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DIVISIBILITY BY 2 OF PARTIAL STIRLING NUMBERS

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Abstract: The partial Stirling numbers $T_n(k)$ used here are defined as $\sum_{i \text{ odd}} \binom{n}{i} i^k$. Their 2-exponents $\nu(T_n(k))$ are important in algebraic topology. We provide many specific results, applying to all values of n, stating that, for all k in a certain congruence class mod 2^t, $\nu(T_n(k)) = \nu(k-k_0)+c_0$, where k_0 is a 2-adic integer and c_0 a positive integer. Our analysis involves several new general results for $\nu(\sum \binom{n}{2i+1} i^j)$, the proofs of which involve a new family of polynomials. Following Clarke ([3]), we interpret T_n as a function on the 2-adic integers, and the 2-adic integers k_0 described above as the zeros of these functions.

Keywords: Stirling number, divisibility, Hensel's Lemma.

1. Main results

The partial Stirling numbers $T_n(k)$ used here are defined, for integers n and k with n positive, by

$$T_n(k) = \sum_{i \text{ odd}} \binom{n}{i} i^k.$$

Other versions can be defined localized at other primes and summed over restricted congruences. Let $\nu(-)$ denote the exponent of 2 in an integer. The numbers $\nu(T_n(k))$ are important in algebraic topology ([1], [4], [6], [8], [9], [12]), and work on evaluating these numbers has appeared in the above papers as well as [3], [5], [11], [14], and [15]. In this paper, we give complete results for $n \leqslant 36$ and also for $n = 2^e + 1$ and $2^e + 2$, and we give two families of results applying to all values of n but with k restricted to certain congruence classes. In [7], some of these results will be applied to obtain new results for v_1-periodic homotopy groups of the special unitary groups. We also present in Section 2 some new results about $\nu(\sum_i \binom{n}{2i+1} i^k)$. The proofs of these, in Section 3, introduce a new family of polynomials $q_m(x)$, which might be of independent interest. Finally, in Section 4 we discuss our results in the context of analytic functions on the 2-adic integers, and Hensel's Lemma.

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We begin with the result which is easiest to state, and hence best illustrates the nature of our results.

Theorem 1.1. *Let $e \geqslant 2$, $n = 2^e + 1$ or $2^e + 2$, and $1 \leqslant i \leqslant 2^{e-1}$.*

1. *There is a 2-adic integer $x_{i,n}$ such that for all integers x*

$$\nu(T_n(2^{e-1}x + i)) = \nu(x - x_{i,n}) + n - 2.$$

Moreover

$$x_{i,2^e+1} \equiv \begin{cases} 1 + 2^i \pmod{2^{i+1}}, & \text{if } i = 2^{e-2} \text{ or } 2^{e-1} \\ 1 \pmod{2^{i+1}}, & \text{otherwise} \end{cases}$$

and

$$x_{i,2^e+2} \equiv \begin{cases} 1 + 2^{i-1} \pmod{2^i}, & \text{if } 1 \leqslant i \leqslant 2^{e-2} \\ 1 + 2^i \pmod{2^{i+1}}, & \text{if } 2^{e-2} < i < 2^{e-1} \\ 1 \pmod{2^{i+1}}, & \text{if } i = 2^{e-1}. \end{cases}$$

2. *Let $g(x) = \nu(T_n(2^{e-1}x + i)) - (n-2)$. Then $x_{i,n} = 2^{t_0} + 2^{t_1} + \cdots$, where $t_0 = g(0)$ and $t_{j+1} = g(2^{t_0} + \cdots + 2^{t_j})$.*

If $e = 1$, the result is different. See Table 1.3.

For example, the last 24 digits of the binary expansion of $x_{4,9}$ are

$$100000000001101010110001,$$

and so we can make the following more explicit statement.

$$\nu(T_9(4x+4)) = \begin{cases} 7, & x \equiv 0 \ (2) \\ 8, & x \equiv 3 \ (4) \\ 9, & x \equiv 5 \ (8) \\ 10, & x \equiv 9 \ (16) \end{cases}$$

and continue indefinitely, just noting the last position in which the binary expansions of x and $x_{4,9}$ differ.

Our next result utilizes `Maple` calculations in its proof. Although each case applies to infinitely many values of x, we will explain in the proof how each case can be reduced to a small number of verifications.

Theorem 1.2. *For each $n \leqslant 36$, there is a partition of $\mathbb{Z}$ into finitely many congruence classes $C = [i \mod 2^m]$ such that, for each, either (a) there exists a*

2-adic integer x_0 and a positive integer c_0 such that $\nu(T_n(2^m x+i)) = \nu(x-x_0)+c_0$ for all integers x, or (b) there exists a positive integer y_0 such that $\nu(T_n(k)) = y_0$ for all k in C. The congruence classes C and integers c_0 and y_0 are as in Tables 1.3 and 1.4.

Let $g(x) = \nu(T_n(2^m x+i)) - c_0$. Then $x_0 = 2^{t_0} + 2^{t_1} + \cdots$, with $t_0 = g(0)$ and $t_{j+1} = g(2^{t_0} + \cdots + 2^{t_j})$.

We conjecture that the general form of the theorem can be extended to all integers n; i.e., that for each n there is a partition of $\mathbb{Z}$ into finitely many congruence classes on each of which either $\nu(T_n(k)) = \nu(k-k_0) + c_0$ for some k_0 and c_0 or else $\nu(T_n(k))$ is constant on C. In the tables, the letter i refers to any integer.

Table 1.3: Values of C, c_0, and y_0 in Theorem 1.2

n	C	c_0	y_0	n	C	c_0	y_0
3	0 (2)		2	4	0 (2)		3
	1 (2)		1		1 (2)		4
5	0, 1 (2)	3		6	0, 1 (2)	4	
7	0 (2)	6		8	0 (2)	7	
	1 (2)	4			1 (2)	9	
9	i (4)	7		10	i (4)	8	
11	0, 2 (4)	9		12	0, 2 (4)	10	
	1, 3 (4)	8			1, 3 (4)	11	
13	1, 2 (4)	10		14	2, 3 (4)	11	
	0, 4 (8)	12			0, 1 (8)	13	
	7 (8)		11		4, 5 (8)	13	
	3 (16)		13				
	11 (16)		15				
15	3 (4)	11		16	0 (4)	15	
	0 (4)	14			1 (4)	18	
	1, 5 (8)	13			2, 6 (8)	17	
	2, 6 (8)	16			3, 7 (8)	20	
17	i (8)	15		18	i (8)	16	
19	0, 2, 4, 6 (8)	17		20	0, 2, 4, 6 (8)	18	
	1, 3, 5, 7 (8)	16			1, 3, 5, 7 (8)	19	
21	1, 2, 5, 6 (8)	18		22	0, 1 (8)	20	
	0, 3 (8)	19			2, 3, 6, 7 (8)	19	
	7, 15 (16)	21			4, 12 (16)	22	
	12 (16)		20		5, 21 (32)	24	
	4 (32)		22		13 (16)		21
	52 (64)		24				
	84 (128)		26				
	20 (256)		28				
	148 (256)		29				

