ for $t \in \mathbb{R}_{<0}$, and (1.4) is obtained by Stirling's formula (see [3, (19)]).

Good [3] gave the approximate functional equation for $L_f(s)$:

$$L_f(\sigma+it) = \sum_{n\leqslant x}\frac{\lambda_f(n)}{n^s} + \chi_f(s)\sum_{n\leqslant y}\frac{\lambda_f(n)}{n^{1-s}} + O(|t|^{\frac{1}{2}-\sigma+\varepsilon})$$

where $\varepsilon \in \mathbb{R}_{>0}$, $s = \sigma + it$ such that $\sigma \in [0,1]$ and $|t| \gg 1$, and $x, y \in \mathbb{R}_{>0}$ satisfying $(2\pi)^2 xy = |t|^2$. The feature of his proof of this equation is to introduce characteristic function and use the residue theorem. Moreover, he gave the mean square formula for $L_f(s)$ using the above equation:

$$\int_1^T |L_f(\sigma+it)|^2 dt = \begin{cases} A_f T\log T + O(T), & \sigma = 1/2, \\ T\displaystyle\sum_{n=1}^{\infty}\frac{|\lambda_f(n)|^2}{n^{2\sigma}} + O(T^{2(1-\sigma)}), & 1/2 < \sigma < 1, \\ T\displaystyle\sum_{n=1}^{\infty}\frac{|\lambda_f(n)|^2}{n^{2\sigma}} + O(\log^2 T), & \sigma = 1, \end{cases} \tag{1.5}$$

where A_f is a positive constant depending on f.

Let $\zeta(s)$ be the Riemann zeta function and $\zeta'(s)$ be its first derivative. Since Speiser [6] proved that the Riemann Hypothesis (for short RH) is equivalent to the non-existence of zeros of $\zeta'(s)$ in $0 < \mathrm{Re}\ s < 1/2$, zeros of $\zeta'(s)$ have been interested by many researchers. Recently Aoki and Minamide [1] studied the density of zeros of $\zeta^{(m)}(s)$ in the right hand side of critical line $\mathrm{Re}\ s = 1/2$ by using Littlewood's method. However there is no result concerning zeros of derivatives of L-functions attached to cusp forms. The m-th derivative of $L_f(s)$ is given by

$$L_f^{(m)}(s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)(-\log n)^m}{n^s} \qquad (\mathrm{Re}\ s > 1).$$

Differentiating both sides of (1.2), we find

$$L_f^{(m)}(s) = \sum_{r=0}^{m}\binom{m}{r}(-1)^r\chi_f^{(m-r)}(s)L_f^{(r)}(1-s). \tag{1.6}$$

In this paper, we shall show the approximate functional equation and the mean value formula for $L_f^{(m)}(s)$ for the purpose of studying the zero-density for $L_f^{(m)}(s)$.

Following [3], we shall introduce characteristic functions. Let φ be the real valued C^∞ function on $[0,\infty)$ satisfying $\varphi(\rho)=1$ for $\rho\in[0,1/2]$ and $\varphi(\rho)=0$ for $\rho\in[2,\infty)$. Let $\mathcal{R}$ be the set of these characteristic functions φ. Write $\varphi_0(\rho)=1-\varphi(1/\rho)$. It is clear to show that if $\varphi\in\mathcal{R}$ then $\varphi_0\in\mathcal{R}$. Let $\varphi^{(j)}$ be the j-th derivative function of $\varphi\in\mathcal{R}$. Then $\varphi^{(j)}$ becomes absolutely integrable function on $[0,\infty)$. Let $\|\varphi^{(j)}\|_1$ be L_1-norm of $\varphi^{(j)}$, that is, $\|\varphi^{(j)}\|_1=\int_0^\infty|\varphi^{(j)}(\rho)|d\rho$. For $r\in\{0,\dots,m\}$, $j\in\mathbb{Z}_{\geqslant 0}$, $\rho\in\mathbb{R}_{>0}$ and $s=\sigma+it$ such that $|t|\gg 1$, let $\gamma_j^{(r)}(s,\rho)$ be

$$\gamma_j^{(r)}(s,\rho)=\frac{1}{2\pi i}\int_{\mathcal{F}}\frac{(\chi_f^{(r)}/\chi_f)(1-s-w)}{w(w+1)\cdots(w+j)}\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}(\rho e^{-i\frac{\pi}{2}\operatorname{sgn}(t)})^w dw$$

where $\mathcal{F}$ is given by $\mathcal{F}=\{-1/2-\sigma+\sqrt{|t|}e^{i\pi\theta}\mid\theta\in(1/2,3/2)\}\cup\{3/2-\sigma+\sqrt{|t|}e^{i\pi\theta}\mid\theta\in(-1/2,1/2)\}\cup\{u\pm\sqrt{|t|}\mid u\in[-1/2-\sigma,3/2-\sigma]\}$.

Then using (1.6) and the approximate formula for $\chi_f^{(r)}(s)$ as $|t|\to\infty$ where $r\in\{0,\dots,m\}$, we obtain the approximate functional equation for $L_f^{(m)}(s)$ with characteristic functions.

Theorem 1.1. *For any $m\in\mathbb{Z}_{\geqslant 0}$, $l\in\mathbb{Z}_{\geqslant(k+1)/2}$, $\varphi\in\mathcal{R}$, $s=\sigma+it$ such that $\sigma\in[0,1]$ and $|t|\gg 1$, and $y_1,y_2\in\mathbb{R}_{>0}$ satisfying $(2\pi)^2y_1y_2=|t|^2$, we have*

$$\begin{aligned}L_f^{(m)}(s)&=\sum_{n=1}^\infty\frac{\lambda_f(n)(-\log n)^m}{n^s}\varphi\left(\frac{n}{y_1}\right)\\&\quad+\sum_{r=0}^m(-1)^r\binom{m}{r}\chi_f^{(m-r)}(s)\sum_{n=1}^\infty\frac{\lambda_f(n)(-\log n)^r}{n^{1-s}}\varphi_0\left(\frac{n}{y_2}\right)+R_\varphi(s),\end{aligned}\tag{1.7}$$

where $R_\varphi(s)$ is given by

$$\begin{aligned}R_\varphi(s)&=\sum_{n=1}^\infty\frac{\lambda_f(n)(-\log n)^m}{n^s}\sum_{j=1}^l\varphi^{(j)}\left(\frac{n}{y_1}\right)\left(-\frac{n}{y_1}\right)^j\gamma_j^{(0)}\left(s,\frac{1}{|t|}\right)\\&\quad+\chi_f(s)\sum_{r=0}^m(-1)^j\binom{m}{r}\sum_{n=1}^\infty\frac{\lambda_f(n)(-\log n)^r}{n^{1-s}}\\&\quad\times\sum_{j=1}^l\varphi_0^{(j)}\left(\frac{n}{y_2}\right)\left(-\frac{n}{y_2}\right)^j\gamma_j^{(m-r)}\left(1-s,\frac{1}{|t|}\right)\\&\quad+O\left(y_1^{1-\sigma}(\log y_1)^m|t|^{-\frac{l}{2}}\|\varphi^{(l+1)}\|_1\right)\\&\quad+O\left(y_2^\sigma\left(\sum_{r=0}^m(\log y_2)^r(\log|t|)^{m-r}\right)|t|^{1-2\sigma-\frac{l}{2}}\|\varphi_0^{(l+1)}\|_1\right).\end{aligned}$$

Introducing new functions $\xi \notin \mathcal{R}$ and $\psi_\alpha \in \mathcal{R}$ for making the main term of without characteristic function and the error term depending on $\alpha \in \mathbb{R}_{\geqslant 0}$ of the approximate functional equation, replacing φ to φ_α in Theorem 1.1, using Deligne's result (see [2]): $|\lambda_f(n)| \leqslant d(n)$ and choosing α to minimize the error term, we obtain the approximate functional equation for $L_f^{(m)}(s)$:

Theorem 1.2. *For any $m \in \mathbb{Z}_{\geqslant 0}$ and $s = \sigma + it$ such that $\sigma \in [0,1]$ and $|t| \gg 1$, we have*

$$L_f^{(m)}(s) = \sum_{n \leqslant \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^m}{n^s} + \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n \leqslant \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} + O(|t|^{1/2-\sigma+\varepsilon}), \tag{1.8}$$

where ε is an arbitrary positive number.

Using Rankin's result (see [5, (4.2.3), p.364]):

$$\sum_{n \leqslant x} |\lambda_f(n)|^2 = C_f x + O(x^{\frac{3}{5}}) \tag{1.9}$$

where C_f is a positive constant depending on f, the approximate formula of $\chi_f^{(r)}(s)$ and the generalizations of Lemmas 6, 7 of [3] to estimate a double sum containing $(\log n_1)^{r_1} (\log n_2)^{r_2}$ where $r_1 + r_2 = r$, we obtain the mean square for $L_f^{(m)}(s)$.

Theorem 1.3. *For any $m \in \mathbb{Z}_{\geqslant 0}$ and large $T \in \mathbb{R}_{>0}$, we have*

$$\int_0^T |L_f^{(m)}(\sigma + it)|^2 dt = \begin{cases} A_{f,m} T(\log T)^{2m+1} + O(T(\log T)^{2m}), & \sigma = 1/2, \\ T \displaystyle\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)}(\log T)^{2m}), & 1/2 < \sigma < 1, \\ T \displaystyle\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma}} + O((\log T)^{2m+2}), & \sigma = 1, \end{cases} \tag{1.10}$$

where $A_{f,m}$ is given by

$$A_{f,m} = \left(\frac{1}{2m+1} + \sum_{r=0}^{2m} \frac{(-2)^{2m-r}}{r+1} \sum_{r_1 + r_2 = r} \binom{m}{r_1} \binom{m}{r_2} \right) C_f.$$

Theorems 1.1–1.3 is applied to the study of zero-density estimate for $L_f^{(m)}(s)$ in [7]. In order to prove Theorems 1.1–1.3, we shall show preliminary lemmas in Section 2. Using these lemmas we shall give proof of Theorems 1.1–1.3 in Sections 3–5 respectively.

2. Preliminary lemmas

To prove Theorem 1.1, we introduce a new function. For $\varphi \in \mathcal{R}$, let $K_\varphi(w)$ be the function

$$K_\varphi(w) = w \int_0^\infty \varphi(\rho)\rho^{w-1} d\rho \qquad (\mathrm{Re}\ w > 0).$$

Then the following fact is known:

Lemma 2.1 ([3, p.335, Lemma 3]). *The function $K_\varphi(w)$ is analytically continued for to the whole w-plane, and has the functional equation*

$$K_\varphi(w) = K_{\varphi_0}(-w). \tag{2.1}$$

Furthermore we have the integral representation

$$\frac{K_\varphi(w)}{w} = \frac{(-1)^{l+1}}{w(w+1)\cdots(w+l)} \int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l} d\rho \tag{2.2}$$

for $l \in \mathbb{Z}_{\geqslant 0}$. Especially $K_\varphi(0) = 1$.

