Functiones et Approximatio
51.2 (2014), 379–391
doi: 10.7169/facm/2014.51.2.9

EXPLICIT RELATIONS BETWEEN PRIMES IN SHORT INTERVALS AND EXPONENTIAL SUMS OVER PRIMES

Alessandro Languasco, Alessandro Zaccagnini

Abstract: Under the assumption of the Riemann Hypothesis, we prove explicit quantitative relations between hypothetical error terms in the asymptotic formulae for truncated mean-square average of exponential sums over primes and in the mean-square of primes in short intervals. We also remark that such relations are connected with a more precise form of Montgomery's pair-correlation conjecture.

Keywords: exponential sum over primes, primes in short intervals, pair-correlation conjecture.

1. Introduction

In many circle-method applications a key role is played by the asymptotic behavior as $X \to \infty$ of the truncated mean square of the exponential sum over primes, i.e. by

$$R(X,\xi) = \int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \,\mathrm{d}\alpha, \qquad \frac{1}{2X} \leqslant \xi \leqslant \frac{1}{2},$$

where $S(\alpha) = \sum_{n\leqslant X} \Lambda(n) e(n\alpha)$, $T(\alpha) = \sum_{n\leqslant X} e(n\alpha)$, $e(x) = e^{2\pi i x}$ and $\Lambda(n)$ is the von Mangoldt function. In 2000 the first author and Perelli [6] studied how to connect, under the assumption of the Riemann Hypothesis (RH) and of Montgomery's pair-correlation conjecture, the behavior as $X \to \infty$ of $R(X,\xi)$ with the one of the mean-square of primes in short intervals, *i.e.*, with

$$J(X,h) = \int_{1}^{X} (\psi(x+h) - \psi(x) - h)^2 \,\mathrm{d}x, \qquad 1 \leqslant h \leqslant X,$$

where $\psi(x) = \sum_{n\leqslant x} \Lambda(n)$. Recalling that Goldston and Montgomery [2] proved that the asymptotic behavior of $J(X,h)$ as $X \to \infty$ is related with Montgomery's pair-correlation function

$$F(X,T) = 4 \sum_{0<\gamma,\gamma'\leqslant T} \frac{X^{i(\gamma-\gamma')}}{4 + (\gamma-\gamma')^2},$$

2010 Mathematics Subject Classification: primary: 11M26; secondary: 11N05, 11M45

where γ, γ' run over the imaginary part of the non-trivial zeros of the Riemann zeta-function, the following result was proved in [6].

Theorem. *Assume RH. As $X \to \infty$, the following statements are equivalent:*

(i) *for every $\varepsilon > 0$, $R(X,\xi) \sim 2X\xi \log X\xi$ uniformly for $X^{-1/2+\varepsilon} \leqslant \xi \leqslant 1/2$;*
(ii) *for every $\varepsilon > 0$, $J(X,h) \sim hX \log(X/h)$ uniformly for $1 \leqslant h \leqslant X^{1/2-\varepsilon}$;*
(iii) *for every $\varepsilon > 0$ and $A \geqslant 1$, $F(X,T) \sim (T/2\pi) \log \min(X,T)$ uniformly for $X^{1/2+\varepsilon} \leqslant T \leqslant X^A$.*

We remark that the uniformity ranges in the previous statement are smaller than the ones in [2]: this is due to the presence of a term $E(X,h)$ which arises from the estimation of some very short integrals naturally arising in applying Gallagher's lemma. In particular in [6] it is proved, for every fixed $\varepsilon > 0$, that

$$E(X,h) \ll \begin{cases} (h+1)^3 (\log X)^2 & \text{(uncond.) for } 0 < h \leqslant X^\varepsilon \\ h^3 & \text{(uncond.) for } X^\varepsilon \leqslant h \leqslant X \\ (h+1) X (\log X)^4 & \text{(under RH) for } 0 < h \leqslant X. \end{cases} \tag{1}$$

Hence it is clear that the above-mentioned limitation in the uniformity ranges comes from the fact that for $h > X^{1/2-\varepsilon}$ the estimates in (1) are too large if compared with the expected main term for $J(X,h)$. In Theorem 1 below we will see that $E(X,h)$ plays an important role here too.