Table 1.4: More values of C, c_0, and y_0 in Theorem 1.2

n	C	c_0	y_0	n	C	c_0	y_0
23	$0,4$ (8)	21		24	$0,4$ (8)	22	
	$3,7$ (8)	19			$1,5$ (8)	24	
	1 (8)	20			2 (8)	23	
	2 (8)	22			3 (8)	25	
	$6,22$ (32)	26			$6,22$ (32)	27	
	$5,21$ (32)	24			$7,23$ (32)	29	
	13 (16)		21		14 (16)		24
	14 (16)		23		15 (16)		26
25	$1,2,3,4$ (8)	22		26	$2,3,4,5$ (8)	23	
	$5,6,7,8$ (16)	24			$0,1,7,8,9,15$ (16)	25	
	$0,13,14,15$ (16)	24			$6,22$ (32)	27	
					14 (16)		24
27	$3,5$ (8)	23		28	$4,6$ (8)	25	
	$4,6$ (8)	24			$5,7$ (8)	26	
	$1,7,9,15$ (16)	25			$0,2,8,10$ (16)	27	
	$0,2,8,10$ (16)	26			$1,3,9,11$ (16)	28	
29	$5,6$ (8)	25		30	$6,7$ (8)	26	
	7 (8)	26			$2,3,10,11$ (16)	28	
	$1,2,9,10$ (16)	27			$0,1,8,9$ (16)	29	
	$0,4,8,12$ (16)	28			$4,5,12,13$ (16)	29	
	11 (16)	29					
	3 (32)		30				
	19 (32)		31				
31	7 (8)	26		32	0 (8)	31	
	0 (8)	30			1 (8)	35	
	$3,11$ (16)	28			$4,12$ (16)	33	
	$1,5,9,13$ (16)	29			$2,6,10,14$ (16)	34	
	$4,12$ (16)	32			$5,13$ (16)	37	
	$2,6,10,14$ (16)	33			$3,7,11,15$ (16)	38	
33	i (16)	31		34	i (16)	32	
35	$2i$ (16)	33		36	$2i$ (16)	34	
	$2i+1$ (16)	32			$2i+1$ (16)	35	

One can notice a lot of nice patterns in these tables, and formulate (and sometimes prove) conjectures about their extension to all values of n. One interesting idea, following Clarke ([3]), is to note that since $T_n(k)$ mod 2^m only depends on k mod 2^{m-1}, $T_n(-)$ extends to a function $T_n : \mathbf{Z}_2 \to \mathbf{Z}_2$, where $\mathbf{Z}_2$ denotes the 2-adic integers. Here the metric on $\mathbf{Z}_2$ is given, as usual, by $d(x,y) = |x-y|$, where $|z| := 1/2^{\nu(z)}$. The 2-adic integers $2^m x_0 + i$ which occur in Theorem 1.2 are just the zeros of the function T_n. We can count the number of zeros to be given as in Table 1.5, and might try to formulate a guess about the general formula for this number of zeros.

Table 1.5: Number of zeros of T_n

n	Number of 0's of T_n
1-4	0
5-8	2
9-13	4
14-16	6
17-21	8
22-24	10
25-29	12
30-32	14
33-36	16

Our second general result establishes for all n, except those 1 less than a 2-power, the values of $\nu(T_n(k))$ for k in the congruence class containing 0. We could almost certainly include $n = 2^e - 1$ into this theorem, but the details of proving that case are so detailed as to be perhaps not worthwhile here.

Theorem 1.6. *Let $n \geqslant 5$ and*

$$(a,b) = \begin{cases} (-2,1) & \text{if } n = 2^e \\ (-1,2) & \text{if } 2^e < n \leqslant 3 \cdot 2^{e-1} \\ (0,1) & \text{if } 3 \cdot 2^{e-1} < n \leqslant 2^{e+1} - 2. \end{cases}$$

Then there exists a 2-adic integer $\overline{x}_n$ such that for all integers x

$$\nu(T_n(2^{e+a}x)) = \nu(x - \overline{x}_n) + n - b.$$

The cases $n = 2^e + 1$ and $2^e + 2$ of this theorem overlap with Theorem 1.1. For these n, we have $\overline{x}_n = 1 + x_{2^{e-1},n}$. For all n in Theorem 1.6, there is an algorithm for $\overline{x}_n$ totally analogous to that of Theorem 1.1.

Our next result is of a similar nature, but applies to many more congruence classes. The cases to which it applies are those in which the 2-exponent of a certain sum (see (2.34) and (2.35)) is determined by exactly one of its summands, and for which the mod 4 result 2.6 suffices to prove it. The algorithm for computing x_0 is like that of Theorem 1.2. Here and throughout, $\alpha(n)$ denotes the number of 1's in the binary expansion of n.

Theorem 1.7. *Suppose $2^e + 2^t \leqslant n < 2^e + 2^{t+1}$ with $0 \leqslant t \leqslant e-1$. Let*

$$S_n = \{p : \ \max(0, n - 2^e - 2^{e-1}) \leqslant p < 2^{e-1} \text{ and } \tbinom{n-1-p}{p} \equiv 1 \ (2)\}.$$

If $p \in S_n$, say that an integer $q < 2^{e-1}$ is associated *to p if $q = p$ or $q = p + 2^w$ with $w = \nu(n) - 1$ or $w > t$. If q is associated to an integer p of S_n, then there exists a 2-adic integer x_0 such that for all integers x*

$$\nu(T_n(2^{e-1}x + q)) = \nu(x - x_0) + n - 2 - \alpha(p_0),$$

where p_0 is the residue of p mod 2^t.

A bit of work is required to get any sort of feel for the complicated condition in this theorem. In Table 1.8, we list for n from 17 to 31, the values of p in S_n, then the additional values of q covered by the theorem, and finally the values of i for which Theorem 1.2, as depicted in Tables 1.3 and 1.4, gives a value for the congruence i mod 8 which is not covered by Theorem 1.7. The strength of the theorem is, of course, that it applies to all values of n (except 2-powers).

Table 1.8: Comparison of Theorems 1.7 and 1.2

n	$p \in S_n$	Additional q	Mod 8 results missed
17	$0,1,2,4$	$3,5,6$	7
18	$0,2,4$	$1,3,5,6$	7
19	$0,1,3,4,5$	7	$2,6$
20	$0,4$	$2,6$	$1,3,5,7$
21	$0,1,2,5,6$		3
22	$0,2,6$	$1,3,7$	
23	$0,1,3,7$		$2,4$
24	0	4	$1,2,3,5$
25	$1,2,4$		3
26	$2,4$	$3,5$	
27	$3,4,5$		6
28	4	6	$5,7$
29	$5,6$		7
30	6	7	
31	7		0

2. Proofs of main theorems

In this section, we prove the four main theorems listed in Section 1. A central ingredient in the proofs is results about $\nu\big(\sum_i \binom{n}{2i+1} i^k\big)$. We begin by providing six results about this, of which all but the first are new. The proofs of most of these appear in Section 3.