Next the following fact is useful for estimating the integrals (3.1), I_1' and I_2' in Section 1.1:

Lemma 2.2 ([3, p.334, Lemma 2]). *Put $s = \sigma + it$ and $w = u + iv$. For $c_1, c_2 \in \mathbb{R}$ let D_1 be the strip such that $\sigma \in [c_1, c_2]$ and $t \in \mathbb{R}$ in s-plane, and D_2 a half-strip such that $\sigma \in (-\infty, -1/2 - (k-1)/2)$ and $t \in (-1, 1)$. For fixed $c_3, c_4 \in \mathbb{R}_{>0}$, there exist $c_5 \in \mathbb{R}_{>0}$ and $c_6 \in \mathbb{R}_{>0}$ such that*

$$\left| \frac{\Gamma(s + w + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})} (e^{-i\frac{\pi}{2}\mathrm{sgn}(t)})^w \right| \\ \leqslant \begin{cases} c_5 \dfrac{(1 + |t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}}}{|t|^{\sigma-\frac{1}{2}+\frac{k-1}{2}}}, & s \in D_1,\ s+w \in D_1 \setminus D_2,\ |t| \geqslant c_3, \\ c_6 |t|^u, & s \in D_1,\ |w| \leqslant c_4 |t|^{1/2}. \end{cases} \tag{2.3}$$

The following fact is required to obtain the approximate formula for $(\chi_f^{(r)}/\chi_f)(s)$:

Lemma 2.3. *Let F and G be holomorphic function in the region D such that $F(s) \neq 0$ and $\log F(s) = G(s)$ for $s \in D$. Then for any fixed $r \in \mathbb{Z}_{\geqslant 1}$, there exist $l_1, \cdots, l_r \in \mathbb{Z}_{\geqslant 0}$ and $C_{(l_1,\cdots,l_r)} \in \mathbb{Z}_{\geqslant 0}$ such that*

$$\frac{F^{(r)}}{F}(s) = \sum_{1l_1+\cdots+rl_r=r} C_{(l_1,\cdots,l_r)} (G^{(1)}(s))^{l_1} \cdots (G^{(r)}(s))^{l_r} \tag{2.4}$$

for $s \in D$. Especially $C_{(r,0,\cdots,0)} = 1$.

Proof. The case $r = 1$ is true because of $(F'/F)(s) = G'(s)$ for $s \in D$. If we assume (2.4) and $C_{(r,0\cdots,0)} = 1$, then we have

$$\begin{aligned}
F^{(r+1)}(s) = &\sum_{1l_1+\cdots+rl_r=r} C_{(l_1,\cdots,l_r)}\Big((F'G^{(1)l_1}\cdots G^{(r)l_r})(s) \\
&+ l_1(FG^{(1)l_1-1}G^{(2)l_2+1}\cdots G^{(r)l_r})(s) + \cdots \\
&+ l_{r-1}(FG^{(1)l_1}\cdots G^{(r-1)l_{r-1}-1}G^{(r)l_r+1})(s) \\
&+ l_r(FG^{(1)l_1}\cdots G^{(r)l_r-1}G^{(r+1)})(s)\Big) \\
= F(s) &\sum_{1l'_1+\cdots+(r+1)l_{r+1}=r+1} C'_{(l'_1,\cdots,l'_{r+1})}(G^{(1)l'_1}\cdots G^{(r+1)l'_{r+1}})(s)
\end{aligned}$$

and $C'_{(r+1,0,\cdots,0)} = 1\cdot C(r,0,\cdots,0) = 1$. Hence (2.4) is true for all $r \in \mathbb{Z}_{\geqslant 1}$. ■

Using Lemma 2.3, we can get the approximate formula for $(\chi_f^{(r)}/\chi_f)(s)$ as follows:

Lemma 2.4. *For any $r \in \mathbb{Z}_{\geqslant 1}$, the function $(\chi_f^{(r)}/\chi_f)(s)$ is holomorphic in $D = \mathbb{C} \setminus \{z \in \mathbb{C} \mid |\sigma| \geqslant k/2-1, |t| \leqslant 1/2\}$. For any $s \in D$ we have*

$$\frac{\chi_f^{(r)}}{\chi_f}(s) = \begin{cases} \left(-2\log\dfrac{|t|}{2\pi}\right)^r + O\left(\dfrac{(\log|t|)^{r-1}}{|t|}\right), & |t| \gg 1, \\ O(1), & |t| \ll 1. \end{cases}$$

Proof. Apply Lemma 2.3 with $F(s) = \chi_f(s)$ and $G(s) = k\log i + (2s-1)\log 2\pi + \log\Gamma(1-s+\frac{k-1}{2}) - \log\Gamma(s+\frac{k-1}{2})$. Then we have

$$\begin{aligned}
&G^{(1)}(s) \\
&= 2\log 2\pi - \frac{\Gamma'}{\Gamma}(1-s+\tfrac{k-1}{2}) - \frac{\Gamma'}{\Gamma}(s+\tfrac{k-1}{2}) \\
&= -\log(s+\tfrac{k-1}{2}) - \log(1-s+\tfrac{k-1}{2}) + \frac{1}{2(s+\frac{k-1}{2})} + \frac{1}{2(1-s+\frac{k-1}{2})} \qquad (2.5)\\
&\quad + 2\log 2\pi + \int_0^\infty \frac{1/2-\{u\}}{(u+s+\frac{k-1}{2})^2}du + \int_0^\infty \frac{1/2-\{u\}}{(u+1-s+\frac{k-1}{2})^2}du \\
&= \begin{cases} -2\log|t| + 2\log 2\pi + O\left(|t|^{-1}\right), & |t| \gg 1, \\ O(1), & |t| \ll 1 \end{cases}
\end{aligned}$$

for $s \in D$ where we used the following formula obtained by Stirling's formula (see [4, p.342, Theorem A.3.5]):

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} - \int_0^\infty \frac{1/2-\{u\}}{(u+s)^2}du$$

and the following the approximate formula (see [3, p.335]):

$$\log s = \log|t| + i\frac{\pi}{2}\mathrm{sgn}\, t + O\left(\frac{1}{|t|}\right), \qquad \frac{1}{s} = -\frac{i}{t} + O\left(\frac{1}{|t|^2}\right).$$

By differentiating both sides of (2.5), for any $j \in \mathbb{Z}_{\geqslant 2}$ and $s \in D$, $G^{(j)}(s)$ is approximated as $G^{(j)}(s) \ll 1/|t|^{j-1}$ when $|t| \gg 1$ or $G^{(j)}(s) \ll 1$ when $|t| \gg 1$. Since $C_{(r,0,\cdots,0)} = 1$, it follows that the main term of $(\chi_f^{(r)}/\chi_f)(s)$ becomes $(G^{(1)}(s))^r$. ■

In order to prove Theorem 1.3, that is, to obtain the approximate formula of the mean square for $L_f^{(m)}(s)$ as sharp as possible, we divide the characteristic function φ as a sum of φ_1 and φ_2. For $\varphi \in \mathcal{R}$, $\delta, \delta_1 \in (0, 1/2)$ such that $\delta < \delta_1 < \delta_2$ where $\delta_2 = 2$, φ_1 and φ_2 are defined by

$$\varphi_1(\rho) = \begin{cases} 1, & \rho \in [0,\delta], \\ 0, & \rho \in [\delta_1, \infty), \end{cases} \qquad \varphi_2(\rho) = \begin{cases} 0, & \rho \in [0,\delta], \\ 1, & \rho \in [\delta_1, 1/2], \\ \varphi(\rho) & \rho \in [1/2, \delta_2], \\ 0, & \rho \in [\delta_2, \infty), \end{cases} \tag{2.6}$$

satisfying $(\varphi_1 + \varphi_2)(\rho) = 1$ for $\rho \in [\delta, \delta_1]$. Similarly for $\varphi_0 \in \mathcal{R}$, φ_{01} and φ_{02} are defined by the above, where $\delta_{01} = \delta_1$ and $\delta_{02} = \delta_2 = 2$. We shall generalize Lemma 7 of p.351 in [3]:

Lemma 2.5. *Fix $\alpha \in \mathbb{Z}_{\geqslant 0}$ and $\beta \in \mathbb{R}_{\geqslant 0}$.*
(a) For $X \in \{1, 01\}$, we have

$$\int_1^T \overline{\varphi_X\left(\frac{2\pi n}{t}\right)}\varphi_X\left(\frac{2\pi n}{t}\right)\frac{\left(\log\frac{t}{2\pi}\right)^\alpha}{t^\beta}dt$$
$$= \begin{cases} T^{1-\beta}(\log T)^\alpha/(1-\beta) + O\left((n^{1-\beta}\log n + T^{1-\beta})(\log T)^{\alpha-1}\right), & \\ & n \in [1, \delta T/2\pi),\ \beta \in [0,1),\ \alpha \in \mathbb{Z}_{\geqslant 1}, \\ T^{1-\beta}/(1-\beta) + O(n^{1-\beta}), & n \in [1, \delta T/2\pi),\ \beta \in [0,1),\ \alpha = 0, \\ O\left(|\log(T/n)|(\log T)^\alpha\right), & n \in [1, \delta T/2\pi),\ \beta = 1, \\ O((\log n)^\alpha/n^{\beta-1}), & n \in [1, \delta T/2\pi),\ \beta \in (1,\infty), \\ O(n^{1-\beta}(\log n)^\alpha), & n \in [\delta T/2\pi, \delta_1 T/2\pi), \\ 0, & n \in [\delta_1 T/2\pi, \infty), \end{cases}$$

(b) For $X \in \{1, 2\}$ and $Y \in \{2, 02\}$, we have

$$\int_1^T \overline{\varphi_X\left(\frac{2\pi n}{t}\right)}\varphi_Y\left(\frac{2\pi n}{t}\right)\frac{\left(\log\frac{t}{2\pi}\right)^\alpha}{t^\beta}dt$$
$$= \begin{cases} O(n^{1-\beta}(\log n)^\alpha), & n \in [1, \delta_X T/2\pi), \\ 0, & n \in [\delta_X T/2\pi, \infty), \end{cases}$$

