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TWINS OF POWERFUL NUMBERS

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Abstract: For $k \geqslant 2$ we bound the number of pairs of consecutive k-full numbers.

Keywords: powerful numbers, additive problem, Thue equation, mixed-integer programming.

1. Introduction

A positive integer n is called k-full for some integer $k \geqslant 2$ if $p \mid n$ implies $p^k \mid n$ for every prime p. This is a natural generalization of k-th powers, and it is easy to see that the sequence of k-full numbers is not much denser than the sequence of k-th powers: the number of k-full integers not exceeding x is $\sim c_k x^{1/k}$ for some constant $c_k > 1$ as $x \to \infty$ (see e.g. [7, Section 14.4]). There are many interesting open questions associated with powerful numbers, and in particular additive problems often turn out to be hard.

In this note we want to consider a binary problem in k-full numbers, and for fixed $l \in \mathbb{Z}\backslash\{0\}$ estimate the number $\mathcal{N}_k(x;l)$ of solutions to the equation $n-m=l$ with k-full numbers $n,m \leqslant x$. The trivial bound is $\mathcal{N}_k(x;l) \ll x^{1/k}$. The usual heuristic arguments based on density considerations predict

$$\mathcal{N}_2(x;l) \ll x^{\varepsilon}, \qquad \mathcal{N}_k(x;l) \ll 1, \qquad k \geqslant 3. \tag{1.1}$$

Using the theory of Pell's equation one can show [8] that $\mathcal{N}_2(x;l) \to \infty$ as $x \to \infty$, for any fixed $l \neq 0$. Proving (1.1) seems extraordinarily hard; it should be noted that even *ternary* additive problems in squarefull numbers are not well-understood (see [2]). The bounds in (1.1) follow essentially from the *abc*-conjecture: Chan has shown [3, Theorem 6] that the *abc*-conjecture implies $\mathcal{N}_2(x;l) \ll_{l,\varepsilon} x^{\varepsilon}$ and hence a fortiori $\mathcal{N}_k(x;l) \ll_{l,\varepsilon} x^{\varepsilon}$ for all $k \geqslant 2$. Recently (August 2012), Mochizuki has announced a proof of the *abc*-conjecture. While this breakthrough result is still

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2010 Mathematics Subject Classification: primary: 11D45; secondary: 90C11 under review, we investigate in this note how far one can get with comparatively elementary methods. We follow a recent paper of Chan who showed [3, Theorem 4]

$$\mathcal{N}_2(x;l) \ll_l x^{7/19} \log x \tag{1.2}$$

where $7/19 = 0.368\ldots$ For $k > 2$, no non-trivial results have been obtained so far. We will improve (1.2) in a moment, but first we give a simple argument that provides non-trivial bounds for *all* $k \geqslant 2$.

Theorem 1. *For $l \neq 0$, $k \geqslant 2$ and $\varepsilon > 0$ one has $\mathcal{N}_k(x;l) \ll_{\varepsilon,k} x^{\frac{2}{2k+1}+\varepsilon}$. The implied constant is independent of l.*

Proof. We can write each k-full number n as $n = n_1^k n_2^{k+1} \cdots n_k^{2k-1}$. This representation is unique if we require $n_2, \ldots, n_k$ to be squarefree and pairwise coprime, but even without this requirement a k-full number n has, by a standard divisor estimate, at most $O(n^\varepsilon)$ representations of this form.

For two k-tuples $\mathbf{N} = (N_1, \ldots, N_k)$, $\mathbf{M} = (M_1, \ldots, M_k)$ of positive real numbers let $\mathcal{N}_k(\mathbf{N}, \mathbf{M}; l)$ be the number of solutions to the equation

$$n_1^k \cdots n_k^{2k-1} - m_1^k \cdots m_k^{2k-1} = l$$

where all variables are restricted to dyadic boxes $n_j \in [N_j, 2N_j]$, $m_j \in [M_j, 2M_j]$. Let $\mathcal{N}_k^*(\mathbf{N}, \mathbf{M}; l)$ denote the number of those $2k$-tuples in $\mathcal{N}_k(\mathbf{N}, \mathbf{M}; l)$ that satisfy in addition $(n_j, m_j) = 1$ for $j = 1, \ldots, k$. Then clearly