The first result was proved in [9, 3.4].

Proposition 2.1 ([9, 3.4]). *For any nonnegative integers n and k,*

$$\nu\Big(\sum_i \binom{n}{2i+1} i^k\Big) \geqslant \nu([n/2]!).$$

In using this, and many times throughout the paper, we use

$$\nu(n!) = n - \alpha(n). \tag{2.2}$$

The next result is a refinement of Proposition 2.1. Here and throughout, $S(n,k)$ denote Stirling numbers of the second kind.

Proposition 2.3. *Mod* 4

$$\frac{1}{n!}\sum_i \binom{2n+\epsilon}{2i+b} i^k \equiv \begin{cases} S(k,n)+2nS(k,n-1) & \epsilon=0,\ b=0\\ (2n+1)S(k,n)+2(n+1)S(k,n-1) & \epsilon=1,\ b=0\\ 2S(k,n-1) & \epsilon=0,\ b=1\\ S(k,n)+2(n+1)S(k,n-1) & \epsilon=1,\ b=1. \end{cases}$$

The proofs of the last three propositions all involve new polynomials $q_m(x)$, which might be of independent interest. See Definition 3.1 for the definition, which pervades Section 3.

Proposition 2.4. *For any nonnegative integers n and k,*

$$\nu\Big(\sum_i \binom{n}{2i+1} i^k\Big) \geqslant n-k-\alpha(n).$$

Proposition 2.5. *For any nonnegative integers n and k with $n>k$,*

$$\nu\Big(\sum_i \binom{n}{2i+1} i^k\Big) \geqslant n-1-k-\alpha(k)$$

with equality iff $\binom{n-1-k}{k}$ is odd.

The final proposition is a refinement of Proposition 2.5.

Proposition 2.6. *If n and k are nonnegative integers with $n>k$, then, mod 4,*

$$\sum_i \binom{n}{2i+1} i^k/(2^{n-1-2k}k!) \equiv \binom{n-1-k}{k} + \begin{cases} 2\binom{n-1-k}{k-2} & \text{if } n-1 \text{ and } k \text{ are even}\\ 0 & \text{otherwise.}\end{cases}$$

The following corollary will also be useful.

Corollary 2.7. *For $n \geqslant 3$, $j>0$, and $p\in\mathbb{Z}$,*

$$\nu\left(\sum \binom{n}{2i+1}(2i+1)^p i^j\right) \geqslant \max(\nu([\tfrac{n}{2}]!), n-\alpha(n)-j)$$

with equality if $n \in \{2^e+1, 2^e+2\}$ and $j=2^{e-1}$.

Proof. The sum equals $\sum_{k\geqslant 0} T_k$, where

$$T_k = 2^k\binom{p}{k}\sum_i \binom{n}{2i+1} i^{j+k}.$$

By Proposition 2.1, $\nu(T_k) \geqslant \nu([\frac{n}{2}]!)$, while by Proposition 2.4, $\nu(T_k) \geqslant n-\alpha(n)-j$, implying the desired inequality. If $n=2^e+1$ and $j=2^{e-1}$, then $\nu(T_0)=2^{e-1}-1$ by Proposition 2.5, while for $k>0$, $\nu(T_k)\geqslant 2^{e-1}$ by 2.1. If $n=2^e+2$ and $j=2^{e-1}$, $\nu(T_0)=2^{e-1}$ by 2.5, $\nu(T_1)>2^{e-1}$ by 2.3, and $\nu(T_k)>2^{e-1}$ for $k>1$ by 2.1. ■

Our proofs of the theorems of Section 1 will make essential use of the following result of [5]. Here and throughout, we will employ the useful notation

$$\min{}'(m,n)=\begin{cases}\min(m,n) & \text{if } m\neq n\\ \text{a number } >m & \text{if } m=n.\end{cases}$$

Note that $\min'(m,m)$ is not a well-defined number, and that

$$\nu(m+n)=\min{}'(\nu(m),\nu(n)).$$

Lemma 2.8 ([5]). *Let* $\mathbf{N}$ *denote the set of nonnegative integers. A function* $f:\mathbb{Z}\to\mathbf{N}\cup\{\infty\}$ *is of the form* $f(n)=\nu(n-E)$ *for some* 2*-adic integer* E *iff it satisfies*

$$f(n+2^d)=\min{}'(f(n),d)$$

for all $d\in\mathbf{N}$ *and all* $n\in\mathbb{Z}$*. In this case,* $E=\sum_{i\geqslant 0}2^{e_i}$*, where* $e_0=f(0)$ *and* $e_{k+1}=f(2^{e_0}+\cdots+2^{e_k})$*.*

We begin the proofs of the theorems of Section 1 by discussing the proof of Theorem 1.2. The way that the cases of Theorem 1.2 are discovered is by having `Maple` compute values of $\nu(T_n(k))$ for ranges of values of k. For example, seeing that $\nu\big(\sum_{i\text{ odd}}\binom{19}{i}i^k\big)$ takes on the values 17, 25, 17, 18, 17, 19, 17, 18, 17, 20, 17, 18 as k goes 10, 18, 26,. . . , 98 makes one pretty sure that for all integers x we have $\nu(T_{19}(8x+2))=\nu(x-x_0)+17$ for some 2-adic integer x_0, and you could even guess that the last 9 digits in the binary expansion of x_0 are 100000010. But to prove it, more is required. This is a case not covered by any of our three general theorems, but the proofs of all four of our theorems have similar structure.

Let $f(x)=\nu(T_{19}(8x+2))-17$. Then

$$\begin{aligned}f(x+2^d)&=-17+\nu\Big(\sum\tbinom{19}{2i+1}(2i+1)^{8x+2}\\&\qquad+\sum\tbinom{19}{2i+1}(2i+1)^{8x+2}((2i+1)^{2^{3+d}}-1)\Big)\\&=\min{}'\left(f(x),\nu\left(\sum\tbinom{19}{2i+1}(2i+1)^{8x+2}((2i+1)^{2^{3+d}}-1)\right)-17\right).\end{aligned}\tag{2.9}$$

Thus the claim that $\nu(T_{19}(8x+2))=\nu(x-x_0)+17$ for some 2-adic integer x_0 will follow from Lemma 2.8 once we show that

$$\nu\left(\sum\tbinom{19}{2i+1}(2i+1)^{8x+2}((2i+1)^{2^{3+d}}-1)\right)=d+17\tag{2.10}$$

for all x and $d\geqslant 0$. We expand the two powers of $(2i+1)$, obtaining terms, for $k\geqslant 0$ and $j>0$, with 2-exponent

$$\nu\tbinom{8x+2}{k}+\nu\tbinom{2^{3+d}}{j}+k+j+\nu\left(\sum\tbinom{19}{2i+1}i^{k+j}\right).\tag{2.11}$$