(c) For $X, Y \in \{1, 2, 01, 02\}$ and $n_1 \neq n_2$, we have

$$\int_1^T \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right)\left(\frac{n_1}{n_2}\right)^{it} \frac{\left(\log\frac{t}{2\pi}\right)^\alpha}{t^\beta} dt$$

$$= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ \dfrac{\left(\log\frac{T}{2\pi}\right)^\alpha}{iT^\beta}\overline{\varphi_X\left(\dfrac{2\pi n_1}{T}\right)}\varphi_Y\left(\dfrac{2\pi n_2}{T}\right)\dfrac{(n_1/n_2)^{iT}}{\log(n_1/n_2)} & \\ +O\left(\dfrac{(\log(\max\{n_1,n_2\}))^\alpha}{(\max\{n_1,n_2\})^{1+\beta}((\log(n_1/n_2))^2}\right), & n_1, n_2\text{: otherwise,} \end{cases}$$

(d) If there exist $\alpha \in \mathbb{Z}_{\geqslant 0}$ and $\beta \in \mathbb{R}_{\geqslant 0}$ such that $M(t) = O((\log t)^\alpha / t^\beta)$, then for $X, Y \in \{1, 2, 01, 02\}$ we have

$$\int_1^T \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right)\left(\frac{n_1}{n_2}\right)^{it} M(t) dt$$

$$= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ O(T^{1-\beta}(\log T)^\alpha), & n_1, n_2\text{: otherwise, } \beta \in [0,1), \\ O\left(|\log(T/\max\{n_1,n_2\})|(\log T)^\alpha\right), & n_1, n_2\text{: otherwise, } \beta = 1, \\ O\left((\log(\max\{n_1,n_2\}))^\alpha/(\max\{n_1,n_2\})^{\beta-1}\right), & n_1, n_2\text{: otherwise, } \beta \in \mathbb{R}_{>1}. \end{cases}$$

(e) For $X \in \{1, 01\}, Y \in \{1, 2, 01, 02\}$, we have

$$\int_1^T \overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)} \varphi_Y\left(\frac{2\pi n_2}{t}\right)(n_1 n_2)^{it} \chi_f^{(\alpha)}(\sigma + it) dt$$

$$= \begin{cases} 0, & n_1 \in [\delta_X T/2\pi, \infty), \\ 0, & n_2 \in [\delta_Y T/2\pi, \infty), \\ O\left(|\log(T/\max\{n_1,n_2\})|(\log T)^\alpha\right), & n_1, n_2\text{: otherwise, } \sigma = \frac{1}{2}, \\ O\left((\log(\max\{n_1,n_2\}))^\alpha/(\max\{n_1,n_2\})^{2\sigma-1}\right), & n_1, n_2\text{: otherwise, } \sigma \in (\frac{1}{2}, 1]. \end{cases}$$

Proof. First we consider the case $n_1 \in [\delta_X T/2\pi, \infty)$ or $n_2 \in [\delta_Y T/2\pi, \infty)$. It is clear that $\varphi_X(2\pi n_1/t) = 0$ or $\varphi_Y(2\pi n_2/t) = 0$ for $t \in [1, T]$. Hence, (a)–(e) are true for the above n_1, n_2. Next we consider the case of $n_1 \in [1, \delta_X T/2\pi)$ and $n_2 \in [1, \delta_Y T/2\pi)$. Then it is clear that $2\pi n_1/\delta_X$, $2\pi n_2/\delta_Y \in [1, T]$. For $t \in [1, 2\pi \max(n_1/\delta_X, n_2/\delta_Y))$, we see that $\varphi_X(2\pi n_1/t) = 0$ (if $n_1/\delta_X \geqslant n_2/\delta_Y$) or $\varphi_Y(2\pi n_2/t) = 0$ (if $n_1/\delta_X \leqslant n_2/\delta_Y$). Hence,

$$\int_1^T \cdots dt = \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^T \cdots dt. \tag{2.7}$$

Later, we shall approximate the right-hand side of (2.7).

First we consider the condition of (a), that is, $X, Y \in \{1, 01\}$ and $n_1 = n_2 =: n$. When $n \in [\delta T/2\pi, \delta_1 T/2\pi)$, we see that $2\pi n/\delta \geqslant T$. Then the right-hand side of (2.7) is estimated as

$$\leqslant \int_{\frac{2\pi n}{\delta_1}}^{\frac{2\pi n}{\delta}} \left|\varphi_X\left(\frac{2\pi n}{t}\right)\right|^2 \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \ll n^{1-\beta}(\log n)^\alpha. \tag{2.8}$$

When $n \in [1, \delta T/2\pi)$, we find that $2\pi n/\delta \in [2\pi n/\delta_1, T]$ and $\varphi_X(2\pi n/t) = 1$ for $t \in [2\pi n/\delta, T]$. Hence the right-hand side of (2.7) is

$$= \int_{\frac{2\pi n}{\delta_1}}^{\frac{2\pi n}{\delta}} \left|\varphi_X\left(\frac{2\pi n}{t}\right)\right|^2 \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt + \int_{\frac{2\pi n}{\delta}}^{T} \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt. \tag{2.9}$$

Here the first term of the right-hand side on (2.9) is estimated as

$$\ll n^{1-\beta}(\log n)^\alpha, \tag{2.10}$$

the second term of the right-hand side on (2.9) is

$$= \begin{cases} O\left(|\log(T/n)|(\log T)^\alpha\right), & \beta = 1, \\ T^{1-\beta}(\log T)^\alpha + O(T^{1-\beta}(\log T)^{\alpha-1}) + O(n^{1-\beta}(\log n)^\alpha), & \beta \in [0, 1), \\ O((\log n)^\alpha/n^{\beta-1}) + O((\log T)^\alpha/T^{\beta-1}), & \beta \in \mathbb{R}_{>1}. \end{cases} \tag{2.11}$$

where the following formula was used:

$$\int_M^N \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt = \begin{cases} \dfrac{(\log \frac{N}{M})\left((\log \frac{N}{2\pi})^\alpha + (\log \frac{N}{2\pi})^{\alpha-1}(\log \frac{M}{2\pi}) + \cdots + (\log \frac{M}{2\pi})^\alpha\right)}{\alpha+1}, & \beta = 1, \\ \displaystyle\sum_{r=0}^{\alpha} \frac{(-1)^r}{(1-\beta)^{r+1}} \frac{\alpha!}{(\alpha-r)!} \left(\frac{(\log \frac{N}{2\pi})^{\alpha-r}}{N^{\beta-1}} - \frac{(\log \frac{M}{2\pi})^{\alpha-r}}{M^{\beta-1}}\right), & \beta \neq 1. \end{cases} \tag{2.12}$$

Therefore combining (2.7)–(2.11), we obtain (a).

Next we consider the condition (b), that is, $Y \in \{2, 02\}$ and $n_1 = n_2 = n$. When $n \in [1, \delta_X T/2\pi) \cap [1, \delta_Y T/2\pi)$, that is, $n \in [1, \delta_X T/2\pi)$, we see that $2\pi n/\delta \in [2\pi n/\delta_X, T]$ and $\varphi_Y(2\pi n/t) = 0$ for $t \in [2\pi n/\delta, T]$. Then the right-hand side of (2.7) is

$$= \int_{\frac{2\pi n}{\delta_X}}^{\frac{2\pi n}{\delta}} \overline{\varphi_X\left(\frac{2\pi n}{t}\right)} \varphi_Y\left(\frac{2\pi n}{t}\right) \frac{(\log \frac{t}{2\pi})^\alpha}{t^\beta} dt \ll n^{1-\beta}(\log n)^\alpha. \tag{2.13}$$

From (2.7) and (2.13), (b) is obtained.

We consider the condition of (c), that is, $n_1 \neq n_2$, $n_1 \in [1, \delta_X T/2\pi)$ and $n_2 \in [1, \delta_Y T/2\pi)$. By integral by parts, the right-hand side of (2.7) is

$$\begin{aligned}
&= \overline{\varphi_X\left(\frac{2\pi n_1}{T}\right)}\varphi_Y\left(\frac{2\pi n_1}{T}\right)\frac{(\log T)^\alpha}{T^\beta}\frac{(n_1/n_2)^{iT}}{i\log(n_1/n_2)} \\
&\quad + \left(\overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)}\varphi_Y\left(\frac{2\pi n_2}{t}\right)\frac{(\log t)^\alpha}{t^\beta}\right)'_{t=T}\frac{(n_1/n_2)^{iT}}{(\log(n_1/n_2))^2} \\
&\quad - \frac{1}{(\log(n_1/n_2))^2}\int_{2\pi\max(\frac{n_1}{\delta_X},\frac{n_2}{\delta_Y})}^{T}\left(\overline{\varphi_X\left(\frac{2\pi n_1}{t}\right)}\varphi_Y\left(\frac{2\pi n_2}{t}\right)\frac{(\log t)^\alpha}{t^\beta}\right)'' \\
&\quad \times \left(\frac{n_1}{n_2}\right)^{it} dt. \qquad (2.14)
\end{aligned}$$

Since $(\varphi_X(2\pi n/t))' = O(n/t^2)$ and $(\varphi_X(2\pi n/t))'' = O(n/t^3) + O(n^2/t^4)$ for $X \in \{1, 2, 01, 02\}$, it follows that

$$\begin{aligned}
(\cdots)'_{t=T} &\ll (n_1+n_2)\frac{(\log T)^\alpha}{T^{\beta+2}} + \frac{(\log T)^{\alpha-1}}{T^{\beta+1}} + \frac{(\log T)^\alpha}{T^{\beta+1}} \ll \frac{(\log T)^\alpha}{T^{\beta+1}}, \\
(\cdots)'' &\ll (n_1+n_2)\frac{(\log t)^\alpha}{t^{\beta+3}} + (n_1^2+n_2^2)\frac{(\log t)^\alpha}{t^{\beta+4}} + n_1 n_2\frac{(\log t)^\alpha}{t^{\beta+4}} \ll \frac{(\log t)^\alpha}{t^{\beta+2}}.
\end{aligned}$$

Hence the second term of the right-hand side of (2.14) is estimated as

$$\ll \frac{(\log T)^\alpha}{T^{\beta+1}(\log(n_1/n_2))^2} \ll \frac{(\log\max(n_1,n_2))^\alpha}{(\max(n_1,n_2))^{\beta+1}(\log(n_1/n_2))^2}, \qquad (2.15)$$

and the third term of the right-hand side of (2.14) is estimated as

$$\ll \frac{1}{(\log(n_1/n_2))^2}\int_{2\pi\max(\frac{n_1}{\delta_X},\frac{n_2}{\delta_Y})}^{T}\frac{(\log t)^\alpha}{t^{\beta+2}}dt \ll \frac{(\log\max(n_1,n_2))^\alpha}{(\max(n_1,n_2))^{\beta+1}(\log(n_1/n_2))^2}. \qquad (2.16)$$

Combining (2.7) and (2.14)–(2.16), we obtain (c).