In 2003 Chan [1] formulated a more precise pair-correlation hypothesis and gave explicit results for the connections between the error terms in the asymptotic formulae for $F(X,T)$ and $J(X,h)$. Such results were recently extended and improved by the authors of this paper in a joint work with Perelli [7]: writing

$$F(X,T) = \frac{T}{2\pi}\Bigl(\log \frac{T}{2\pi} - 1\Bigr) + R_F(X,T), \tag{2}$$

$$J(X,h) = hX\Bigl(\log \frac{X}{h} + c'\Bigr) + R_J(X,h) \tag{3}$$

and

$$c' = -\gamma - \log(2\pi) \tag{4}$$

(γ is Euler's constant), they gave explicit relations between (2), (3) and error terms essentially of type

$$R_F(X,T) \ll \frac{T^{1-a}}{(\log T)^b} \qquad \text{and} \qquad R_J(X,h) \ll \frac{hX}{(\log X)^b}\Bigl(\frac{h}{X}\Bigr)^a,$$

with X, T and h in suitable ranges and $a, b \geqslant 0$. According to the heuristics in Montgomery-Soundararajan [9] (see p.511) it appears that such bounds are both reasonable ones if $0 \leqslant a \leqslant 1/2 - \varepsilon$, $b \geqslant 0$ and, respectively, uniformly for $T^{1+\varepsilon} \leqslant X \leqslant T^A$ and $X^\varepsilon \leqslant h \leqslant X^{1-\varepsilon}$.

Our aim here is to investigate the connections between (2)-(3) with an asymptotic formula of the type

$$R(X,\xi) = 2X\xi \log X\xi + cX\xi + W(X,\xi), \tag{5}$$

say, where the expected value for c is given by

$$c = 2(c' - 2 + \gamma + \log(2\pi)) \tag{6}$$

(which, by (4), gives $c = -4$), and to prove explicit connections between the error terms involved. The heuristics in [9] suggests that a reasonable estimate should be

$$W(X, \xi) \ll \frac{(X\xi)^{1-a}}{(\log X\xi)^b}, \tag{7}$$

with $0 \leqslant a \leqslant 1/2 - \varepsilon,\ b \geqslant 0$, uniformly for $X^{-1+\varepsilon} \leqslant \xi \leqslant X^{-\varepsilon}$. Unfortunately the presence of the above-mentioned term $E(X, h)$ forces us, as in [6], to restrict our attention to the range $X^{-1/2+\varepsilon} \leqslant \xi \leqslant 1/2$ (or, equivalently, to $1 \leqslant h \leqslant X^{1/2-\varepsilon}$).

In what follows the implicit constants may depend on a, b. Our first result is

Theorem 1. *Assume RH and let $1 \leqslant h \leqslant X^{1/2-\varepsilon}$, $X^{-1/2+\varepsilon} \leqslant \xi \leqslant 1/2$. Let further $0 \leqslant a < 1$, $b \geqslant 0$, $(a, b) \neq (0, 0)$ be fixed. If (7) holds uniformly for*

$$\frac{1}{h}\Big(\frac{h}{X}\Big)^a (\log X)^{-b-4} \leqslant \xi \leqslant \frac{1}{h}\Big(\frac{X}{h}\Big)^a (\log X)^{b+4}, \tag{8}$$

then

$$R_J(X, h) \ll X + E(X, h) + R_{a,b}(X, h)$$

uniformly for

$$X\Big(\frac{1}{X\xi(\log X)^{b+4}}\Big)^{1/(1-a)} \leqslant h \leqslant X\Big(\frac{(\log X)^{b+4}}{X\xi}\Big)^{1/(a+1)},$$

where $E(X, h)$ is defined in (1), and

$$R_{a,b}(X, h) = \begin{cases} hX \log\log X (\log X)^{-b} & \text{if } a = 0 \\ hX\big(h/X\big)^a (\log X)^{-b} & \text{if } a > 0. \end{cases} \tag{9}$$

We explicitly remark that the conditions $\xi \leqslant 1/2$ and (8) imply

$$h \gg X^{a/(a+1)} (\log X)^{(b+4)/(a+1)}$$

which also leads to $R_{a,b}(X, h) \gg X$. It is also useful to remark that $E(X, h) \ll R_{a,b}(X, h)$ only for $h \ll X^{(1-a)/(2+a)} (\log X)^{-b/(2+a)}$.

The technique used to prove Theorem 1 is similar to the one in Lemma 2 in [7]; the main difference is in the presence of the terms $E(X, h)$ (which comes from Lemma 3) and $\mathcal{O}(X)$ (which comes from the term $\mathcal{O}(1)$ in (12)). We further remark that eq. (12) of Lemma 1 is directly connected to the ability of detecting the second order term in (5) and to establish the relation (4), which leads to the expected value of c in (5)-(6).