Let $\psi(s)=s+\nu(\sum\binom{19}{2i+1}i^s)$. Since $\nu\binom{2^{3+d}}{j}=3+d-\nu(j)$, it will suffice to show that the minimum value of $\psi(k+j)-\nu(j)+\nu\binom{8x+2}{k}$ is 14, and that this value occurs for an odd number of pairs (k, j). `Maple` computes that the minimum value of $\psi(s)$ is 16, which occurs when $s = 3$, 5, 7, or 9, and that $\psi(s) = 17$ for $s = 1$, 4, 6, 8, and 10. For $s \geqslant 11$, $\psi(s) \geqslant 7 + s$ by 2.1. This information makes it easy to check that the minimum value of $\psi(k+j) - \nu(j) + \nu\binom{8x+2}{k}$ is indeed 14, and this value occurs exactly when $(k, j) = (0, 8)$, $(2, 8)$, or $(1, 8)$. This completes the proof that for all integers x we have $\nu(T_{19}(8x + 2)) = \nu(x - x_0) + 17$ for some 2-adic integer x_0. Each of the cases of Theorem 1.2 can be established in this manner, although many of the cases are covered by our general theorems 1.1, 1.6, and 1.7.

The cases in which $T_n(k)$ is constant on a congruence class are proved similarly, although Lemma 2.8 need not be used. For example, to show $\nu(T_{13}(8x + 7)) = 11$ for all x, we first define

$$\theta(k) = 2^k \sum \tbinom{13}{2i+1} i^k.$$

`Maple` and 2.1 show

$$\nu(\theta(k)) \begin{cases} = 10, & k = 5, 6 \\ = 11, & k = 1, 2, 7 \\ > 11, & \text{other } k. \end{cases}$$

Since $T_{13}(8x + 7) = \sum \binom{8x+7}{k}\theta(k)$, and $\binom{8x+7}{k}$ is odd for $k \in \{5, 6, 1, 2, 7\}$, we obtain, mod 2^{12},

$$T_{13}(8x + 7) \equiv \tbinom{8x+7}{5}\theta(5) + \tbinom{8x+7}{6}\theta(6) + 2^{11}.$$

`Maple` shows $\theta(5) \equiv \theta(6) \equiv 3 \cdot 2^{10} \bmod 2^{12}$. Since

$$\tbinom{8x+7}{5} + \tbinom{8x+7}{6} = \tbinom{8x+7}{5}(1 + \tfrac{8x+2}{6}) \equiv 0 \pmod 4,$$

we obtain $\binom{8x+7}{5}\theta(5) + \binom{8x+7}{6}\theta(6) \equiv 0 \bmod 2^{12}$, from which our desired conclusion follows. This concludes our comments regarding the proof of Theorem 1.2.

Now we work toward proofs of the more general results, Theorems 1.1, 1.6, and 1.7. First we recall some background information. We will often use that

$$(-1)^j j! S(k, j) = \sum \tbinom{j}{2i}(2i)^k - T_j(k). \tag{2.12}$$

Sometimes we have $k < j$, in which case $S(k, j) = 0$, and so $T_j(k) = \sum \binom{j}{2i}(2i)^k$ when $k < j$. Other times we use (2.12) to say that $T_j(k) \equiv \pm j! S(k, j) \bmod 2^k$.

Many times we will use without comment the fact, related to (2.2), that

$$\nu\big(\tbinom{m}{n}\big) = \alpha(n) + \alpha(m - n) - \alpha(m).$$

Closely related is the fact that $\binom{m}{n}$ is odd iff each digit in the binary expansion of m is at least as large as the corresponding digit of n. We will sometimes say that $\binom{m}{n}$ is even due to the 2^t-position, meaning that in this position m has a 0 and n has a 1. Other basic formulas that we use without comment are

$$\alpha(n - 1) = \alpha(n) - 1 + \nu(n)$$

and, if $0 < \Delta < 2^t$, then

$$\alpha(2^{t+1}A + 2^t - \Delta) = \alpha(A) + t - \alpha(\Delta - 1).$$

We also use the well-known formula

$$S(k+i,k) \equiv \tbinom{k+2i-1}{k-1} \mod 2. \tag{2.13}$$

A generalization to mod 4 values was given in [2] and will be used several times. We will not bother to state all eight cases of that theorem—just those that we need.

The proof of Theorem 1.1 utilizes the following two lemmas.

Lemma 2.14. *Let $1 \leqslant i \leqslant 2^{e-1}$ with $e \geqslant 2$. Then*

$$\nu(T_{2^e+1}(2^{e-1}+i)) \begin{cases} = 2^e - 1 + i, & i \in \{2^{e-2}, 2^{e-1}\} \\ \geqslant 2^e + i, & \textit{otherwise}, \end{cases}$$

while

$$\nu(T_{2^e+2}(2^{e-1}+i)) \begin{cases} = 2^e - 1 + i, & 1 \leqslant i \leqslant 2^{e-2} \\ = 2^e + i, & 2^{e-2} < i < 2^{e-1} \\ > 2^e + i, & i = 2^{e-1}, \end{cases}$$

Proof. For the first part, by the remarks following (2.12), we must show

$$\nu\left(\sum \tbinom{2^e+1}{2j} j^{2^{e-1}+i}\right) \begin{cases} = 2^{e-1} - 1, & i \in \{2^{e-2}, 2^{e-1}\} \\ \geqslant 2^{e-1}, & \text{otherwise}, \end{cases}$$

or equivalently

$$\tfrac{1}{(2^{e-1})!} \sum \tbinom{2^e+1}{2j} j^{2^{e-1}+i} \equiv \begin{cases} 1 \mod 2, & i \in \{2^{e-2}, 2^{e-1}\} \\ 0 \mod 2, & \text{otherwise}. \end{cases}$$

By Proposition 2.3, the LHS is congruent mod 2 to $S(2^{e-1}+i, 2^{e-1})$, and by (2.13) this is $\binom{2^{e-1}-1+2i}{2^{e-1}-1}$, which is as required.

The second part of the lemma reduces similarly to showing

$$\tfrac{1}{(2^{e-1}+1)!} \sum_j \tbinom{2^e+2}{2j} j^{2^{e-1}+i} \equiv \begin{cases} 1 \mod 2, & 1 \leqslant i \leqslant 2^{e-2} \\ 2 \mod 4, & 2^{e-2} < i < 2^{e-1} \\ 0 \mod 4, & i = 2^{e-1}. \end{cases} \tag{2.15}$$

By 2.3, the LHS is congruent mod 4 to

$$S(2^{e-1}+i, 2^{e-1}+1) + 2S(2^{e-1}+i, 2^{e-1}). \tag{2.16}$$

Mod 2, this is $\binom{2^{e-1}+2i-2}{2^{e-1}}$ which is odd if $1 \leqslant i \leqslant 2^{e-2}$. Now assume $2^{e-2} < i < 2^{e-1}$. The second term of (2.16) is easily seen to be 0 mod 4 using (2.13). For the first term of (2.16), we use part of [2, Thm 3.3], which relates mod 4 values of $S(n,k)$ to binomial coefficients. It implies that, if $e \geqslant 3$, the mod 4 value of the first term is $\binom{2^{e-2}+k}{2^{e-3}}$, where $0 \leqslant k < 2^{e-3}$. The 2-exponent in this number is $1+\alpha(2^{e-3}+k)-\alpha(2^{e-2}+k)=1$, as desired. If $i=2^{e-1}$, both terms of (2.16) are 2 mod 4, by a similar analysis. ■