Next we consider the condition of (d), that is, $n_1 \in [1, \delta_X T/2\pi)$ and $n_2 \in [1, \delta_Y T/2\pi)$. Then (2.12) gives that the right-hand side of (2.7) is estimated as

$$\begin{aligned}
&\ll \int_{2\pi\max(\frac{n_1}{\delta_X},\frac{n_2}{\delta_Y})}^{T}\frac{(\log t)^\alpha}{t^\beta}dt \\
&\ll \begin{cases} T^{1-\beta}(\log T)^\alpha, & \beta \in [0,1), \\ |\log(T/\max(n_1,n_2))|(\log T)^\alpha, & \beta = 1, \\ (\log\max(n_1,n_2)^\alpha)/(\max(n_1,n_2))^{\beta-1}, & \beta \in \mathbb{R}_{>1}. \end{cases}
\end{aligned}$$

Thus (d) is obtained.

Finally we consider the condition of (e), that is, $X \in \{1, 01\}$, $n_1 \in [1, \delta_X T/2\pi)$ and $n_2 \in [1, \delta_Y T/2\pi)$. Using (1.4) and Lemma 2.4, we have

$$(n_1 n_2)^{it} \chi_f^{(\alpha)}(s) = (-1)^{-\frac{k}{2}} (-2)^{\alpha} (2\pi)^{2\sigma-1} e^{i\frac{\pi}{2}(1-k)\mathrm{sgn}(t)} \times e^{-2t \log \frac{|t|}{2\pi e \sqrt{n_1 n_2}}} |t|^{1-2\sigma} \left(\log \frac{|t|}{2\pi}\right)^{\alpha} + M_1(t) \tag{2.17}$$

where $M_1(t) = O((\log|t|)^{\alpha}/|t|^{2\sigma})$. Since we have $\delta_X \delta_Y < 1$, it follows that

$$2\pi \max(n_1/\delta_X, n_2/\delta_Y) \geqslant 2\pi \sqrt{(n_1 n_2)/(\delta_X \delta_Y)} > 2\pi \sqrt{n_1 n_2}.$$

Therefore we see that $|\log(2\pi\sqrt{n_1 n_2}/t)| > -\log(\sqrt{\delta_X \delta_Y}) > 0$ and

$$e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}} = \left(\frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} \right)' - \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2it (\log \frac{2\pi \sqrt{n_1 n_2}}{t})^2} \tag{2.18}$$

for $t \in [2\pi \max(n_1/\delta_X, n_2/\delta_Y), T]$. By (2.17) and (2.18), the right-hand side of (2.7) is estimated as

$$\begin{aligned} &= (-1)^{-\frac{k}{2}} (-2)^{\alpha} (2\pi)^{2\sigma-1} e^{i\frac{\pi}{2}(1-k)} \\ &\quad \times \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^{T} \overline{\varphi_X\left(\frac{2\pi n}{t}\right)} \varphi_Y\left(\frac{2\pi n}{t}\right) \frac{(\log \frac{t}{2\pi})^{\alpha}}{t^{2\sigma-1}} \left(\frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} \right)' dt \\ &\quad + O\left(\int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^{T} \overline{\varphi_X\left(\frac{2\pi n}{t}\right)} \varphi_Y\left(\frac{2\pi n}{t}\right) M_2(t) dt \right), \end{aligned} \tag{2.19}$$

where $M_2(t) = O((\log t)^{\alpha}/t^{2\sigma})$. From (d), the second term of the right-hand side of (2.19) is estimated as

$$\ll \begin{cases} |\log(T/\max(n_1, n_2))| (\log T)^{\alpha}, & \sigma = 1/2, \\ (\log \max(n_1, n_2))^{\alpha} / (\max(n_1, n_2))^{2\sigma-1}, & \sigma \in (1/2, 1]. \end{cases} \tag{2.20}$$

Integration by parts and (2.12) give that the first term of the right-hand side of (2.19) is

$$\begin{aligned} &= \frac{(-2)^{\alpha} (2\pi)^{2\sigma-1}}{(-1)^{\frac{k}{2}} e^{i\frac{\pi}{2}(k-1)}} \left(\overline{\varphi_X\left(\frac{2\pi n}{T}\right)} \varphi_Y\left(\frac{2\pi n}{T}\right) \frac{(\log \frac{T}{2\pi})^{\alpha}}{T^{2\sigma-1}} \frac{e^{-i2t \log \frac{T}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{T}} \right. \\ &\quad \left. - \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^{T} \left(\overline{\varphi_X\left(\frac{2\pi n}{t}\right)} \varphi_Y\left(\frac{2\pi n}{t}\right) \frac{(\log \frac{t}{2\pi})^{\alpha}}{t^{2\sigma-1}} \right)' \frac{e^{-i2t \log \frac{t}{2\pi e \sqrt{n_1 n_2}}}}{2i \log \frac{2\pi \sqrt{n_1 n_2}}{t}} dt \right) \\ &\ll \frac{(\log T)^{\alpha}}{T^{2\sigma-1}} + \int_{2\pi \max(\frac{n_1}{\delta_X}, \frac{n_2}{\delta_Y})}^{T} \frac{(\log t)^{\alpha}}{t^{2\sigma}} dt \\ &\ll \begin{cases} |\log(T/\max(n_1, n_2))| (\log T)^{\alpha}, & \sigma = 1/2, \\ (\log \max(n_1, n_2))^{\alpha} / (\max(n_1, n_2))^{2\sigma-1}, & \sigma \in (1/2, 1], \end{cases} \end{aligned} \tag{2.21}$$

where the following estimate was used:

$$(\cdots)' \ll (n_1+n_2)\frac{(\log t)^{\alpha}}{t^{2\sigma+1}} + \frac{(\log t)^{\alpha}}{t^{2\sigma}} + \frac{(\log t)^{\alpha-1}}{t^{2\sigma}} \ll \frac{(\log t)^{\alpha}}{t^{2\sigma}}.$$

Combining (2.7) and (2.19)–(2.21), we get (e). ∎

After using Lemma 2.5, we shall estimate the following sums:

Lemma 2.6. *For $x \in \mathbb{R}_{\geqslant 2}$, $r_1, r_2 \in \mathbb{Z}_{\geqslant 0}$ and complex valued arithmetic functions α, β such that $\alpha(n) \ll |\lambda_f(n)|$, $\beta(n) \ll |\lambda_f(n)|$, we have*

(a) $$\sum_{n_1 \leqslant n_2 \leqslant x} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{\sigma}} \ll \begin{cases} x^{2(1-\sigma)}(\log x)^{r_1+r_2}, & \sigma \in [1/2,1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1, \end{cases}$$

(b) $$\sum_{n_1 \leqslant n_2 \leqslant x} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{\sigma}} \left|\log \frac{x}{n_2}\right| \ll \frac{(\log x)^{r_1+r_2}}{x^{2(\sigma-1)}}$$

for $\sigma \in [1/2, 1)$,

(c) $$\sum_{n_1 < n_2 \leqslant x} \frac{|\alpha(n_1)\beta(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{\sigma} n_2 (\log(n_1/n_2))^2} \ll \begin{cases} x^{2(1-\sigma)}(\log x)^{r_1+r_2}, & \sigma \in [1/2,1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1, \end{cases}$$

(d) $$\sum_{\substack{n_1, n_2 \leqslant x, \\ n_1 \neq n_2}} \frac{\overline{\alpha(n_1)}\beta(n_2)(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1 n_2)^{\sigma} \log(n_1/n_2)} \ll \begin{cases} x^{2(1-\sigma)}(\log x)^{r_1+r_2}, & \sigma \in [1/2,1), \\ (\log x)^{r_1+r_2+2}, & \sigma = 1. \end{cases}$$

Proof. Using the fact $(\log n_1)^{r_1}(\log n_2)^{r_2} \ll (\log x)^{r_1+r_2}$ for $n_1, n_2 \leqslant x$ and the estimates of $R_{\sigma}(x)$ and $S_{\sigma}(x)$ in [3, p.348, Lemma 6], we obtain (a) and (b). By the same discussion for $T_{\sigma}(x)$ and $U_{\sigma}(x)$ with $\alpha_{n_1} = \alpha(n_1)(\log n_1)^{r_1}$, $\beta_{n_2} = \beta(n_2)(\log n_2)^{r_2}$, $a_{n_1} = \lambda_f(n_1)(\log n_2)^{r_1}$, $b_{n_2} = \lambda_f(n_2)(\log n_2)^{r_2}$ in [3, p.348, Lemma 6], (c) and (d) are obtained. ∎

3. Proof of Theorem 1.1

First we shall show the following formula:

Proposition 3.1. *For $s = \sigma + it$ such that $\sigma \in [0,1]$ and $|t| \gg 1$, $\varphi \in \mathcal{R}$, $x \in \mathbb{R}_{>0}$, and fixed $l \in \mathbb{Z}_{\geqslant (l+1)/2}$, we have*

$$L_f^{(m)}(s) = G_m(s, x; \varphi) + \chi_f(s) \sum_{r=0}^{m} (-1)^r \binom{m}{r} G_r\left(1-s, \frac{1}{x}; \varphi_0\right),$$

where $G_r(s,x;\varphi)$ $(r\in\{0,\dots,m\})$ *are given by*

$$G_r(s,x;\varphi)=\frac{1}{2\pi i}\int_{(\frac{3}{2}-\sigma)}\frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w)L_f^{(r)}(s+w)\frac{K_\varphi(w)}{w}\\ \times\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn}t}\right)^w dw.$$

Proof. First we shall show that the integral

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-\sigma\pm iv}^{\frac{3}{2}-\sigma\pm iv}L_f^{(m)}(s+w)\frac{K_\varphi(w)}{w}\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn}t}\right)^w dw \tag{3.1}$$

vanishes as $|v|\to\infty$ for $l\in\mathbb{Z}_{\geqslant(k+1)/2}$. Write $w=u+iv$ and choose $|v|\gg|t|+1$, then $|s+w|\gg|t+v|\gg1$. Using (1.4), (1.6) and Lemma 2.4 we have

$$L_f^{(m)}(s)\ll\sum_{r=0}^{m}|t|^{1-2\sigma}(\log|t|)^{m-r}|L_f^{(r)}(1-s)|\ll|t|^{1-2\sigma}(\log|t|)^m\qquad(|t|\to\infty)$$

for Re $s<0$. Hence the Phragmén-Lindelöf theorem gives

$$L_f^{(m)}(s+w)\ll|t+v|^{\frac{3}{2}-(\sigma+u)}(\log|t+v|)^m\\ \ll|v|^{\frac{3}{2}-(\sigma+u)}(\log|v|)^m\quad(|v|\to\infty) \tag{3.2}$$

uniformly for $\sigma+u\in[-1/2,3/2]$. Using (2.2) and (2.3) we see that

$$\frac{K_\varphi(w)}{w}\times\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn}t}\right)^w\\ \ll\frac{\|\varphi^{(l+1)}\|_1}{|v|^{l+1}}\times\frac{(1+|t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}}}{|t|^{\sigma-\frac{1}{2}+\frac{k-1}{2}}}\ll|v|^{\sigma+u-\frac{3}{2}+\frac{k-1}{2}-l}\quad(|v|\to\infty) \tag{3.3}$$

uniformly for $\sigma+u\in[-1/2,3/2]$. From (3.2) and (3.3), the integral (3.1) is $\ll|v|^{\frac{k-1}{2}-l}(\log|v|)^m$, that is, (3.1) tends to 0 as $|v|\to\infty$ when $l\in\mathbb{Z}_{\geqslant(k+1)/2}$.