$$\mathcal{N}_k(x;l) \ll x^\varepsilon \max_{\substack{N_1^k \cdots N_k^{2k-1} \leqslant x \\ M_1^k \cdots M_k^{2k-1} \leqslant x}} \mathcal{N}_k(\mathbf{N}, \mathbf{M}; l) \ll (xl)^\varepsilon \max_{\substack{N_1^k \cdots N_k^{2k-1} \leqslant x \\ M_1^k \cdots M_k^{2k-1} \leqslant x}} \max_{d \mid l} \mathcal{N}_k^*(\mathbf{N}, \mathbf{M}; d). \tag{1.3}$$

Here and in the following, all estimates are uniform in l, and implied constants depend on ε and k at most. Obviously

$$\mathcal{N}_k^*(\mathbf{N}, \mathbf{M}; d) \ll \min(N_1 \cdots N_k, M_1 \cdots M_k)^{1+\varepsilon}, \tag{1.4}$$

since fixing $(n_1, \ldots, n_k)$ leaves $O((m_1 \cdots m_k)^\varepsilon)$ choices for $m_1, \ldots, m_k$, and the same argument holds with the roles of n_j and m_j reversed.

Alternatively, let us fix $n_2, \ldots, n_k, m_2, \ldots m_k$. In the case $k \geqslant 3$ we are left with a Thue equation in n_1, m_1, and by the main theorem in [1] the number of primitive (i.e. with n_1 and m_1 coprime) solutions is $O(l^\varepsilon)$, uniformly in the other variables. In the case $k = 2$, we obtain a Pell-type equation $n_1^2 n_2^3 - m_1^2 m_2^3 = d$. Since $(n_2, m_2) = 1$, the product $n_2^3 m_2^3$ is a square if and only if n_2 and m_2 are squares in which case there are $O(d^\varepsilon)$ solutions. If $n_2^3 m_2^3$ is not a square, then we bound the number of solutions by $O((xd)^\varepsilon)$, again uniformly in the other variables, since the fundamental unit of real quadratic fields is bounded below by an absolute constant, see e.g. [5, Hilfssatz 2] for a detailed proof. Hence we have the additional bound

$$\mathcal{N}_k^*(\mathbf{N}, \mathbf{M}; d) \ll (xd)^\varepsilon N_2 \cdots N_k M_2 \cdots M_k. \tag{1.5}$$

Combining (1.3) – (1.5) we obtain

$$\begin{aligned}\mathcal{N}_k(x;l) &\ll (xl)^{\varepsilon} \max_{\substack{N_1^k\cdots N_k^{2k-1}\leqslant x\\ M_1^k\cdots M_k^{2k-1}\leqslant x}} \min\big(N_1\cdots N_k,\ M_1\cdots M_k,\ N_2\cdots N_k M_2\cdots M_k\big)\\ &\leqslant (xl)^{\varepsilon} \max_{N^k,M^k\leqslant x} \min\left(N\Big(\frac{x}{N^k}\Big)^{\frac{1}{k+1}}, M\Big(\frac{x}{M^k}\Big)^{\frac{1}{k+1}}, \Big(\frac{x^2}{N^kM^k}\Big)^{\frac{1}{k+1}}\right)\end{aligned}$$

since $N_2\cdots N_k \leqslant (x/N_1^k)^{1/(k+1)}$ and similarly for $M_2\cdots M_k$. Using $\min(a,b,c)\leqslant (ab)^{\frac{k}{2k+1}}c^{\frac{1}{2k+1}}$ we complete the proof of Theorem 1. ■

The bounds are uniform in l and the argument is not sensitive to signs (the number of solutions to $n_1^2n_2^3+m_1^2m_2^3=d$ is still $O(d^{\varepsilon})$), hence the same argument shows:

Corollary 2. *The number of representations of an integer N as a sum of two k-full numbers is $O(N^{\frac{2}{2k+1}+\varepsilon})$.*

This seems to be the first non-trivial result of this kind.