Lemma 2.17. *If $p \in \mathbb{Z}$, $\delta = 1$ or 2, and $\nu(n) = e+\Delta$ with $\Delta \geqslant -1$, then*

$$\nu\left(\sum_i \tbinom{2^e+\delta}{2i+1}(2i+1)^p((2i+1)^n-1)\right) = 2^e+\Delta+\delta-1.$$

Proof. The sum equals $\sum\limits_{j>0} T_j$, where

$$T_j := 2^j \tbinom{n}{j} \sum_i \tbinom{2^e+\delta}{2i+1}(2i+1)^p i^j.$$

For evaluation of the 2-exponent of the i-sum here, we use Corollary 2.7. We obtain that if $j \leqslant 2^{e+\Delta}$, then

$$\nu(T_j) \geqslant j+e+\Delta-\nu(j)+\begin{cases} 2^e+\delta-2-j, & 1 \leqslant j \leqslant 2^{e-1} \\ 2^{e-1}-1, & j > 2^{e-1}, \end{cases}$$

with equality if $j=2^{e-1}$. This is $\geqslant 2^e+\Delta+\delta-1$ with equality iff $j=2^{e-1}$. If $j > 2^{e+\Delta}$, then $\nu(T_j) > 2^e+\Delta$ since $2^{e+\Delta}+2^{e-1} > 2^e+\Delta$ for $\Delta \geqslant -1$. ■

Now we easily prove Theorem 1.1.

Proof of Theorem 1.1. Let $\delta \in \{1,2\}$, $1 \leqslant i \leqslant 2^{e-1}$, and let

$$g(x) = \nu(T_{2^e+\delta}(2^{e-1}x+2^{e-1}+i)) - 2^e + 2 - \delta.$$

Note that the expression that we wish to evaluate for Theorem 1.1 is $g(x-1)+2^e-2+\delta$.

For $d \geqslant 0$, writing $T_n(-)$ as a sum of two parts as we did in (2.9),

$$g(x+2^d) = \min{}'\big(g(x), -2^e+2-\delta+\nu(\sum \tbinom{2^e+\delta}{2j+1}(2j+1)^p((2j+1)^{2^{d+e-1}}-1))\big),$$

where $p = 2^{e-1}x+2^{e-1}+i$. By Lemma 2.17, the RHS equals $\min'(g(x),d)$, and so $g(x)=\nu(x-E_\delta)$ for some E_δ by Lemma 2.8. By Lemma 2.14

$$\nu(E_\delta) = g(0)\begin{cases} = i, & \delta=1,\ i \in \{2^{e-2},2^{e-1}\} \\ > i, & \delta=1,\ i \notin \{2^{e-2},2^{e-1}\} \\ = i-1, & \delta=2,\ 1 \leqslant i \leqslant 2^{e-2} \\ = i, & \delta=2,\ 2^{e-2} < i < 2^{e-1} \\ > i, & \delta=2,\ i=2^{e-1}. \end{cases}$$

Our desired $g(x-1)+2^e-2+\delta$ equals $\nu(x-1-E_\delta)+2^e-2+\delta$, and $x_{i,2^e+\delta} := 1+E_\delta$ is as claimed. ■

The proof of Theorem 1.6 is similar in nature, but longer.

Proof of Theorem 1.6. Using Lemma 2.8 and arguing as in (2.10), it suffices to prove that for $d \geqslant 0$ and any integer x

$$\nu\left(\sum_i \tbinom{n}{2i+1}(2i+1)^{2^{e+a}x}((2i+1)^{2^{e+a+d}}-1)\right) = n-b+d. \tag{2.18}$$

Indeed, if

$$g(x) = \nu\left(\sum_i \tbinom{n}{2i+1}(2i+1)^{2^{e+a}x}\right) - n + b,$$

then (2.18) implies $g(x+2^d) = \min'(g(x), d)$ and Theorem 1.6 then follows from Lemma 2.8.

We write the sum in (2.18) as $\sum T_j$ with

$$\nu(T_j) = j+e+a+d-\nu(j)+\nu\left(\sum_i \tbinom{n}{2i+1}(2i+1)^{2^{e+a}x} i^j\right). \tag{2.19}$$

We will show that in all cases $\nu(T_j)$ is minimized for a unique value of j.

The second case of the theorem will follow from proving that if $2^e < n \leqslant 3\cdot 2^{e-1}$ and $\nu(p) \geqslant e-1$, then

$$j+e-1-\nu(j)+\nu\left(\sum \tbinom{n}{2i+1}(2i+1)^p i^j\right) \geqslant n-2$$

with equality iff $j = 2^{e-1}$. Expanding $(2i+1)^p$ as $\sum_{k\geqslant 0} 2^k \binom{p}{k} i^k$ leads us to needing that for $j > 0$

$$j+e-\nu(j)+\nu\left(\sum \tbinom{n}{2i+1} i^j\right) \geqslant n-1 \tag{2.20}$$

with equality iff $j = 2^{e-1}$, and

$$j+k+2e-\nu(j)-\nu(k)+\nu\left(\sum \tbinom{n}{2i+1} i^{j+k}\right) > n \tag{2.21}$$

for $j, k > 0$.

The equality in (2.20) when $j = 2^{e-1}$ follows easily from Proposition 2.5. Also by 2.5, the difference in (2.20) becomes

$$j+e+1-\nu(j)+\nu\left(\sum \tbinom{n}{2i+1} i^j\right) - n \geqslant e-\alpha(j)-\nu(j) = e-1-\alpha(j-1). \tag{2.22}$$

This is > 0 if $j \neq 2^{e-1}$ and $j < 3\cdot 2^{e-2}$, while if $j = 3\cdot 2^{e-2}$, then $\binom{n-1-j}{j} = 0$ and so (2.22) is > 0 by 2.5.

Now suppose $j > 3\cdot 2^{e-2}$. Then $j - \nu(j) > 3\cdot 2^{e-2}$, and since $n \leqslant 3\cdot 2^{e-1}$,

$$n - \nu([\tfrac{n}{2}]!) \leqslant 3\cdot 2^{e-2} + e - 1. \tag{2.23}$$

Thus, using Proposition 2.1, we obtain

$$j+e-1+\nu(j)+\nu\left(\sum \tbinom{n}{2i+1} i^j\right) > 3\cdot 2^{e-2}+e+1+\nu([\tfrac{n}{2}]!) \geqslant n,$$

establishing strict inequality in (2.20).