Concerning the opposite direction, we have

Theorem 2. *Assume RH and let* $1 \leqslant h \leqslant X^{1/2-\varepsilon}$, $X^{-1/2+\varepsilon} \leqslant \xi \leqslant 1/2$. *Let further* $0 \leqslant a < 1$, $b \geqslant 0$, $(a,b) \neq (0,0)$ *be fixed. If we have*

$$R_J(X,h) \ll \frac{hX}{(\log X)^b}\Big(\frac{h}{X}\Big)^a$$

uniformly for

$$\frac{1}{\xi}\frac{(X\xi)^{-a/(2a+6)}}{(\log X)^{(a+b+4)/(2a+6)}} \leqslant h \leqslant \frac{1}{\xi}(X\xi)^{4a/(a+3)}(\log X)^{(3a+4b+13)/(a+3)},$$

then

$$W(X,\xi) \ll \frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}, \tag{10}$$

uniformly for

$$\frac{1}{h}\Big(\frac{h}{X}\Big)^{a/(3a+6)}(\log X)^{-(a+b+4)/(3a+6)} \leqslant \xi \leqslant \frac{1}{h}\Big(\frac{X}{h}\Big)^{4a/(3-3a)}(\log X)^{(3a+4b+13)/(3-3a)}.$$

Note that for $a = 0$ we have to take $b > 2$ to get that the error term in (10) is $o(X\xi)$. The technique used to prove Theorem 2 is similar to the one in Lemma 5 of [7]; the main difference is in the use of Lemma 4 which is needed to provide pair-correlation independent estimates of the involved quantities.

We remark that results similar to Theorems 1-2 can be proved for the weighted quantities

$$\begin{aligned}
\widetilde{S}(\alpha) &= \sum_{n=1}^{\infty} \Lambda(n) e^{-n/X} e(n\alpha),\\
\widetilde{T}(\alpha) &= \sum_{n=1}^{\infty} e^{-n/X} e(n\alpha),\\
\widetilde{R}(X,\xi) &= \int_{-\xi}^{\xi} |\widetilde{S}(\alpha) - \widetilde{T}(\alpha)|^2 \,\mathrm{d}\alpha,\\
\widetilde{J}(X,h) &= \int_0^{\infty} (\psi(x+h) - \psi(x) - h)^2 \, e^{-2x/X} \,\mathrm{d}x.
\end{aligned}$$

The proofs are similar; in the analogue of Theorem 1 the main difference is in using the second part of Lemma 3 thus replacing $E(X,h)$ with the sharper quantity $\widetilde{E}(X,h)$ defined in (15). Concerning the analogue of Theorem 2, the key point is in Eq. (33): in this case we will be able to extend its range of validity to $\xi \leqslant x \leqslant \xi X^{1-\varepsilon}$ and to get rid of the term $(x^3/\xi)(\log X)^2$. These remarks lead to results which hold in wider ranges: $1 \leqslant h \leqslant X^{1-\varepsilon}$ and $X^{-1+\varepsilon} \leqslant \xi \leqslant 1/2$.

The order of magnitude of $\widetilde{J}(X,h)$ can be directly deduced from the one of $J(X,h)$ via partial integration, see *e.g.* eq. (18). Unfortunately, the vice-versa seems to be very hard to achieve; this depends on the fact that we do not have sufficiently strong Tauberian theorems to get rid of the exponential weight in the definition of $\widetilde{J}(X,h)$. Such a phenomenon is well known in the literature, see, *e.g.*, Heath-Brown's remark on pages 385-386 of [4].

Acknowledgments. We wish to thank the referee for pointing out inaccuracies and suggesting improvements at several places.

2. Some lemmas

In the following we will need two weight functions and, in particular, precise information on their total mass and size of the derivatives. For $h > 0$ we let

$$K(\alpha,h) = \sum_{-h\leqslant n\leqslant h} (h-|n|)\, e(n\alpha) \qquad \text{and} \qquad U(\alpha,h) = \Big(\frac{\sin(\pi h\alpha)}{\pi\alpha}\Big)^2. \tag{11}$$

Lemma 1. *For $h>0$, we have $\int_0^{1/2} K(\alpha,h)\,\mathrm{d}\alpha = h/2$ and $\int_0^{+\infty} U(\alpha,h)\,\mathrm{d}\alpha = h/2$. Moreover we also have*

$$\int_0^{1/2} \log(h\alpha)\, K(\alpha,h)\,\mathrm{d}\alpha = -\frac{h}{2}(\log(2\pi)+\gamma-1) + \mathcal{O}(1), \tag{12}$$
$$\int_0^{+\infty} \log(h\alpha)\, U(\alpha,h)\,\mathrm{d}\alpha = -\frac{h}{2}(\log(2\pi)+\gamma-1).$$

Before the proof, we remark that this lemma is consistent with the constant in Lemma 2 of Languasco, Perelli and Zaccagnini [7], taking into account the fact that our variable h here corresponds to $\pi\kappa$ there.