The rest of the paper is devoted to refinements of Theorem 1. Following [3], we change the argument leading to (1.5) by fixing only $2k-3$ variables and considering the remaining count as a problem of bounding the number of rational points close to an (algebraic, but irrational) curve. To this end we use a result of Huxley which finally leads to an explicit, but non-trivial optimization problem. For $k=2$ this can be solved by hand, for $k>2$ we transform it into a linear mixed integer optimization problem. For $3\leqslant k\leqslant 5$ we use FICO Xpress to find a numerical solution. This gives the following sharpening of Theorem 1.

Theorem 3. *For fixed $l\neq 0$ and $2\leqslant k\leqslant 5$ we have $\mathcal{N}_k(x;l)\ll x^{\gamma_k}$ for $\gamma_2>61/180$, $\gamma_3=0.2665$, $\gamma_4=0.21$, $\gamma_5=0.174$.*

Note that $61/180=0.338\ldots$, so that Theorem 3 improves the main result of [3], cf. (1.2) above. As mentioned above, the bound for $k=2$ does not use a computer search. For $k>5$ the same method works, but the improvement compared to Theorem 1 becomes marginal. For relatively large values of k, recent work of Heath-Brown and Salberger on the Bombieri-Pila determinant method should be able to provide stronger improvements.

Remark. While the present article was in press, a paper by Chan [4] appeared in which the bound $\mathcal{N}_3(x;l)\ll x^{45/139}(\log x)^2$ is proved. This is weaker than the corresponding bounds in Theorems 1 and 3 of the present paper. A recent preprint of Reuss [9] improves our value of γ_2 in Theorem 3 to 29/100.

2. Rational points close to a curve

For positive real numbers M,T,Δ and $s\in\mathbb{N}$ define

$$\begin{aligned}H_s(M,T,\Delta) := (M^{-s-1}T)^{\frac{1}{2s+1}} + (\Delta^{\frac{1}{s}}M^{-s}T)^{\frac{1}{2s+1}} + \Delta^{\frac{1}{2s+1}}\\ + (\Delta^{s^2+2s-1}T^{s(s-1)})^{\frac{1}{s(s+1)(2s-1)}}.\end{aligned}\tag{2.1}$$

For positive real numbers $\lambda > 0$, $M \geqslant 2$, $\alpha \notin \mathbb{Z}$ let $f : [0, M] \to \mathbb{R}$ be defined by

$$f(x) = \lambda(1 + x/M)^{\alpha}.$$

We quote the following theorem of Huxley [6, Theorem 1] whose assumptions are easily verified for the present choice of f.

Proposition 1. *Let λ, M, α and f be as above. Fix $s \in \mathbb{N}$. Let $0 < \Delta < 1/2$ and $Q \geqslant 2 + 4/\lambda^{1/2}$. Then the number of integer triples (m, r, q) with $0 \leqslant m \leqslant M$, $1 \leqslant q \leqslant Q$, $(r, q) = 1$ satisfying*

$$\left| f(m) - \frac{r}{q} \right| \leqslant \frac{\Delta}{Q^2}$$

is at most

$$\ll_{s,\alpha} (M(1+\lambda)Q)^{\varepsilon} \cdot M \cdot H_s(M, \lambda Q^2, \Delta).$$

Now let $(n_1, \ldots, n_k, m_1, \ldots, m_k)$ be a $2k$-tuple counted by $\mathcal{N}^*(\mathbf{N}, \mathbf{M}; d)$ and let us write

$$X = N_1^k \cdots N_k^{2k-1}$$

for notational simplicity. We recall that $n_j \in [N_j, 2N_j]$ and $m_j \in [M_j, 2M_j]$. We can assume without loss of generality that X is sufficiently large. The equation

$$n_1^k \cdots n_k^{2k-1} - m_1^k \cdots m_k^{2k-1} = d \tag{2.2}$$

implies $M_1^k \cdots M_k^{2k-1} \asymp X$. Fix an index $j \in \{1, \ldots, k\}$. Then we conclude from (2.2) that

$$\frac{n_j^{k+j-1}}{m_j^{k+j-1}} - \frac{m_1^k \cdots \widehat{m_j}^{k+j-1} \cdots m_k^{2k-1}}{n_1^k \cdots \widehat{n_j}^{k+j-1} \cdots n_k^{2k-1}} = \frac{d}{m_j^{k+j-1} n_1^k \cdots \widehat{n_j}^{k+j-1} \cdots n_k^{2k-1}}$$