Now we verify (2.21). By 2.5, (2.21) is satisfied if

$$\nu(j)+\nu(k)+\alpha(j+k) \leqslant 2e-2 \tag{2.24}$$

or if

$$\nu(j)+\nu(k)+\alpha(j+k) = 2e-1 \text{ and } \tbinom{n-1-j-k}{j+k} \equiv 0 \pmod 2. \tag{2.25}$$

By 2.1 and (2.23), (2.21) is also satisfied if

$$j+k-\nu(j)-\nu(k) \geqslant 3\cdot 2^{e-2}-e. \tag{2.26}$$

If $j+k > 3\cdot 2^{e-2}$, then (2.26) is satisfied. If $\{j,k\} = \{2^{e-1}, 2^{e-2}\}$, then (2.25) is satisfied. Assume WLOG that $\nu(j) \geqslant \nu(k)$. Then (2.24) is implied by $\nu(j)+1+\alpha(j+k-1) \leqslant 2e-2$ and this is satisfied whenever $j+k \leqslant 3\cdot 2^{e-2}$ and $(j,k) \neq (2^{e-1}, 2^{e-2})$.

The third case of the theorem will follow from proving that, referring to (2.19), if $2^{e+1}-2^{t+1} < n \leqslant 2^{e+1}-2^t$ with $1 \leqslant t < e-1$ and $\nu(p) \geqslant e$, then

$$j+e+d-\nu(j)+\nu\left(\sum \tbinom{n}{2i+1}(2i+1)^p i^j\right) \geqslant n-1+d$$

with equality iff $j = 2^e - 2^t$. Expanding $(2i+1)^p$, this reduces to showing if $j>0$ then

$$j+e+1-\nu(j)+\nu\left(\sum \tbinom{n}{2i+1} i^j\right) \geqslant n \tag{2.27}$$

with equality iff $j = 2^e-2^t$, and if $j,k>0$, then

$$j+k+2e-\nu(j)-\nu(k)+\nu\left(\sum \tbinom{n}{2i+1} i^{j+k}\right) \geqslant n. \tag{2.28}$$

If $j > 2^e$, since $n \leqslant 2^{e+1}-2$,

$$j+e+1-\nu(j) \geqslant 2^e+e+1 > n-\nu([\tfrac{n}{2}]!),$$

and so strict inequality holds in (2.27) by 2.1.

By Theorem 2.5, (2.27) is satisfied if

$$e \geqslant \nu(j)+\alpha(j) = \alpha(j-1)+1, \tag{2.29}$$

and equality holds in (2.27) iff equality holds in (2.29) and $\binom{n-1-j}{j}$ is odd. If $j<2^e$, then $\alpha(j-1) \leqslant e-1$ with equality iff $j = 2^e-2^r$ for some r. Thus (2.29) holds with equality iff $j = 2^e-2^t$ by Lemma 2.32.

If $j = 2^e$, by Proposition 2.6 the LHS of (2.27) is $\geqslant n+1$. Thus (2.27), including consideration of equality, has been established for all j.

By 2.1, (2.28) is satisfied if

$$j + k + 2e - \nu(j) - \nu(k) \geqslant n - \nu([\tfrac{n}{2}]!),$$

and hence, since $n \leqslant 2^e - 2$, it is satisfied if

$$j + k + 2e - \nu(j) - \nu(k) \geqslant 2^e + e - 1.$$

This is satisfied if $j + k > 2^e$.

By 2.5, (2.28) is also satisfied if

$$\nu(j) + \nu(k) + \alpha(j + k) \leqslant 2e - 1.$$

This is satisfied if $j = k = 2^{e-1}$ and if $\nu(j) + \alpha(j + k - 1) \leqslant 2e - 2$, which is true for all other (j, k) with $j + k \leqslant 2^e$.

The first case, $n = 2^e$, will follow similarly from

$$j + e - 1 - \nu(j) + \nu\left(\sum \tbinom{2^e}{2i+1} i^j\right) \geqslant 2^e \tag{2.30}$$

for $j > 0$ with equality iff $j = 2^{e-2}$, while if $j, k > 0$, then

$$j + k + 2e - 2 - \nu(j) - \nu(k) + \nu\left(\sum \tbinom{2^e}{2i+1} i^{j+k}\right) \geqslant 2^e + 2. \tag{2.31}$$

Equality in (2.30) with $j = 2^{e-2}$ follows from Proposition 2.6 since $\binom{2^e-1-2^{e-2}}{2^{e-2}} \equiv 2 \bmod 4$. If $j > 2^{e-1}$, then strict inequality in (2.30) is implied by 2.1. If $j = 2^{e-1}$, it is implied by Proposition 2.3. It is implied by Theorem 2.4 if $\nu(j) \leqslant e - 3$, which is true for $j < 2^{e-1}$ provided $j \neq 2^{e-2}$.

Similarly, (2.31) is implied by 2.1 if $j + k \geqslant 2^{e-1}$ unless $j = k = 2^{e-2}$, in which case it is implied by 2.3. If $j + k < 2^{e-1}$, then $\nu(j) + \nu(k) \leqslant 2e - 5$, and so the claim follows from Proposition 2.4. ∎

The following lemma was used in the above proof.

Lemma 2.32. *If $2^{e+1} - 2^{t+1} \leqslant m < 2^{e+1} - 2^t$ with $0 \leqslant t < e$ and $j = 2^e - 2^r$ with $0 \leqslant r < e$, then $\binom{m-j}{j}$ is odd iff $r = t$.*

Proof. If $r < t$, then $0 \leqslant m - j < j$, so $\binom{m-j}{j} = 0$. If $r = t$, then $\binom{m-j}{j} = \binom{2^e-2^t+d}{2^e-2^t}$ with $0 \leqslant d < 2^t$ and hence is odd. If $r > t$, then the binary expansion of $m - j$ has a 0 in the 2^r position, while j has a 1 there. ∎

The following lemma will be useful in the proof of Theorem 1.7.

Lemma 2.33. *In the notation of Theorem 1.7, if $p \in S_n$, then $\alpha(p - p_0) \leqslant 1$ and $\binom{n-p_0-1}{p_0}$ and $\binom{n-2^{e-1}-p_0-1}{2^{e-1}+p_0}$ are odd.*

Proof. Let $p = A2^t + p_0$ and $n = 2^e + 2^t + \Delta$ with both p_0 and Δ nonnegative and less than 2^t. If $\Delta - p_0 - 1 < 0$, then $\binom{2^e - A2^t}{A2^t}$ is odd, which easily implies $\alpha(A) \leqslant 1$, while if $\Delta - p_0 - 1 \geqslant 0$, then $\binom{2^e+2^t-A2^t}{A2^t}$ is odd. This implies that A is 0 or an even 2-power.

Now, if $p \neq p_0$, we can write $p = p_0 + 2^{t+r}$ with $r \geqslant 0$. If $r = 0$, then $\binom{2^e+\Delta-1-p_0}{2^t+p_0}$ odd implies $\Delta - 1 - p_0 < 0$ and $\binom{2^t+\Delta-1-p_0}{p_0}$ odd, which implies $\binom{n-p_0-1}{p_0}$ odd. If $r > 0$, then the odd binomial coefficient can be considered mod 2 as $\binom{2^e-2^{t+r}}{2^{t+r}}\binom{2^t+\Delta-1-p_0}{p_0}$, which implies $\binom{n-p_0-1}{p_0}$ is odd.