Using the above fact, $K_\varphi(0)=1$ and applying Cauchy's residue theorem, we have

$$L_f^{(m)}(s)=\frac{1}{2\pi i}\left(\int_{(\frac{3}{2}-\sigma)}-\int_{(-\frac{1}{2}-\sigma)}\right)L_f^{(m)}(s+w)\frac{K_\varphi(w)}{w}\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\\ \times\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn}t}\right)^w dw. \tag{3.4}$$

for $l\in\mathbb{Z}_{\geqslant(k+1)/2}$. Clearly, the first term of the right-hand side of (3.4) is

$$=G_m(s,x;\varphi). \tag{3.5}$$

We consider the second term of the right-hand side of (3.4). Now we can calculate

$$\begin{aligned}
&L_f^{(m)}(s+w)\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\\
&=\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\chi_f(s+w)\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{\chi_f^{(m-r)}}{\chi_f}(s+w)L_f^{(r)}(1-s-w)\\
&=\chi_f(s)(2\pi)^{2w}\frac{\Gamma(1-s+w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{\chi_f^{(m-r)}}{\chi_f}(s+w) \qquad (3.6)\\
&\quad\times L_f^{(r)}(1-s-w)
\end{aligned}$$

where we used (1.3) and (1.6) which give that

$$\frac{\chi_f(s+w)}{\chi_f(s)}=(2\pi)^{2w}\frac{\Gamma(s+w)}{\Gamma(s+w+\frac{k-1}{2})}\frac{\Gamma(1-s-w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}.$$

Using (2.1), (3.6) and transforming $w\mapsto -w$, we see that the second term of the right-hand side of (3.4) is

$$\begin{aligned}
&=-\frac{\chi_f(s)}{2\pi i}\int_{-(\frac{1}{2}+\sigma)}\frac{K_\varphi(-w)}{-w}\frac{\Gamma(1-s+w+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}\left(2\pi x e^{-i\frac{\pi}{2}\operatorname{sgn}(t)}\right)^{-w}\\
&\quad\times\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{\chi_f^{(m-r)}}{\chi_f}(s-w)L_f^{(r)}(1-s+w)(-dw)\\
&=\chi_f(s)\sum_{r=0}^{m}(-1)^r\binom{m}{r}G_r\left(1-s,\frac{1}{x};\varphi_0\right). \qquad (3.7)
\end{aligned}$$

By (3.4)–(3.7) Proposition 3.1 is showed. ■

Next, the approximate formula of $G_r(s,x;\varphi)$ is written as follows:

Proposition 3.2. *For $s=\sigma+it$ such that $\sigma\in[0,1]$ and $|t|\gg 1$, $\varphi\in\mathcal{R}$, $x,y\in\mathbb{R}_{>0}$ satisfying $x/(2\pi y)=1/|t|$, fixed $r\in\{0,\cdots,m\}$ and $l\in\mathbb{Z}_{\geqslant(k+1)/2}$, we have*

$$\begin{aligned}
G_r(s,x;\varphi)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)(-\log n)^r}{n^s}\sum_{j=0}^{l}\varphi^{(j)}\left(\frac{n}{y}\right)\left(-\frac{n}{y}\right)^j\gamma_j^{(m-r)}\left(s,\frac{1}{|t|}\right)\\
+O(y^{1-\sigma}(\log y)^r(\log|t|)^{m-r}|t|^{-\frac{l}{2}}\|\varphi^{(l+1)}\|_1).
\end{aligned}$$

Proof. First using (2.2) and dividing the series $L_f^{(r)}(s+w)$ into two path at ρy, we can write

$$G_r(s,x;\varphi)=I_1+I_2, \qquad (3.8)$$

where I_1 and I_2 are given by

$$I_1 = \frac{1}{2\pi i}\int_{(\frac{3}{2}-\sigma)} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w \frac{(-1)^l}{w(w+1)\cdots(w+l)}$$
$$\times \frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w)\left(\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l}\sum_{n\leqslant\rho y}\frac{\lambda_f(n)(-\log n)^r}{n^{s+w}}d\rho\right)dw,$$
$$I_2 = \frac{1}{2\pi i}\int_{(\frac{3}{2}-\sigma)} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w \frac{(-1)^l}{w(w+1)\cdots(w+l)}$$
$$\times \frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w)\left(\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l}\sum_{n>\rho y}\frac{\lambda_f(n)(-\log n)^r}{n^{s+w}}d\rho\right)dw.$$

Let $L_{\pm1}, L_{\pm2}, C_1, C_2$ be paths of integration defined by

$$\begin{aligned} L_{\pm1} &= \{-1/2-\sigma\pm iv \mid v\in(\sqrt{|t|},\infty)\},\\ L_{\pm2} &= \{3/2-\sigma\pm iv \mid v\in(\sqrt{|t|},\infty)\},\\ C_1 &= \{-1/2-\sigma+\sqrt{|t|}e^{-i\pi\theta} \mid \theta\in(1/2,3/2)\},\\ C_2 &= \{3/2-\sigma+\sqrt{|t|}e^{i\pi\theta} \mid \theta\in(-1/2,1/2)\}. \end{aligned}$$

Then by the residue theorem, we have

$$I_1 = I_1' + \operatorname{Res}\mathcal{F}, \quad I_2 = I_2', \tag{3.9}$$

where $I_1', I_2', \operatorname{Res}\mathcal{F}$ are given by

$$\begin{aligned} I_1' = \frac{1}{2\pi i}&\int_{L_{-1}+C_1+L_{+1}} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w\\ &\times \frac{(-1)^l}{w(w+1)\cdots(w+l)}\frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w)\\ &\times\left(\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l}\sum_{n\leqslant\rho y}\frac{\lambda_f(n)(-\log n)^r}{n^{s+w}}d\rho\right)dw,\\ I_2' = \frac{1}{2\pi i}&\int_{L_{-2}+C_2+L_{+2}} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})}\left(\frac{x}{2\pi}e^{-i\frac{\pi}{2}\operatorname{sgn} t}\right)^w\\ &\times \frac{(-1)^l}{w(w+1)\cdots(w+l)}\frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w)\\ &\times\left(\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l}\sum_{n>\rho y}\frac{\lambda_f(n)(-\log n)^r}{n^{s+w}}d\rho\right)dw, \end{aligned}$$

and

$$\mathrm{Res}\,\mathcal{F} = \sum_{w=0,-1,\dots,-l} \frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left(\frac{x}{2\pi} e^{-i\frac{\pi}{2}\mathrm{sgn}\, t}\right)^{w} \frac{(-1)^l}{w(w+1)\cdots(w+l)}$$
$$\times \frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w) \left(\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l} \sum_{n\leqslant\rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \right)$$

By the same way to [3, p.337, Lemma 4 (ii)], Res $\mathcal{F}$ is written by

$$\mathrm{Res}\,\mathcal{F} = \sum_{n\leqslant 2y} \frac{\lambda_f(n)(-\log n)^r}{n^s} \sum_{j=0}^{l} \varphi^{(j)}\left(\frac{n}{y}\right)\left(-\frac{n}{y}\right)^j \gamma_j^{(m-r)}\left(s, \frac{1}{|t|}\right) \tag{3.10}$$

under the condition $x/(2\pi y) = 1/|t|$.

Next to estimate I_1' and I_2', we consider these integral. Clearly (2.3) gives

$$\frac{\Gamma(s+w+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} \left(\frac{x}{2\pi} e^{-i\frac{\pi}{2}\mathrm{sgn}\, t}\right)^{w}$$
$$\ll \begin{cases} |t|^{\frac{1}{2}-\sigma-\frac{k-1}{2}}(1+|t+v|)^{\sigma+u-\frac{1}{2}+\frac{k-1}{2}}(x/2\pi)^u, & w\in L_{\pm1,\pm2}, \\ |t|^u (x/2\pi)^u, & w \in \mathcal{F} \end{cases} \tag{3.11}$$

as $|t|\to\infty$. Using Cauchy's inequality and (1.9), we have

$$\sum_{n\leqslant\rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} \ll \sqrt{\sum_{n\leqslant\rho y} |\lambda_f(n)|^2} \sqrt{\sum_{n\leqslant\rho y} \frac{(\log n)^{2r}}{n^{2(\sigma+u)}}}$$
$$\ll (\rho y)^{1-(\sigma+u)}(\log \rho y)^r, \qquad w\in L_{\pm1}\cup C_1,$$
$$\sum_{n>\rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} \ll \int_{\rho y}^\infty \left(\frac{(\log\mu)^r}{\mu^{\sigma+u}}\right)' \sum_{n\leqslant\mu} |\lambda_f(n)| d\mu$$
$$\ll (\rho y)^{1-(\sigma+u)}(\log \rho y)^r, \qquad w\in L_{\pm2}\cup C_2.$$

Hence we obtain

$$\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l} \sum_{n\leqslant\rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \ll y^{1-(\sigma+u)}(\log y)^r \|\varphi^{(l+1)}\|_1,$$
$$w\in L_{\pm1}\cup C_1, \tag{3.12}$$
$$\int_0^\infty \varphi^{(l+1)}(\rho)\rho^{w+l} \sum_{n>\rho y} \frac{\lambda_f(n)(-\log n)^r}{n^{s+w}} d\rho \ll y^{1-(\sigma+u)}(\log y)^r \|\varphi^{(l+1)}\|_1,$$
$$w\in L_{\pm2}\cup C_2. \tag{3.13}$$

Therefore Lemma 2.4 gives

$$\frac{(-1)^l}{w\cdots(w+l)} \frac{\chi_f^{(m-r)}}{\chi_f}(1-s-w) \ll \begin{cases} |v|^{-(l+1)}(\log|v|)^{m-r}, & w\in L_{\pm1,\pm2}, \\ |t|^{-\frac{l+1}{2}}(\log|t|)^{m-r}, & w\in\mathcal{F}. \end{cases} \tag{3.14}$$

Remark 3.3. Note that

$$\gamma_j^{(r)}(s,1/|t|) = \begin{cases} O\left(\dfrac{(\log|t|)^r}{|t|^{j/2}}\right), & j\in\mathbb{Z}_{\geqslant 0},\\ \dfrac{\chi_f^{(r)}}{\chi_f}(1-s) = \left(-2\log\dfrac{|t|}{2\pi}\right)^r + O\left(\dfrac{(\log|t|)^{r-1}}{|t|}\right), & j=0,\\ \dfrac{\chi_f^{(r)}}{\chi_f}(1-s) - \dfrac{\chi_f^{(r)}}{\chi_f}(-s)\dfrac{it}{s+\frac{k-1}{2}} = O\left(\dfrac{(\log|t|)^r}{|t|}\right), & j=1,\end{cases}$$

by using (3.14), the residue theorem and Lemma 2.4.