where a hat denotes omission of the respective factor. Since

$$|a - b| \leqslant |a^{k+j-1} - b^{k+j-1}| a^{2-j-k} \qquad \text{for any } \ a, b > 0,$$

we obtain

$$\left| \frac{n_j}{m_j} - \frac{(m_1^k \cdots \widehat{m_j}^{k+j-1} \cdots m_k^{2k-1})^{\frac{1}{k+j-1}}}{(n_1^k \cdots \widehat{n_j}^{k+j-1} \cdots n_k^{2k-1})^{\frac{1}{k+j-1}}} \right| \leqslant \frac{2d}{M_j^2} \cdot \frac{N_j M_j}{X}. \tag{2.3}$$

Now fix another index $i \neq j$ and write

$$\frac{(m_1^k \cdots \widehat{m_j}^{k+j-1} \cdots m_k^{2k-1})^{\frac{1}{k+j-1}}}{(n_1^k \cdots \widehat{n_j}^{k+j-1} \cdots n_k^{2k-1})^{\frac{1}{k+j-1}}} = \lambda \left(1 + \frac{\tilde{m}_i}{M_i} \right)^{\frac{k+i-1}{k+j-1}}$$

where

$$\tilde{m}_i = m_i - M_i, \qquad \lambda = \left(\frac{M_i}{m_i} \right)^{\frac{k+i-1}{k+j-1}} \frac{(m_1^k \cdots \widehat{m_j}^{k+j-1} \cdots m_k^{2k-1})^{\frac{1}{k+j-1}}}{(n_1^k \cdots \widehat{n_j}^{k+j-1} \cdots n_k^{2k-1})^{\frac{1}{k+j-1}}}.$$

Without loss of generality we can assume that M_i is an integer. Then $\tilde{m}_i \in [0, M_i]$ is an integer, and we can count the number of triples $(n_j, m_j, \tilde{m}_i)$ satisfying (2.3) using Proposition 1 with

$$\alpha = \frac{k+i-1}{k+j-1} \notin \mathbb{Z}, \qquad Q = 2M_j, \qquad M = M_i, \qquad \Delta \asymp \frac{N_j M_j}{X}, \qquad \lambda \asymp \frac{N_j}{M_j}$$

(where implicit constants may depend on d and k). The assumptions $M \geqslant 2$, $Q \geqslant 2 + 4/\lambda^{1/2}$ can be satisfied by multiplying Q and/or M with a fixed constant if necessary. If X is sufficiently large, the condition $\Delta < 1/2$ will be satisfied automatically, unless possibly $k = 2$ and $N_2 M_2 \ll 1$, in which case Theorem 3 is trivial by (1.5). Proposition 1 now implies

$$\mathcal{N}^*(\mathbf{N}, \mathbf{M}; d) \ll_{\varepsilon,d,k,s} X^{\varepsilon} \frac{N_1 \cdots N_k M_1 \cdots M_k}{N_j M_j} H_s\Big(M_i, N_j M_j, \frac{N_j M_j}{X}\Big) \tag{2.4}$$

for any choice of $i \neq j \in \{1, \ldots, k\}$ and any fixed $s \in \mathbb{N}$. By symmetry we can interchange the roles of N and M and obtain in the same way

$$\mathcal{N}^*(\mathbf{N}, \mathbf{M}; d) \ll_{\varepsilon,d,k,s} X^{\varepsilon} \frac{N_1 \cdots N_k M_1 \cdots M_k}{N_j M_j} H_s\Big(N_i, N_j M_j, \frac{N_j M_j}{X}\Big). \tag{2.5}$$

Combining (1.3), (1.4), (2.4) and (2.5), we obtain $\mathcal{N}_k(x; l) \ll_{k,l,s_0,\varepsilon} x^{\varepsilon} \max\limits_{X \leqslant x} \mathcal{M}_k(X)$ where

$$\begin{aligned} \mathcal{M}_k(X) = \max_{\substack{N_1^k \cdots N_k^{2k-1} = X \\ M_1^k \cdots M_k^{2k-1} = X}} \min\Big(& N_1 \cdots N_k,\ M_1 \cdots M_k, \\ & \min_{1 \leqslant i \neq j \leqslant k} \min_{s \leqslant s_0} \frac{N_1 \cdots N_k M_1 \cdots M_k}{N_j M_j} H_s\Big(\max(N_i, M_i), N_j M_j, \frac{N_j M_j}{X}\Big)\Big) \end{aligned} \tag{2.6}$$

for any fixed s_0. Here we used that H_s is decreasing in the first variable. We are now left with solving the minimax problem (2.6).