Proof. The results on $U(\alpha,h)$ can be immediately obtained by integrals n.3.821.9 and n.4.423.3, respectively on pages 460 and 594 of Gradshteyn and Ryzhik [3]. The first identity for $K(\alpha,h)$ immediately follows by isolating the contribution of $n=0$ in its definition and making a trivial computation. Now we prove (12). Separating again the contribution of the term $n=0$, a straightforward computation gives

$$\begin{aligned} I(h) &:= 2\int_0^{1/2} \log(h\alpha)\, K(\alpha,h)\,\mathrm{d}\alpha \\ &= h\log h - h(\log 2+1) + 2\sum_{1\leqslant n\leqslant h} (h-n)\int_0^1 \log\Big(\frac{h\beta}{2}\Big)\cos(\pi n\beta)\,\mathrm{d}\beta. \end{aligned}$$

A standard argument lets us write

$$
\begin{aligned}
I(h) &= h\log h - h(\log 2+1) + 2\sum_{1\leqslant n\leqslant h}(h-n)\int_0^1 \log\beta\,\cos(\pi n\beta)\,\mathrm{d}\beta \\
&= h\log h - h(\log 2+1) - \sum_{1\leqslant n\leqslant h}\frac{h-n}{n} - 2\sum_{1\leqslant n\leqslant h}(h-n)\frac{\mathrm{si}(\pi n)}{\pi n},
\end{aligned}
$$

by Formula 4.381.2 on page 581 of [3], where the sine integral function is defined by

$$
\mathrm{si}(x) = -\int_x^{+\infty}\frac{\sin t}{t}\,\mathrm{d}t \tag{13}
$$

for $x>0$. The elementary relation $\sum_{1\leqslant n\leqslant h} 1/n = \log h + \gamma + \mathcal{O}\bigl(h^{-1}\bigr)$ shows that

$$
I(h) = -h(\log 2+\gamma) + \mathcal{O}(1) - \frac{2h}{\pi}\sum_{1\leqslant n\leqslant h}\frac{\mathrm{si}(\pi n)}{n} + \frac{2}{\pi}\sum_{1\leqslant n\leqslant h}\mathrm{si}(\pi n).
$$

Finally we remark that Eq. (13) implies, by means of a simple integration by parts, that $\mathrm{si}(x)\ll x^{-1}$ as $x\to+\infty$. Hence

$$
\sum_{1\leqslant n\leqslant h}\frac{\mathrm{si}(\pi n)}{n} = \sum_{n\geqslant 1}\frac{\mathrm{si}(\pi n)}{n} + \mathcal{O}\bigl(h^{-1}\bigr) = \frac{\pi}{2}(\log\pi - 1) + \mathcal{O}\bigl(h^{-1}\bigr),
$$

by Formula 6.15.2 on page 154 of [10]. Moreover, by a double partial integration in (13) we get

$$
\sum_{1\leqslant n\leqslant h}\mathrm{si}(\pi n) = \sum_{1\leqslant n\leqslant h}\frac{(-1)^{n+1}}{\pi n} + \mathcal{O}\Bigl(\sum_{1\leqslant n\leqslant h}\frac{1}{n^2}\Bigr) \ll 1.
$$

In conclusion

$$
I(h) = -h(\log 2+\gamma) - \frac{2h}{\pi}\Bigl(\frac{\pi}{2}(\log\pi - 1) + \mathcal{O}\bigl(h^{-1}\bigr)\Bigr) + \mathcal{O}(1),
$$

and Lemma 1 is proved. ■

Lemma 2. *For $h\geqslant 1$ we have*

$$
K(\alpha,h) \ll \min\Bigl(h^2, \|\alpha\|^{-2}\Bigr),
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha}K(\alpha,h) \ll h\|\alpha\|\min\Bigl(h^3, \|\alpha\|^{-3}\Bigr).
$$

The proof of Lemma 2 is standard and hence we omit it. We also remark that estimates similar to the ones in Lemma 2 hold for $U(\alpha,h)$ too; since they immediately follow from the definition we omit their proofs too.

We need the following auxiliary result which is based on Gallagher's lemma.

Lemma 3. *Let* $1 \leqslant h \leqslant X$,

$$R(\alpha) = S(\alpha) - T(\alpha) \qquad \text{and} \qquad \widetilde{R}(\alpha) = \widetilde{S}(\alpha) - \widetilde{T}(\alpha). \tag{14}$$

Then

$$\int_{-1/2}^{1/2} |R(\alpha)|^2 K(\alpha,h)\,\mathrm{d}\alpha = \int_{-\infty}^{+\infty} |R(\alpha)|^2 U(\alpha,h)\,\mathrm{d}\alpha = J(X,h) + \mathcal{O}(E(X,h)),$$

where $E(X,h)$ *is defined in* (1). *Moreover we have,*

$$\int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2 K(\alpha,h)\,\mathrm{d}\alpha = \int_{-\infty}^{+\infty} |\widetilde{R}(\alpha)|^2 U(\alpha,h)\,\mathrm{d}\alpha = \widetilde{J}(X,h) + \mathcal{O}\big(\widetilde{E}(X,h)\big),$$

where, for every fixed $\varepsilon > 0$, *we define*

$$\widetilde{E}(X,h) = \begin{cases} (h+1)^3(\log X)^2 & \text{(uncond.) for } 0 < h \leqslant X^{\varepsilon} \\ h^3 & \text{(uncond.) for } X^{\varepsilon} < h \leqslant X \\ (h+1)^2(\log X)^4 & \text{(under RH) for } 0 < h \leqslant X. \end{cases} \tag{15}$$