Finally combining (3.11)–(3.14) and using the same way to [3, p.343–344], we find that I_1', I_2' are estimated as

$$\begin{aligned} I_1' &\ll y^{1-\sigma}(\log y)^r\|\varphi^{(l+1)}\|_1 \\ &\quad\times\int_{L_{\pm1}}|t|^{\frac12-(\sigma+u)-\frac{k-1}{2}}(1+|t+v|)^{\sigma+u-\frac12+\frac{k-1}{2}}\frac{(\log|v|)^{m-r}}{|v|^{l+1}}dv \\ &\quad+ y^{1-\sigma}(\log y)^r(\log|t|)^{m-r}\|\varphi^{(l+1)}\|_1|t|^{-\frac{l+1}{2}}\int_{C_1}|t|^u\left(\frac{x}{2\pi y}\right)^u|dw| \\ &\ll y^{1-\sigma}(\log y)^r(\log|t|)^{m-r}|t|^{-\frac{l}{2}}\|\varphi^{(l+1)}\|_1, \qquad (3.15)\\ I_2' &\ll y^{1-\sigma}(\log y)^r(\log|t|)^{m-r}|t|^{-\frac{l}{2}}\|\varphi^{(l+1)}\|_1, \qquad (3.16)\end{aligned}$$

under the condition $x/(2\pi y) = 1/|t|$. From (3.8)–(3.10), (3.15) and (3.16), the proof of Proposition 3.2 is completed. ■

We use (1.4) and combine the result Propositions 3.1 and 3.2. Let y_1, y_2 be the positive numbers satisfying $x/(2\pi y_2) = 1/|t|$, $(1/x)/(2\pi y_2) = 1/|t|$ respectively. Using Remark 3.3, the main term of (1.7) is obtained. Then under the condition $(2\pi)^2 y_1 y_2 = |t|^2$, the proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

To get the approximate functional equation for $L_f^{(m)}(s)$ without characteristic functions, we introduce new functions ξ, ψ_α and $\psi_{0\alpha}$. Let ξ be the function defined by $\xi(\rho) = 1$ when $\rho\in[0,1]$ and $\xi(\rho)=0$ when $\rho\in[1,\infty)$. For $\alpha\in\mathbb{R}_{\geqslant 0}$ and $\varphi\in\mathcal{R}$, let ψ_α be the function defined by

$$\psi_\alpha(\rho) = \begin{cases} 1, & \rho\in[0,1-1/(2|t|^\alpha)],\\ \varphi(1+(\rho-1)|t|^\alpha), & \rho\in[1-1/(2|t|^\alpha),1+1/|t|^\alpha],\\ 0, & \rho\in[1+1/|t|^\alpha,\infty),\end{cases}$$

and $\psi_{0\alpha}$ is defined by $\psi_{0\alpha}(\rho) = 1-\psi_\alpha(1/\rho)$.

Remark 4.1. From [3, (12)–(15)], we see that $\psi_\alpha, \psi_{0\alpha} \in \mathcal{R}$, $\xi \notin \mathcal{R}$,

$$(\psi_\alpha - \xi)(\rho) = 0, \qquad (\psi_{0\alpha} - \xi)(\rho) = 0, \qquad \psi_\alpha^{(j)}(\rho) = 0, \qquad \psi_{0\alpha}^{(j)}(\rho) = 0.$$

for $j \in \mathbb{Z}_{\geqslant 1}$ and $\rho \in [0, 1 - 1/(2|t|^\alpha)] \cup [1 + 1/|t|^\alpha, \infty)$, and

$$\psi_\alpha^{(j)}(\rho) \ll |t|^{\alpha j}, \quad \psi_{0\alpha}^{(j)}(\rho) \ll |t|^{\alpha j}, \quad \|\psi_\alpha^{(j)}\|_1 \ll |t|^{\alpha(j-1)}, \quad \|\psi_{0\alpha}^{(j)}\|_1 \ll |t|^{\alpha(j-1)}$$

for $j \in \mathbb{Z}_{\geqslant 0}$ and $\rho \in [0, \infty)$.

Let $M_\varphi(s)$ be the first sum on the right-hand side of (1.7). Setting $y_1 = y_2 = |t|/(2\pi)$ and replacing $\varphi \mapsto \psi_\alpha$ in Theorem 1.1, we can write

$$L_f^{(m)}(s) = M_\xi(s) + O(M_{\psi_\alpha - \xi}(s) + R_{\psi_\alpha}(s)). \tag{4.1}$$

Then we have

$$\begin{aligned} M_\xi(s) = &\sum_{n \leqslant \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^m}{n^s} \\ &+ \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n \leqslant \frac{|t|}{2\pi}} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &M_{\psi_\alpha - \xi}(s) + R_{\psi_\alpha}(s) \\ &\quad \ll \sum_{\frac{|t|}{2\pi}\frac{1}{1+\frac{1}{|t|^\alpha}} \leqslant n \leqslant \frac{|t|}{2\pi}(1+\frac{1}{|t|^\alpha})} \frac{|\lambda_f(n)|(\log n)^m}{n^\sigma} |S_{\psi_\alpha}^{(0)}(s)| \\ &\qquad + \sum_{r=0}^{m} \sum_{\frac{|t|}{2\pi}\frac{1}{1+\frac{1}{|t|^\alpha}} \leqslant n \leqslant \frac{|t|}{2\pi}(1+\frac{1}{|t|^\alpha})} \frac{|\lambda_f(n)|(\log n)^r}{n^\sigma} |S_{\psi_{0\alpha}}^{(m-r)}(1-s)| \\ &\qquad + |t|^{1-\sigma+(\alpha-\frac{1}{2})l}(\log|t|)^m. \end{aligned} \tag{4.3}$$

where $S_{\psi_\alpha}^{(r)}(s)$ is given by

$$\begin{aligned} S_{\psi_\alpha}^{(r)}(s) = &(\psi_\alpha - \xi)\left(\frac{2\pi n}{|t|}\right) \frac{\chi_f^{(r)}}{\chi_f}(1-s) \\ &+ \sum_{j=1}^{l} \psi_\alpha^{(j)}\left(\frac{2\pi n}{|t|}\right)\left(-\frac{2\pi n}{|t|}\right)^j \gamma_j^{(r)}\left(s, \frac{1}{|t|}\right), \end{aligned}$$

and we used Remarks 3.3, 4.1, (1.4) and the fact $1 - 1/(2|t|^\alpha) \geqslant 1/(1 + 1/|t|^\alpha)$ for $\alpha \in \mathbb{R}_{\geqslant 0}$. Using Remarks 3.3 and 4.1, in the case of $n \in [|t|/(2\pi(1 + |t|^{-\alpha})),$

$(1+|t|^{-\alpha})|t|/(2\pi)]$ the sum $S_{\psi_\alpha}^{(r)}(s)$ is estimated as follows under the condition $\alpha \leqslant 1/2$:

$$S_{\psi_\alpha}^{(r)}(s) \ll (\log|t|)^r + \sum_{j=1}^{l} |t|^{(\alpha-\frac{1}{2})j}(\log|t|)^r \ll (\log|t|)^r \ll |t|^\varepsilon. \qquad (4.4)$$

Deligne's estimate $|\lambda_f(n)| \leqslant d(n) \ll n^\varepsilon$ (see [2]) gives

$$\sum_{\frac{|t|}{2\pi}\frac{1}{1+\frac{1}{|t|^\alpha}} \leqslant n \leqslant \frac{|t|}{2\pi}(1+\frac{1}{|t|^\alpha})} \frac{|\lambda_f(n)|(\log n)^r}{n^\sigma} \ll |t|^{1-\sigma-\alpha+\varepsilon}. \qquad (4.5)$$

Therefore combining (4.3)–(4.5), we obtain the following estimate:

$$M_{\psi_\alpha-\xi}(s) + R_{\psi_\alpha}(s) = O(|t|^{1-\sigma-\alpha+\varepsilon}) + O(|t|^{1-\sigma+(\alpha-\frac{1}{2})l+\varepsilon}) = O(|t|^{\frac{1}{2}-\sigma+\varepsilon}), \quad (4.6)$$

where we put $\alpha = 1/2 - \varepsilon$ and take $l \geqslant 1/(2\varepsilon)$. Combining (4.1), (4.2) and (4.6), we obtain the assertion of Theorem 1.2.