3. A minimax problem with disjunctive constraints

For $k = 2$ it is easy to solve (2.6) by hand. We substitute $N_2 = X^{1/3} N_1^{-2/3}$, $M_2 = X^{1/3} M_1^{-2/3}$ and write for notational simplicity $N = N_1$, $M = M_1$. We consider only the case $s = 4$ and obtain

$$\begin{aligned} \mathcal{M}_2(X) \leqslant \max_{N, M \leqslant X^{1/2}} \min\Bigg(& (XN)^{1/3}, (XM)^{1/3}, NMH_4\Big(\max(N, M), \frac{X^{2/3}}{(NM)^{2/3}}, \\ & \frac{1}{X^{1/3}(NM)^{2/3}}\Big), \frac{X^{2/3}}{(NM)^{2/3}} H_4\Big(\frac{X^{1/3}}{\min(N^{2/3}, M^{2/3})}, NM, \frac{NM}{X}\Big)\Bigg). \end{aligned}$$

On any given hyperbola $NM = Y$, the expression $\min(\ldots)$ takes its maximum at $N = M = Y^{1/2}$. Hence

$$\mathcal{M}_2(X) \leqslant \max_{N \leqslant X^{1/2}} \min\Bigg((XN)^{1/3}, N^2 H_4\Big(N, \frac{X^{2/3}}{N^{4/3}}, \frac{1}{X^{1/3}N^{4/3}}\Big), \frac{X^{2/3}}{N^{4/3}} H_4\Big(\frac{X^{1/3}}{N^{2/3}}, N^2, \frac{N^2}{X}\Big)\Bigg).$$

A lengthy, but straightforward calculation shows that the expression $\min(\ldots)$ is given by

$$\begin{cases} N^2, & N \leqslant X^{2/19}, \\ X^{2/27}N^{35/27}, & X^{2/19} \leqslant N \leqslant X^{1/8}, \\ X^{7/108}N^{37/27}, & X^{1/8} \leqslant N \leqslant X^{1/5}, \\ X^{53/108}N^{-41/54}, & X^{1/5} \leqslant N \leqslant X^{19/62}, \\ X^{2/3}N^{-4/3}, & X^{19/62} \leqslant N \leqslant N^{1/2}. \end{cases}$$

and hence $\mathcal{M}_2(X) \leqslant X^{61/180}$ as claimed.

For $k > 2$ it becomes complicated to solve (2.6) by hand. In order to prepare for a computer search we linearize the problem by writing $N_j = X^{\nu_j}$, $M_j = X^{\mu_j}$. For notational simplicity we write $\nu := \nu_1 + \ldots + \nu_k$, $\mu := \mu_1 + \ldots + \mu_k$. This gives $\mathcal{M}_k(X) \ll X^\beta$ where

$$\beta = \max_{\substack{k\nu_1+\ldots+(2k-1)\nu_k=1 \\ k\mu_1+\ldots+(2k-1)\mu_k=1 \\ \nu_j,\mu_j \geqslant 0}} \min\Big(\nu, \mu, \min_{1\leqslant i\neq j\leqslant k} \min_{s\leqslant s_0} \big(\nu+\mu-\nu_j-\mu_j + \min(L_{i,j,s}(\boldsymbol{\nu},\boldsymbol{\mu}), L_{i,j,s}(\boldsymbol{\mu},\boldsymbol{\nu}))\big)\Big) \tag{3.1}$$

with

$$L_{i,j,s}(\boldsymbol{\nu},\boldsymbol{\mu}) = \max\Bigg(\frac{-(s+1)\nu_i+\nu_j+\mu_j}{2s+1}, \frac{(s+1)(\nu_j+\mu_j)-s-s^2\nu_i}{s(2s+1)}, \frac{\nu_j+\mu_j-1}{2s+1}, \frac{(s^2+2s-1)(\nu_j+\mu_j-1)+s(s-1)(\nu_j+\mu_j)}{s(s+1)(2s-1)}\Bigg)$$