Now we may assume $\binom{n-p_0-1}{p_0}$ is odd. Thus $\binom{2^t+\Delta-1-p_0}{p_0}$ is odd and hence so is $\binom{2^{e-1}+2^t+\Delta-1-p_0}{2^{e-1}+p_0}$. ■

The proof of Theorem 1.7 is similar to the others, but longer yet.

Proof of Theorem 1.7. Similarly to the proofs of the other three theorems, it suffices to prove for $d \geqslant 0$ and any integer x

$$\nu\left(\sum \binom{n}{2i+1}(2i+1)^{2^{e-1}x+q}\left((2i+1)^{2^{e-1+d}} - 1\right)\right) = d + n - 2 - \alpha(p_0). \quad (2.34)$$

Here, and for the remainder of this section, n, e, t, q, p, and p_0 are as in Theorem 1.7. To prove (2.34), it suffices to show for $k \geqslant 0$ and $j > 0$

$$\nu\binom{2^{e-1}x+q}{k} + e - 1 - \nu(j) + j + k + \nu\left(\sum \binom{n}{2i+1} i^{j+k}\right) \geqslant n - 2 - \alpha(p_0) \quad (2.35)$$

with equality iff $j = 2^{e-1}$ and $k = p_0$.

We first prove the equality. Note that if q is associated to $p \in S_n$, then $\binom{2^{e-1}x+q}{p_0}$ is odd. We must show that

$$\nu\left(\sum \binom{n}{2i+1} i^{2^{e-1}+p_0}\right) = n - 2 - p_0 - 2^{e-1} - \alpha(p_0).$$

This follows from Proposition 2.5 since $\binom{n-2^{e-1}-1-p_0}{2^{e-1}+p_0}$ is odd by Lemma 2.33.

Strict inequality in (2.35) when $j = 2^{e-1}$ and $k \neq p_0$ follows from Lemma 2.36 using Propositions 2.1 and 2.6. Here we also use that if $p \in S_n$ and k satisfies (2.37) then $k < 2^{e-1}$ and hence the x in $\binom{2^{e-1}x+q}{k}$ does not play an essential role.

Lemma 2.36. *If n, e, t, q, p, and p_0 are as in Theorem 1.7 and*

$$0 \leqslant k \leqslant n - \nu([\tfrac{n}{2}]!) - 2 - 2^{e-1} - \alpha(p_0), \quad (2.37)$$

then

$$\alpha(q-k) + \alpha(p_0) - \alpha(q) \geqslant -1. \quad (2.38)$$

If the LHS of (2.38) equals -1, then either n is even and $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} \equiv 0$ mod 4 or $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} = 0 = \binom{n-1-2^{e-1}-k}{2^{e-1}+k-2}$. If the LHS of (2.38) equals 0, then either $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} \equiv 0$ mod 2 or $k = p_0$.

Proof. We begin by proving (2.38). Using Lemma 2.33 and that q is associated to p, we have

$$q = p_0 + \delta 2^{t+r} + \epsilon 2^w \tag{2.39}$$

with δ and ϵ equal to 0 or 1, $r \geqslant 0$, and $w = \nu(n) - 1$ or $w > t$. The only way that (2.38) could fail is if $k = q = p_0 + 2^{t+r} + 2^w$. But (2.37) implies $k \leqslant 2^t + t - 2$, which is inconsistent with $w > t$ and with $r > 0$. Thus n is even and $w = \nu(n) - 1$ and $r = 0$. Let $n = 2^e + 2^t + c2^{t-1} + 2b$ with $c \in \{0, 1\}$ and $b \leqslant 2^{t-2} - 1$. If $c = 0$, then (2.37) reduces to $p_0 + \alpha(p_0) + 2^{t-2} + 2^w \leqslant t - 3$, which is false. If, on the other hand, $c = 1$, the assumption that $\binom{n-p-1}{p}$ is odd and $p \geqslant 2^t$ implies that $p_0 \geqslant 2^{t-1} + 2b$, so $k > 2^t + 2^{t-1}$ contradicts $k \leqslant 2^t + t - 2$.

There are three conceivable ways in which equality could hold in (2.38). One is $\epsilon = 0$, $\delta = 1$, and $k = q$; i.e., $k = q = p \geqslant 2^t$. But $k \geqslant 2^t$ implies $2^{e-1} + k > n - 1 - 2^{e-1} - k$ and hence $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} = 0$. We also have $\binom{n-1-2^{e-1}-k}{2^{e-1}+k-2} = 0$; the only way this could fail is if $n = 2^e + 2^{t+1} - 1$ and $k = p = 2^t$, but then $\binom{n-p-1}{p}$ is not odd. Another is $\epsilon = \delta = 1$ and $\alpha(q - k) = 1$. In this case, the only way to have $k < 2^t$ is if $w = \nu(n) - 1$ and $k = q - 2^{t+r}$, where r is as in (2.39). In this case, $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} \equiv 0 \bmod 4$, using the result that $\binom{a+b}{b}$ is divisible by 2^t if there are at least t carries in the binary addition of a and b. In this case, either the binomial coefficient equals 0, or else, if $v = \nu(n)$, there will be carries in the 2^{v-1} and 2^v positions in the relevant binary addition. The third possibility, $\epsilon = 1$, $\delta = 0$, and $k = q$, implies that n is even and $\nu(q) = \nu(n) - 1$ and leads to $\binom{n-1-2^{e-1}-k}{2^{e-1}+k} \equiv 0 \bmod 4$, exactly as above.

Finally we show that if $\binom{n-1-2^{e-1}-k}{2^{e-1}+k}$ is odd and the LHS of (2.38) equals 0, then $k = p_0$. It is not difficult to see that if $0 \leqslant k < n - 2^{e-1}$ and $\binom{n-1-2^{e-1}-k}{2^{e-1}+k}$ is odd, then $k < 2^t$ and $\binom{n-1-k}{k}$ is odd.

First suppose $\alpha(q - k) = 2$ and $\alpha(q) - \alpha(p_0) = 2$. If $q = p_0 + 2^{t+r} + 2^{t+s}$ with $0 \leqslant r < s$, then to keep $k < 2^t$, we must have $k = p_0$. If $q = p_0 + 2^{t+r} + 2^{\nu(n)-1}$ with $r \geqslant 0$, then we must have $k = p_0 + 2^{\nu(n)-1} - 2^u$ for some u. If $u \neq \nu(n) - 1$, then $\binom{n-1-2^{e-1}-k}{2^{e-1}+k}$ is even due to the $2^{\min(u,\nu(n)-1)}$-position.