5. Proof of Theorem 1.3

Putting $y_1 = y_2 = |t|/(2\pi)$ in Theorem 1.1 and writing $\varphi = \varphi_1 + \varphi_2$, $\varphi_0 = \varphi_{01} + \varphi_{02}$ where $\varphi_1, \varphi_2, \varphi_{01}, \varphi_{02}$ are defined by (2.6), we obtain the following formula:

$$\int_0^T |L_f^{(m)}(s)|^2 dt = \int_1^T \left|\sum_{r=1}^{5} S_r(s)\right|^2 dt + O(1) = \sum_{1\leqslant \mu,\nu \leqslant 5} I_{\mu,\nu} + O(1), \qquad (5.1)$$

where $S_r(s)$ are given by

$$S_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^s} \varphi_1\left(\frac{2\pi n}{t}\right),$$

$$S_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^m}{n^{1-s}} \varphi_2\left(\frac{2\pi n}{t}\right),$$

$$S_3(s) = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \varphi_{01}\left(\frac{2\pi n}{t}\right),$$

$$S_4(s) = \sum_{r=0}^{m} (-1)^r \binom{m}{r} \chi_f^{(m-r)}(s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)(-\log n)^r}{n^{1-s}} \varphi_{02}\left(\frac{2\pi n}{t}\right),$$

$$S_5(s) = R_\varphi(s),$$

and $I_{\mu,\nu}$ $(\mu,\nu \in \{1,\dots,5\})$ are given by

$$I_{\mu,\nu} = \int_1^T S_\mu(s)\overline{S_\nu(s)}dt.$$

First we consider the integral $I_{\mu,\nu}$ in the case of $\mu=\nu$. In the case of $(\mu,\nu)=(1,1)$, applying (a), (c) of Lemma 2.5, we get

$$\begin{aligned}
I_{1,1} &= \sum_{n_1,n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)}\lambda_f(n_2)(\log n_1 \log n_2)^m}{(n_1 n_2)^{\sigma}} \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)}\varphi_1\left(\frac{2\pi n_2}{t}\right)\left(\frac{n_1}{n_2}\right)^{it} dt \\
&= T\sum_{n\leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O\left(\sum_{n\leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma-1}}\right) \\
&\quad + \frac{1}{i}\sum_{\substack{n_1,n_2<\frac{\delta_1}{2\pi}T,\\ n_1\neq n_2}} \frac{\overline{\lambda_f(n_1)\varphi_1(2\pi n_1/T)n_1^{-iT}}\lambda_f(n_2)\varphi_1(2\pi n_2/T)n_2^{-iT}}{(n_1n_2)^{\sigma}} \\
&\quad \times \frac{\log(n_1/n_2)}{(\log n_1 \log n_2)^m} + O\left(\sum_{n_1<n_2\leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1\log n_2)^m}{(n_1n_2)^{\sigma}n_2(\log(n_1/n_2))^2}\right) \\
&=: U_1 + O(U_2) + U_3 + O(U_4). \qquad (5.2)
\end{aligned}$$

Here we shall calculate the right-hand side of (5.2). Using partial summation and (1.9), we obtain the approximate formula for U_1 as

$$U_1 = \begin{cases} \dfrac{C_f}{2m+1}T(\log T)^{2m+1} + O(T), & \sigma = 1/2, \\ T\displaystyle\sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma}} + O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma\in(1/2,1]. \end{cases} \qquad (5.3)$$

The result (1.9), the estimates (d), (c) of Lemma 2.6 imply that

$$U_j = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma\in[1/2,1), \\ O((\log T)^{2m+2}), & \sigma = 1 \end{cases} \qquad (5.4)$$

for $j=2,3,4$ respectively. From (5.2)–(5.4), the error term and the main term of $I_{1,1}$ correspond to those of the right-hand side of (1.10) when $\sigma\in(1/2,1]$. However, the main term of the right-hand side of (1.10) is not obtained completely when $\sigma=1/2$. In the case of $(\mu,\nu)=(2,2)$, applying (b), (c) of Lemma 2.5 and (a), (c), (d) of Lemma 2.6, we obtain

$$
\begin{aligned}
I_{2,2} &= \frac{1}{i} \sum_{\substack{n_1, n_2 < \frac{T}{\pi}, \\ n_1 \neq n_2}} \frac{\overline{\lambda_f(n_1)\varphi_2(2\pi n_1/T) n_1^{-iT}} \lambda_f(n_2)\varphi_2(2\pi n_2/T) n_2^{-iT}}{(n_1 n_2)^{\sigma}} \\
&\quad \times \frac{(\log n_1 \log n_2)^m}{\log(n_1/n_2)} + O\left(\sum_{n_1 < n_2 \leqslant \frac{T}{\pi}} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1 \log n_2)^m}{(n_1 n_2)^{\sigma} n_2 (\log(n_1/n_2))^2} \right) \\
&\quad + O\left(\sum_{n \leqslant \frac{T}{\pi}} \frac{|\lambda_f(n)|^2 (\log n)^{2m}}{n^{2\sigma - 1}} \right) \\
&= \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+1}), & \sigma = 1. \end{cases} \qquad (5.5)
\end{aligned}
$$

Next we consider the case $(\mu, \nu) = (3, 3)$. Using (2.17) and the condition $r_1 + r_2 = r$, we obtain the following formula:

$$
(\overline{\chi_f^{(m-r_1)}} \chi_f^{(m-r_2)})(s) = (2\pi)^{4\sigma-2}(-2)^{2m-r} \frac{\left(\log \frac{t}{2\pi}\right)^{2m-r}}{t^{4\sigma-2}} + M(t).
$$

where $M(t)$ is given by $M(t) = O((\log t)^{2m-r}/t^{4\sigma-1})$. Then $I_{3,3}$ is written as

$$
\begin{aligned}
I_{3,3} &= \sum_{r=0}^{2m} \sum_{r_1+r_2=r} (-1)^r \binom{m}{r_1}\binom{m}{r_2} \sum_{n_1, n_2 = 1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \\
&\quad \times \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} (\overline{\chi_f^{(m-r_1)}} \chi_f^{(m-r_2)})(s) dt \\
&= I_{3,3}^{+} + I_{3,3}^{-}, \qquad (5.6)
\end{aligned}
$$

where $I_{3,3}^{+}$, $I_{3,3}^{-}$ are defined by

$$
\begin{aligned}
I_{3,3}^{+} &:= (2\pi)^{4\sigma-2} \sum_{r=0}^{2m} (-2)^{2m-r} \sum_{r_1+r_2=r} \binom{m}{r_1}\binom{m}{r_2} \\
&\quad \times \sum_{n_1, n_2 = 1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2) (\log n_1)^{r_1} (\log n_2)^{r_2}}{(n_1 n_2)^{1-\sigma}} \\
&\quad \times \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} \frac{\left(\log \frac{t}{2\pi}\right)^{2m-r}}{t^{4\sigma-2}} dt, \\
I_{3,3}^{-} &:= \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \binom{m}{r_1}\binom{m}{r_2} \sum_{n_1, n_2 = 1}^{\infty} \frac{\overline{\lambda_f(n_1)} \lambda_f(n_2)}{(n_1 n_2)^{1-\sigma}} \\
&\quad \times (\log n_1)^{r_1} (\log n_2)^{r_2} \int_1^T \overline{\varphi_{01}\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right) \left(\frac{n_1}{n_2}\right)^{it} M(t) dt.
\end{aligned}
$$

respectively. Here we shall approximate $I_{3,3}^{+}$ and $I_{3,3}^{-}$.In order to estimate $I_{3,3}^{-}$, we use the fact that $(n_1n_2)^{1-\sigma}n_2^{4\sigma-2} = (n_1n_2)^{\sigma}(n_2/n_1)^{2\sigma-1} \gg (n_1n_2)^{\sigma}$ for $\sigma \in \mathbb{R}_{\geqslant 1/2}$ and $n_1 \leqslant n_2$. Then using (d) of Lemma 2.5 and (a), (b) of Lemma 2.6, we see that

$$
\begin{aligned}
I_{3,3}^{-} \ll & \sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 \leqslant n_2 \leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{1-\sigma}} \\
& \times \begin{cases} |\log(T/n_2)|(\log T)^{2m-r}, & \sigma = 1/2, \\ (\log n_2)^{2m-r}/n_2^{4\sigma-2}, & \sigma \in (1/2, 1] \end{cases} \ll \begin{cases} T^{2(1-\sigma)}(\log T)^{2m}, & \sigma \in [1/2, 1), \\ (\log T)^{2m+2}, & \sigma = 1. \end{cases}
\end{aligned}
\tag{5.7}
$$

The formula (a), (c) of Lemma 2.5 imply that

$$
\begin{aligned}
I_{3,3}^{+} = & \begin{cases} \dfrac{(2\pi)^{4\sigma-2}}{3-4\sigma} T^{3-4\sigma} \displaystyle\sum_{r=0}^{2m} \left(2\log \tfrac{T}{2\pi}\right)^{2m-r} \times \\ \times \displaystyle\sum_{r_1+r_2=r} \binom{m}{r_1}\binom{m}{r_2} \sum_{n \leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n)|^2(\log n)^r}{n^{2(1-\sigma)}}, & \sigma \in [1/2, 4/3), \\ 0, & \sigma \in [3/4, 1], \end{cases} \\
& + O\left(\sum_{r=0}^{2m} \sum_{n \leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n)|^2(\log n)^r}{n^{2(1-\sigma)}} \times \begin{cases} T^{3-4\sigma}(\log T)^{2m-r}, \\ \sigma \in [1/2, 3/4), \\ |\log(T/n)|(\log T)^{2m-r}, \\ \sigma = 3/4, \\ (\log T)^{2m-r}/n^{4\sigma-3}, \\ \sigma \in (3/4, 1]. \end{cases} \right) \\
& + O\left(\sum_{r=0}^{2m} \sum_{\frac{\delta}{2\pi}T < n \leqslant \frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n)|^2(\log n)^r}{n^{2(1-\sigma)}} \frac{(\log T)^{2m-r}}{n^{4\sigma-3}} \right) \\
& + \frac{(2\pi)^{4-2\sigma}}{i} \sum_{r=0}^{2m} \left(2\log \tfrac{T}{2\pi}\right)^{2m-r} \sum_{r_1+r_2=r} \binom{m}{r_1}\binom{m}{r_2} \\
& \times \sum_{\substack{n_1, n_2 \leqslant \frac{\delta_1}{2\pi}T, \\ n_1 \neq n_2}} \frac{(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{\sigma}\log(n_1/n_2)} \\
& \times \overline{\frac{\lambda_f(n_1)\varphi_{01}(2\pi n_1/T)(n_1/T)^{2\sigma-1}}{n_1^{iT}}} \, \frac{\lambda_f(n_2)\varphi_{01}(2\pi n_2/T)(n_2/T)^{2\sigma-1}}{n_2^{iT}} \\
& + O\left(\sum_{r=0}^{2m} \sum_{r_1+r_2=r} \sum_{n_1 < n_2 \leqslant \frac{\delta_1 T}{2\pi}} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{1-\sigma}} \times \right. \\
& \left. \times \frac{(\log n_2)^{2m-r}}{n_2^{4\sigma-1}(\log(n_1/n_2))^2} \right) \\
=: & \, V_1 + O(V_2) + O(V_3) + V_4 + O(V_5),
\end{aligned}
\tag{5.8}
$$