Proof. The first part is Lemma 1 of [6], so we skip the proof. For the second part, we start arguing as in Lemma 1 of [6] thus getting

$$\begin{aligned} \int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2\ K(\alpha,h)\,\mathrm{d}\alpha &= \int_{-\infty}^{+\infty} |\widetilde{R}(\alpha)|^2 U(\alpha,h)\,\mathrm{d}\alpha \\ &= \int_{-\infty}^{+\infty} \Big| \sum_{\substack{|n-x|<h/2 \\ n\geqslant 1}} (\Lambda(n)-1)e^{-n/X} \Big|^2\ \mathrm{d}x. \end{aligned}$$

A standard computation hence gives

$$\begin{aligned} \int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2\ K(\alpha,h)\,\mathrm{d}\alpha &= \int_{0}^{+\infty} \Big| \sum_{x<n\leqslant x+h} (\Lambda(n)-1)e^{-n/X} \Big|^2\ \mathrm{d}x \\ &\quad + \mathcal{O}\big((h+1)^2(\log(h+1))^4\big), \end{aligned} \tag{16}$$

where in the last estimate we assumed RH and we used the asymptotic formula

$$\psi(y) = y + \mathcal{O}\Big(y^{1/2}(\log y)^2\Big) \tag{17}$$

on a interval of length $\leqslant h$. Noting that

$$\sum_{x<n\leqslant x+h} (\Lambda(n)-1)e^{-n/X} = e^{-x/X}(\psi(x+h)-\psi(x)-h)\Big(1+\mathcal{O}\Big(\frac{h+1}{X}\Big)\Big)$$

and recalling that $h \leqslant X$, from (16) we have

$$\int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2 \, K(\alpha, h) \, \mathrm{d}\alpha = \widetilde{J}(X, h)\Big(1 + \mathcal{O}\Big(\frac{h+1}{X}\Big)\Big) + \mathcal{O}\big((h+1)^2 (\log X)^4\big).$$

To estimate the last error term we connect $\widetilde{J}(X, h)$ to $J(X, h)$. A partial integration immediately gives

$$\widetilde{J}(X, h) = \frac{2}{X} \int_0^\infty J(t, h) e^{-2t/X} \, \mathrm{d}t. \tag{18}$$

Splitting the range of integration on the right-hand side of (18) into $[0, h] \cup [h, +\infty)$, a direct computation using (17) shows that $\int_0^h J(t, h) e^{-2t/X} \, \mathrm{d}t \ll h^3 (\log h)^4$ while, still assuming RH, in the remaining range we use the Selberg [11] estimate

$$J(t, h) \ll ht(\log t)^2 \qquad \text{for } \ 1 \leqslant h \leqslant t, \tag{19}$$

and so we get

$$\int_h^{+\infty} J(t, h) e^{-2t/X} \, \mathrm{d}t \ll h \int_h^{+\infty} t(\log t)^2 \; e^{-2t/X} \, \mathrm{d}t \ll hX^2 (\log X)^2.$$

Summing up, under RH we have

$$\widetilde{J}(X, h) \ll (h+1) X (\log X)^4$$

and we can finally write

$$\int_{-1/2}^{1/2} |\widetilde{R}(\alpha)|^2 \, K(\alpha, h) \, \mathrm{d}\alpha = \widetilde{J}(X, h) + \mathcal{O}\big((h+1)^2 (\log X)^4\big).$$

The unconditional cases follow by replacing (17) with the Brun-Titchmarsh inequality and (19) with the estimate $J(t, h) \ll h^2 t + ht \log t$ (see the Lemma in [5]). ∎

In the next sections we will also need the following remark. Let $\xi > 0$ and $\delta\xi = 1/2$. In this case $U(\alpha, \delta) \gg \delta^2$ for $|\alpha| \leqslant \xi$; hence by the first equation in Lemma 3 we obtain

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 \, \mathrm{d}\alpha \ll \xi^2 \Big(J\Big(X, \frac{1}{2\xi}\Big) + E\Big(X, \frac{1}{2\xi}\Big)\Big).$$

By (19) and (1), under RH we immediately obtain, for every $1/(2X) \leqslant \xi \leqslant 1/2$, that

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 \, \mathrm{d}\alpha \ll X\xi (\log X)^4. \tag{20}$$