by (2.6) and (2.1). Using $k\nu_1 + \ldots + (2k-1)\nu_k = k\mu_1 + \ldots + (2k-1)\mu_k = 1$ we can eliminate the variables ν_k and μ_k and define the polyhedron

$$D := \{(\nu_1, \ldots, \nu_{k-1}, \mu_1, \ldots, \mu_{k-1}) \in [0,1]^{2k-2} \mid k\nu_1 + \ldots + (2k-2)\nu_{k-1} \leqslant 1, k\mu_1 + \ldots + (2k-2)\mu_{k-1} \leqslant 1\}.$$

An inspection of (3.1) shows that we need to maximize a function of the type

$$\min_{i\in I} \max_{j\in J} f_{ij}(\boldsymbol{\nu},\boldsymbol{\mu}) \longrightarrow \max, \qquad (\boldsymbol{\nu},\boldsymbol{\mu}) \in D, \tag{3.2}$$

where $f_{ij}(\boldsymbol{\nu},\boldsymbol{\mu}) = (a_{ij})^t\boldsymbol{\nu} + (b_{ij})^t\boldsymbol{\mu} + c_{ij}$ are certain linear functions with matrices $(a_{ij}), (b_{ij}), (c_{ij})$ with $1 \leqslant j \leqslant 4$, $1 \leqslant i \leqslant 2k(k-1)s_0 + 2$ that are determined by (3.1). The nested occurrence of min and max requires some preparation. It is not hard to see that $0 \leqslant f_{ij}(\boldsymbol{\nu},\boldsymbol{\mu}) \leqslant 1$ for $(\boldsymbol{\nu},\boldsymbol{\mu}) \in D$ and each pair (i,j). For each pair (i,j) we introduce an integer (in fact boolean) variable $z_{ij} \in \{0,1\}$. Then (3.2) is equivalent to the following linear mixed-integer program

$$\begin{aligned}
&\text{(MIP)}\ \max\ f\\
\text{s.t.}\ & f_{ij}(\boldsymbol{\nu},\boldsymbol{\mu}) = a_{ij}\boldsymbol{\nu} + b_{ij}\boldsymbol{\mu} + c_{ij} \quad \text{for all } i \in I, j \in J\\
& f_i \geqslant f_{ij} \quad \text{for all } i \in I, j \in J\\
& f_i \leqslant f_{ij} + M(1 - z_{ij}) \quad \text{for all } i \in I, j \in J\\
& f \geqslant f_i \quad \text{for all } i \in I\\
& \sum_{j\in J} z_{ij} \geqslant 1 \quad \text{for all } i \in I\\
& (\boldsymbol{\nu},\boldsymbol{\mu}) \in D\\
& z_{ij} \in \{0,1\} \quad \text{for all } i \in I, j \in J\\
& f, f_i, f_{ij} \ \text{free}
\end{aligned}$$

if $M \geqslant f_{ij}(\nu,\mu) - f_{i'j'}(\nu,\mu)$ for all $i,i' \in I, j,j' \in J$ and for all $(\boldsymbol{\nu},\boldsymbol{\mu}) \in D$. In our case we chose $M = 1$ and $s_0 = 20$.

In order to solve the above mixed-integer program (MIP) we used FICO Xpress v7.2. Without any help, Xpress was not even able to find a feasible solution. From symmetry considerations and motivated by the case $k = 2$, one may conjecture that the optimal solution should occur on the diagonal $\boldsymbol{\nu} = \boldsymbol{\mu}$. In order to generate a feasible solution we did a pre-run in which we added this additional symmetry condition. This type of variable fixing made the problem tractable and generated an optimal solution under the extra condition $\boldsymbol{\nu} = \boldsymbol{\mu}$. Using this solution as starting solution Xpress was then able to solve (MIP) and justified that the symmetry we required indeed holds.

We remark on the side that it would be interesting to find a proof for the general validity of this symmetry assumption. In this case the optimization problem (3.2) for $k = 3$ can be solved by hand rather easily leading to the exponent $105/394 \approx 0.26649\ldots$

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