A similar discussion to U_3 gives that V_1 is approximated as

$$V_1 = \begin{cases} (A_{f,m} - C_f/(2m+1))T(\log T)^{2m+1} + O(T(\log T)^{2m}), & \sigma = 1/2, \\ O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in (1/2, 1]. \end{cases} \tag{5.9}$$

To estimate V_4 and V_5, we use the fact that

$$(n_1 n_2)^{1-\sigma} n_2^{4\sigma-1} = (n_1 n_2)^{\sigma} \qquad n_2(n_2/n_1)^{2\sigma-1} \gg (n_1 n_2)^{\sigma} n_2$$

for $\sigma \in \mathbb{R}_{\geqslant 1/2}$ and $n_1 \leqslant n_2$. Then the estimates (d), (c) of Lemma 2.6 give that

$$V_j = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1 \end{cases} \tag{5.10}$$

for $j = 4, 5$ respectively. By the fact that

$$n^{2(1-\sigma)} \gg n^{2(1-\sigma)} n^{4\sigma-3} = n^{2\sigma-1}$$

for $\sigma \in \mathbb{R}_{\leqslant 3/4}$, the estimate (b) of Lemma 2.6 when $\sigma = 3/4$ and the formula (1.9), the sum V_2 and V_3 are estimated as

$$V_j = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+1}), & \sigma = 1 \end{cases} \tag{5.11}$$

for $j = 2, 3$. Therefore, from (5.6)–(5.11) the approximate formula for $I_{3,3}$ is obtained. In the case of $(\mu, \nu) = (4, 4)$, by a similar discussion to the case of $(\mu, \nu) = (3, 3)$ the integral $I_{4,4}$ is approximated as

$$
\begin{aligned}
I_{4,4} &= O\left(\sum_{r=0}^{2m}\sum_{r_1+r_2=r}\sum_{n\leqslant\frac{T}{\pi}}\frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma-1}}\right)\\
&\quad+\frac{(2\pi)^{4-2\sigma}}{i}\sum_{r=0}^{2m}\left(2\log\tfrac{T}{2\pi}\right)^{2m-r}\sum_{r_1+r_2=r}\binom{m}{r_1}\binom{m}{r_2}\\
&\quad\times\sum_{\substack{n_1,n_2\leqslant\frac{T}{\pi},\\ n_1\neq n_2}}\frac{(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{\sigma}\log(n_1/n_2)}\\
&\quad\times\overline{\frac{\lambda_f(n_1)\varphi_{02}(2\pi n_1/T)(n_1/T)^{2\sigma-1}}{n_1^{iT}}}\,\frac{\lambda_f(n_2)\varphi_{02}(2\pi n_2/T)(n_2/T)^{2\sigma-1}}{n_2^{iT}}\\
&\quad+O\left(\sum_{r=0}^{2m}\sum_{r_1+r_2=r}\sum_{n_1<n_2\leqslant\frac{T}{\pi}}\frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{1-\sigma}}\right.\\
&\quad\left.\times\frac{(\log n_2)^{2m-r}}{n_2^{4\sigma-1}(\log(n_1/n_2))^2}\right)\\
&\quad+O\left(\sum_{r=0}^{2m}\sum_{r_1+r_2=r}\sum_{n_1\leqslant n_2\leqslant\frac{T}{\pi}}\frac{|\lambda_f(n_1)\lambda_f(n_2)|^2(\log n_1)^{r_1}(\log n_2)^{r_2}}{(n_1n_2)^{1-\sigma}}\right.\\
&\quad\left.\times\begin{cases}|\log(T/n)|(\log T)^{2m-r}, & \sigma=1/2,\\ (\log n_2)^{2m-r}/n_2^{4\sigma-2}, & \sigma\in(1/2,1]\end{cases}\right)\\
&=\begin{cases}O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma\in[1/2,1),\\ O((\log T)^{2m+2}), & \sigma=1,\end{cases}
\end{aligned}
\tag{5.12}
$$

where (b)–(d) of Lemma 2.5, the formla (1.9) and (b)–(d) of Lemma 2.6 were used. Finally we consider the case $(\mu,\nu)=(5,5)$. Remarks 3.3, 4.1 and the formula (1.4) imply that

$$
\begin{aligned}
R_\varphi(s) &\ll \sum_{\frac{t}{4\pi}\leqslant n\leqslant\frac{t}{\pi}}\frac{|\lambda_f(n)|(\log n)^m}{n^\sigma}\left(\frac{1}{|t|}+\sum_{j=2}^{l}\frac{1}{|t|^{\frac{j}{2}}}\right)+|\chi_f(s)|\sum_{r=0}^{m}\sum_{\frac{t}{4\pi}\leqslant n\leqslant\frac{t}{\pi}}1\\
&\quad\times\frac{|\lambda_f(n)|(\log n)^r}{n^{1-\sigma}}\left(\frac{(\log|t|)^{m-r}}{|t|}+\sum_{j=2}^{l}\frac{(\log|t|)^{m-r}}{|t|^{\frac{j}{2}}}\right)+\frac{(\log|t|)^m}{|t|^{\sigma-1+\frac{l}{2}}}\\
&\ll\frac{(\log t)^m}{t^\sigma}.
\end{aligned}
\tag{5.13}
$$

Hence we get

$$
I_{5,5}\ll\int_1^T\frac{(\log t)^{2m}}{t^{2\sigma}}dt\ll\begin{cases}(\log T)^{2m+1}, & \sigma=1/2,\\ 1, & \sigma\in(1/2,1].\end{cases}
\tag{5.14}
$$

Lastly we consider $I_{\mu,\nu}$ in the case of $\mu \neq \nu$. Since $I_{1,1}$ contains the main term of the mean value formula for $L_f^{(m)}(s)$, and Cauchy's inequality implies that $|I_{\mu,\nu}| \leqslant I_{\mu,\mu} I_{\nu,\nu}$ for $\mu, \nu \in \{1, \ldots, 5\}$, it follows that it is enough to consider $I_{\mu,\nu}$ in the case of $(\mu,\nu) = (1,2)$, $(1,3)$, $(1,4)$, $(1,5)$. First in the case of $(\mu,\nu) = (1,2)$, using (b), (c) of Lemma 2.5, (c), (d) of Lemma 2.6 and the estimate (5.3), we obtain

$$
\begin{aligned}
I_{1,2} &= \sum_{n_1,n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)}\lambda_f(n_2)(\log n_1)^m(\log n_2)^m}{(n_1n_2)^\sigma} \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)} \varphi_2\left(\frac{2\pi n_2}{t}\right)\left(\frac{n_1}{n_2}\right)^{it} dt \\
&= \frac{1}{i} \sum_{\substack{n_1,n_2<\frac{T}{\pi},\\ n_1\neq n_2}} \frac{\overline{\lambda_f(n_1)\varphi_1(2\pi n_1/T)n_1^{-iT}}\lambda_f(n_2)\varphi_2(2\pi n_2/T)n_2^{-iT}}{(n_1n_2)^\sigma} \\
&\quad \times \frac{(\log n_1 \log n_2)^m}{\log(n_1/n_2)} + O\left(\sum_{n_1<n_2\leqslant\frac{T}{\pi}} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1 \log n_2)^m}{(n_1n_2)^\sigma n_2(\log(n_1/n_2))^2}\right) \\
&\quad + O\left(\sum_{n<\frac{T}{\pi}} \frac{|\lambda_f(n)|^2(\log n)^{2m}}{n^{2\sigma-1}}\right) \\
&= \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2,1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases}
\end{aligned}
\tag{5.15}
$$

Next we consider the case $(\mu,\nu) = (1,3)$. From (e) of Lemma 2.5 and (a), (b) of Lemma 2.6, the integral $I_{1,3}$ is estimated as

$$
\begin{aligned}
I_{1,3} &= \sum_{r=0}^{m}(-1)^m\binom{m}{r} \sum_{n_1,n_2=1}^{\infty} \frac{\overline{\lambda_f(n_1)}\lambda_f(n_2)(\log n_1)^m(\log n_2)^m}{n_1^\sigma n_2^{1-\sigma}} \\
&\quad \times \int_1^T \overline{\varphi_1\left(\frac{2\pi n_1}{t}\right)} \varphi_{01}\left(\frac{2\pi n_2}{t}\right)(n_1n_2)^{it}\chi_f^{(m-r)}(s)dt \\
&= O\left(\sum_{r=0}^{2m}\sum_{r_1+r_2=r}\sum_{n_1\leqslant n_2\leqslant\frac{\delta_1}{2\pi}T} \frac{|\lambda_f(n_1)\lambda_f(n_2)|(\log n_1)^{r_1}(\log n_2)^{r_2}}{n_1^\sigma n_2^{1-\sigma}}\right. \\
&\quad \left. \times \begin{cases} |\log(T/n_2)|(\log T)^{2m-r}, & \sigma = 1/2, \\ (\log n_2)^{2m-r}/n_2^{2\sigma-1}, & \sigma \in (1/2,1] \end{cases}\right) \\
&= \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2,1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases}
\end{aligned}
\tag{5.16}
$$

In the case of $(\mu,\nu) = (1,4)$, a similar discussion to the case of $(\mu,\nu) = (1,3)$ gives that

$$I_{1,4} = \begin{cases} O(T^{2(1-\sigma)}(\log T)^{2m}), & \sigma \in [1/2, 1), \\ O((\log T)^{2m+2}), & \sigma = 1. \end{cases} \tag{5.17}$$

Finally we consider the case $(\mu, \nu) = (1, 5)$. The formula (1.9) and Cauchy's inequality imply that $\sum_{n \leqslant x} |\lambda_f(n)| = O(x)$. Then using the estimate (5.13) and partial summation we get

$$\begin{aligned} I_{1,5} &\ll \int_1^T \frac{(\log t)^m}{t^\sigma} \sum_{n \leqslant \frac{\delta_1}{2\pi} t} \frac{|\lambda_f(n)|(\log n)^m}{n^\sigma} dt \\ &\ll \int_1^T \frac{(\log t)^m}{t^\sigma} \begin{cases} t^{1-\sigma}(\log t)^m, & \sigma \in [1/2, 1), \\ (\log t)^{m+1}, & \sigma = 1 \end{cases} dt \\ &\ll \begin{cases} T^{2(1-\sigma)}(\log T)^{2m}, & \sigma \in [1/2, 1), \\ (\log T)^{2m+2}, & \sigma = 1. \end{cases} \end{aligned} \tag{5.18}$$

Therefore combining (5.1)–(5.18), we complete the proof of Theorem 1.3.

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