3. Proof of Theorem 1

We use Lemma 3 in the form

$$J(X,h) = \int_{-1/2}^{1/2} |R(\alpha)|^2 \, K(\alpha,h) \, \mathrm{d}\alpha + \mathcal{O}(E(X,h)), \tag{21}$$

where $R(\alpha)$ is defined in (14). Observe that both $|R(\alpha)|^2$ and $K(\alpha,h)$ are even functions of α, and hence we may restrict our attention to $\alpha \in [0,1/2]$. Recalling (6) and writing

$$f(X,\alpha) = X\log(X\alpha) + \Bigl(\frac{c}{2}+1\Bigr)X = X\log\frac{X}{h} + X\log(h\alpha) + \Bigl(\frac{c}{2}+1\Bigr)X, \tag{22}$$

we can approximate $|R(\alpha)|^2$ as $|R(\alpha)|^2 = f(X,\alpha) + \bigl(|R(\alpha)|^2 - f(X,\alpha)\bigr)$. Using Lemma 1 and (22), we obtain

$$\int_0^{1/2} f(X,\alpha)K(\alpha,h)\,\mathrm{d}\alpha = \frac{h}{2}X\log\frac{X}{h} + c'\frac{h}{2}X + \mathcal{O}(X), \tag{23}$$

where c' is defined in (4).

Let now $U_1 < 1/h < U_2 \leqslant 1$ be two parameters to be chosen later. By Lemma 2, (20) and a partial integration we immediately obtain

$$\Bigl(\int_0^{U_1} + \int_{U_2}^{1/2}\Bigr)\bigl(|R(\alpha)|^2 - f(X,\alpha)\bigr)K(\alpha,h)\,\mathrm{d}\alpha \ll h^2U_1X(\log X)^4 + \frac{X(\log X)^4}{U_2}. \tag{24}$$

From (24) it is clear that the optimal choice is $h^2U_1 = 1/U_2$. We now evaluate the integral over $[U_1,U_2]$. A direct computation and the hypothesis show that

$$\int_0^{\xi} \bigl(|R(\alpha)|^2 - f(X,\alpha)\bigr)\,\mathrm{d}\alpha \ll \frac{(X\xi)^{1-a}}{(\log X\xi)^b},$$

and hence, by partial integration and Lemma 2, we obtain

$$\begin{aligned}\int_{U_1}^{U_2} \bigl(|R(\alpha)|^2 - f(X,\alpha)\bigr)K(\alpha,h)\,\mathrm{d}\alpha &\ll h^2\frac{(XU_1)^{1-a}}{(\log X)^b} + \frac{X^{1-a}U_2^{-1-a}}{(\log X)^b} \\ &\quad + \frac{hX^{1-a}}{(\log X)^b}\int_{U_1}^{U_2} \xi^{2-a}\min\bigl(h^3,\xi^{-3}\bigr)\,\mathrm{d}\xi.\end{aligned}$$

Using the constraints $h^2U_1 = 1/U_2$ and $U_1 < 1/h$, the right-hand side is

$$\ll \frac{h^{1+a}X^{1-a}}{(\log X)^b} + \frac{hX^{1-a}}{(\log X)^b}\int_{1/h}^{U_2} \xi^{-1-a}\,\mathrm{d}\xi \ll R_{a,b}(X,h,U_2), \tag{25}$$

where

$$R_{a,b}(X,h,U_2) = \begin{cases} hX\log(hU_2)(\log X)^{-b} & \text{if } a=0 \\ h^{1+a}X^{1-a}\,(\log X)^{-b} & \text{if } a>0. \end{cases}$$

Hence, by (24)-(25) and $h^2 U_1 = 1/U_2$ we get

$$\int_0^{1/2} \bigl(|R(\alpha)|^2 - f(X,\alpha)\bigr) K(\alpha,h)\,\mathrm{d}\alpha \ll \frac{X(\log X)^4}{U_2} + R_{a,b}(X,h,U_2). \tag{26}$$

Choosing

$$U_2 = \frac{X^a(\log X)^{b+4}}{h^{1+a}} \qquad \text{and} \qquad U_1 = \frac{h^{a-1}}{X^a(\log X)^{b+4}},$$

by (23) and (26) we finally get

$$\int_0^{1/2} |R(\alpha)|^2\, K(\alpha,h)\,\mathrm{d}\alpha = \frac{h}{2} X \log\frac{X}{h} + c'\frac{h}{2}X + \mathcal{O}(X + R_{a,b}(X,h))$$

where c' and $R_{a,b}(X,h)$ are defined in (4) and (9). Theorem 1 follows from (21).

4. Proof of Theorem 2

We adapt the proof of Lemma 5 of [7], which is an explicit form of Lemma 4 of [2]. We recall that $0 < \eta < 1/4$ is a parameter to be chosen later and

$$K_\eta(x) = \frac{\sin(2\pi x) + \sin(2\pi(1+\eta)x)}{2\pi x(1-4\eta^2x^2)},$$

so that its Fourier transform becomes

$$\widehat{K}_\eta(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1 \\ \cos^2\Bigl(\dfrac{\pi(|t|-1)}{2\eta}\Bigr) & \text{if } 1 \leqslant |t| \leqslant 1+\eta \\ 0 & \text{if } |t| \geqslant 1+\eta \end{cases}$$

and

$$K''_\eta(x) \ll \min(1; (\eta x)^{-3}), \tag{27}$$

see Eqs. (3.14)-(3.15) and Lemma 4 of [7]. Moreover, by Lemma 3 of [7], we also have

$$\widehat{K}_\eta(t) = \int_0^\infty K''_\eta(x) U(t,x)\,\mathrm{d}x. \tag{28}$$

Hence, again considering only positive values of α, we have

$$\int_0^\infty |R(\alpha)|^2\, \widehat{K}_\eta\Bigl(\frac{\alpha}{\xi}(1+\eta)\Bigr)\,\mathrm{d}\alpha \leqslant \frac{R(X,\xi)}{2} \leqslant \int_0^\infty |R(\alpha)|^2\, \widehat{K}_\eta\Bigl(\frac{\alpha}{\xi}\Bigr)\,\mathrm{d}\alpha \tag{29}$$

where $R(\alpha)$ is defined in (14). Writing $f(X,\alpha)$ as in (22), we approximate $|R(\alpha)|^2$ as $|R(\alpha)|^2 = f(X,\alpha) + (|R(\alpha)|^2 - f(X,\alpha))$. Observing that $U(\alpha/\xi, x) = \xi^2 U(\alpha, x/\xi)$, letting

$$g(x,\xi) = \xi^2 \int_0^\infty (|R(\alpha)|^2 - f(X,\alpha)) U\Bigl(\alpha, \frac{x}{\xi}\Bigr)\,\mathrm{d}\alpha$$

and using (28), we get

$$\int_0^{\infty} |R(\alpha)|^2\, \widehat{K}_\eta\Big(\frac{\alpha}{\xi}\Big)\,\mathrm{d}\alpha = \int_0^{\infty} f(X,\alpha)\widehat{K}_\eta\Big(\frac{\alpha}{\xi}\Big)\,\mathrm{d}\alpha + \int_0^{\infty} K_\eta''(x) g(x,\xi)\,\mathrm{d}x = J_1 + J_2, \tag{30}$$

say. A direct computation and (6) show that

$$J_1 = X\xi \log X\xi + \frac{c}{2} X\xi + \mathcal{O}(\eta X\xi \log X\xi). \tag{31}$$

In order to estimate J_2 we first remark that by Lemma 1, (22) and (4), we have

$$\xi^2 \int_0^{\infty} f(X,\alpha) U\Big(\alpha, \frac{x}{\xi}\Big)\,\mathrm{d}\alpha = \frac{xX\xi}{2} \log \frac{X\xi}{x} + \frac{c'}{2} xX\xi. \tag{32}$$

Now we need the following Lemma whose proof follows the line of Lemma 2 of [6].

Lemma 4. *Assume RH and let $\varepsilon > 0$. We have*

$$g(x,\xi) \ll \begin{cases} X\xi^2 \log X & \text{if } 0 < x \leqslant \xi \\ xX\xi(\log X)^2 & \text{if } \xi \leqslant x \leqslant \xi X^{1/2-\varepsilon} \\ xX\xi(\log X)^4 & \text{if } x \geqslant \xi X^{1/2-\varepsilon}. \end{cases}$$

Assume further the hypothesis of Theorem 2. We have

$$g(x,\xi) \ll x^{1+a}\frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi}(\log X)^2 \qquad \text{if } \xi \leqslant x \leqslant \xi X^{1/2-\varepsilon}. \tag{33}$$

Choosing now V_1, V_2 such that $\xi < V_1 < 1/\eta < V_2 < \xi X^{1/2-\varepsilon}$, we split J_2's integration range into six subintervals. We obtain

$$\begin{aligned} J_2 &= \Big(\int_0^{\xi} + \int_{\xi}^{V_1} + \int_{V_1}^{1/\eta} + \int_{1/\eta}^{V_2} + \int_{V_2}^{\xi X^{1/2-\varepsilon}} + \int_{\xi X^{1/2-\varepsilon}}^{+\infty}\Big) K_\eta''(x) g(x,\xi)\,\mathrm{d}x \\ &= M_1 + M_2 + M_3 + M_4 + M_5 + M_6, \end{aligned} \tag{34}$$

say. By Lemma 4 and (27), we obtain

$$\begin{aligned} M_1 &\ll X\xi^2 \log X \int_0^{\xi} \mathrm{d}x \ll X\xi^3 \log X, \\ M_2 &\ll X\xi(\log X)^2 \int_{\xi}^{V_1} x\,\mathrm{d}x \ll X\xi V_1^2 (\log X)^2, \\ M_3 &\ll \int_{V_1}^{1/\eta} \Big(x^{1+a}\frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi}(\log X)^2\Big)\,\mathrm{d}x \ll \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b} + \frac{(\log X)^2}{\xi\eta^4}, \\ M_4 &\ll \frac{1}{\eta^3}\int_{1/\eta}^{V_2} \Big(x^{a-2}\frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{(\log X)^2}{\xi}\Big)\,\mathrm{d}x \ll \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b} + \frac{V_2(\log X)^2}{\xi\eta^3}, \\ M_5 &\ll \frac{X\xi(\log X)^2}{\eta^3}\int_{V_2}^{\xi X^{1/2-\varepsilon}} \frac{\mathrm{d}x}{x^2} \ll \frac{X\xi(\log X)^2}{V_2\eta^3}, \end{aligned}$$

and

$$M_6 \ll \frac{X\xi(\log X)^4}{\eta^3}\int_{\xi X^{1/2-\varepsilon}}^{+\infty}\frac{\mathrm{d}x}{x^2} \ll \frac{X^{1/2+\varepsilon}(\log X)^4}{\eta^3}.$$

Hence, recalling $\xi > X^{-1/2+\varepsilon}$, by (34) and the definitions of V_1 and V_2 we get

$$J_2 \ll X\xi(\log X)^2\Big(V_1^2 + \frac{(\log X)^2}{V_2\eta^3}\Big) + \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b}. \tag{35}$$

Choosing $V_1 = \eta^{1/2}/\log X$ and $V_2 = \log^3 X/\eta^4$, by (30)-(31) and (35), we obtain

$$\int_0^{\infty} |R(\alpha)|^2\, \widehat{K}_\eta\Big(\frac{\alpha}{\xi}\Big)\,\mathrm{d}\alpha = X\xi\log X\xi + \frac{c}{2}X\xi + \mathcal{O}\Big(\eta X\xi\log X + \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b}\Big). \tag{36}$$

To optimize the error term we choose $\eta^{3+a} = (X\xi)^{-a}(\log X)^{-b-1}$, so that (36) becomes

$$\int_0^{\infty} |R(\alpha)|^2\, \widehat{K}_\eta\Big(\frac{\alpha}{\xi}\Big)\,\mathrm{d}\alpha = X\xi\log X\xi + \frac{c}{2}X\xi + \mathcal{O}\Big(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\Big). \tag{37}$$

Finally, by (29) and (37), we obtain

$$R(X,\xi) \leqslant 2X\xi\log X\xi + cX\xi + \mathcal{O}\Big(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\Big).$$

In a similar way we also get that

$$R(X,\xi) \geqslant 2X\xi\log X\xi + cX\xi + \mathcal{O}\Big(\frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}}\Big),$$

and Theorem 2 follows.

References

[1] T.H. Chan, *More precise pair correlation of zeros and primes in short intervals*, J. London Math. Soc. **68** (2003), 579–598.

[2] D.A. Goldston and H.L. Montgomery, *Pair correlation of zeros and primes in short intervals*, Analytic Number Theory and Diophantine Problems (A.C. Adolphson et al., eds.), Proc. of a Conference at Oklahoma State University (1984), Birkhäuser Verlag, 1987, pp. 183–203.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, series, and products*, Academic Press, 2007.

[4] D.R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. Lon. Math. Soc. **38** (1979), 385–422.

[5] A. Languasco, *A note on primes and Goldbach numbers in short intervals*, Acta Math. Hungar. **79** (1998), 191–206.

[6] A. Languasco and A. Perelli, *Pair correlation of zeros, primes in short intervals and exponential sums over primes*, J. Number Theory **84** (2000), 292–304.

[7] A. Languasco, A. Perelli, and A. Zaccagnini, *Explicit relations between pair correlation of zeros and primes in short intervals*, J. Math. Anal. Appl. **394** (2012), 761–771.

[8] H.L. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics, Springer Verlag, 1971.

[9] H.L. Montgomery, K. Soundararajan, *Beyond pair correlation*, in *Paul Erdős and his Mathematics, Bolyai Soc. Math. Stud.* **11** (2002), 507–514.

[10] F.W.J. Olver, D.W. Lozer, R.F. Boisvert, and C.W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge U. P., 2010.

[11] A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, Arch. Math. Naturvid. **47** (1943), 87–105.

Addresses: Alessandro Languasco: Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy;
Alessandro Zaccagnini: Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/a, 43124 Parma, Italy.

E-mail: languasco@math.unipd.it, alessandro.zaccagnini@unipr.it

Received: 5 November 2013; **revised:** 21 November 2013