Now suppose $\alpha(q - k) = 1$ and $\alpha(q) - \alpha(p_0) = 1$. If $q = p_0 + 2^{t+r}$ with $r \geqslant 0$, then $k = p_0$ is the only way to have $k < 2^t$. If $q = p_0 + 2^{\nu(n)-1}$, then the argument at the end of the preceding paragraph applies. This completes the proof of Lemma 2.36, and hence the proof that when $j = 2^{e-1}$, (2.35) holds with equality exactly as claimed there. ∎

We continue the proof of Theorem 1.7 by establishing strict inequality in (2.35) when $0 < j < 2^{e-1}$ and $0 \leqslant k < 2^{e-1}$. The following elementary lemma will be useful.

Lemma 2.40. *Suppose $0 < j < 2^{e-1}$ and $0 \leqslant k < 2^{e-1}$. Let $\phi(j, k) = \alpha(j + k) + \nu(j) - \alpha(k)$. Then*

1. $\phi(j, k) \leqslant e - 1$*;*
2. $\phi(j, k) = e - 1$ *iff* $j = 2^{e-1} - 2^h$ *and* $0 \leqslant k < 2^h$ *for some* $0 \leqslant h < e - 1$*;*

3. *$\phi(j,k)=e-2$ iff either $j=2^{e-1}-2^h$ and $2^{e-2}\leqslant k<2^{e-2}+2^h$ for some $0\leqslant h<e-1$, or $j=2^{e-1}-2^\ell-2^h$, $0\leqslant h\leqslant \ell<e-1$, and $0\leqslant k<2^h$ or $2^\ell\leqslant k<2^\ell+2^h$.*

Proof. Let $h=\nu(j)$, and let $k_0=k-(k \mod 2^h)$. Then $\phi(j,k)=h+\alpha(j+k_0)-\alpha(k_0)$. The only way to get $\alpha(j+k_0)-\alpha(k_0)=e-h-1$ is if $j+k_0=2^e-2^h$ and $\alpha(k_0)=1$, or $j+k_0=2^{e-1}-2^h$ and $k_0=0$. But the first is impossible since $k_0<2^{e-1}$. Similarly the only ways to get $\alpha(j+k_0)-\alpha(k_0)=e-h-2$ is if $(\alpha(j+k_0),\alpha(k_0))=(e-h-1,1)$ or $(e-h,2)$, and these can only be accomplished in the ways listed in part (3). ■

Let

$$\binom{m-s}{s}'=\binom{m-s}{s}+\begin{cases}2\binom{m-s}{s-2}, & \text{if } m \text{ and } s \text{ are even}\\ 0, & \text{otherwise}\end{cases}$$

and

$$\widetilde{\nu}_2\binom{m-s}{s}=\min\left(2,\nu\left(\binom{m-s}{s}'\right)\right). \tag{2.41}$$

Let $\phi(j,k)$ be as in Lemma 2.40. The desired strict inequality in (2.35) when $0<j<2^{e-1}$ and $0\leqslant k<2^{e-1}$ follows from the following result using Proposition 2.6.

Theorem 2.42. *If n,e,t,q, p, and p_0 are as in Theorem 1.7, $0<j<2^{e-1}$, and $0\leqslant k<2^{e-1}$, then*

$$\alpha(q-k)+e-1+\widetilde{\nu}_2\binom{n-1-j-k}{j+k}\geqslant \alpha(q)-\alpha(p_0)+\phi(j,k). \tag{2.43}$$

Proof. By Lemma 2.33, $p=p_0$ or p_0+2^{t+s} with $s\geqslant 0$. Hence $\alpha(q)-\alpha(p_0)\leqslant 2$. Also $\nu(p)\geqslant\nu(n)$, a consequence of the oddness of $\binom{n-1-p}{p}$, will be used often without comment. The theorem will follow from showing:

- if $\phi(j,k)=e-1$, then

$$\widetilde{\nu}_2\binom{n-1-j-k}{j+k}\geqslant\begin{cases}2, & \text{if } \alpha(q)-\alpha(p_0)=2 \text{ and } k=q\\ 1, & \text{if } \alpha(q)-\alpha(p_0)=2 \text{ and } \alpha(q-k)=1\\ 1, & \text{if } \alpha(q)-\alpha(p_0)=1 \text{ and } k=q,\end{cases}$$

- and if $\phi(j,k)=e-2$, then

$$\nu\binom{n-1-j-k}{j+k}\geqslant 1 \qquad \text{if } \alpha(q)-\alpha(p_0)=2 \text{ and } k=q.$$

We call these cases 1 through 4. Let $n=2^e+2^t+\Delta$ with $0\leqslant\Delta<2^t$. Our hypothesis is that $\binom{2^e+2^t-\epsilon 2^{t+s}+\Delta-p_0-1}{\epsilon 2^{t+s}+p_0}$ is odd.

Case 3: We have $q=p_0+2^r$ with $r\geqslant t$ or $r=\nu(n)-1$, in which latter case Δ and p_0 are divisible by 2^{r+1}. We must show that $\binom{2^{e-1}+2^t+2^h+\Delta-1-p_0-2^r}{2^{e-1}-2^h+p_0+2^r}$ is even. Here $2^h>p_0+2^r$. If $r\geqslant t$, then the binomial coefficient is even due to the 2^r- or 2^h-position, while if $r=\nu(n)-1$, it is even due to the $2^{\nu(n)-1}$-position.

Case 2: Here $q = p_0 + 2^s + 2^r$ with $s \geqslant t$ and $r = \nu(n) - 1$ or $r > s$. Also $k = q - 2^v$ and $2^h > k$. The binomial coefficient which we must show is even is

$$C := \binom{2^{e-1} + 2^h + 2^t + \Delta - 1 - p_0 - 2^s - 2^r + 2^v}{2^{e-1} - 2^h + p_0 + 2^s + 2^r - 2^v}.$$

If $v = r$ or s, it reduces to Case 3, just considered. If $r = \nu(n) - 1$, then C is even due to the $2^{\min(v,\nu(n)-1)}$-position. Otherwise C is even due to the 2^h-position, since $2^t + \Delta - 1 - p_0 - 2^s - 2^r + 2^v$ is negative.

Case 1: Now q is as in Case 2, but $k = q$. We must show that there are at least two carries in the binary addition of $2^{e-1} - 2^h + p_0 + 2^s + 2^r$ and $2^{h+1} + 2^t + \Delta - 1 - 2p_0 - 2^{s+1} - 2^{r+1}$. If $r = \nu(n) - 1$, carries occur in positions 2^r and 2^{r+1}. If $r > s$, carries occur in 2^r and 2^h. The second term in the definition of $\binom{m-s}{s}'$ is easily seen to be inconsequential here.

Case 4: Again q is as in Case 2, $k = q$, and (j,k) is one of the two types in Theorem 2.42. For the first type of (j,k), if $r > s$, then $j + k > n - 1 - j - k$, so $\binom{n-1-j-k}{j+k} = 0$, while if $r = \nu(n) - 1$, the binomial coefficient is even due to the $2^{\nu(n)-1}$-position. If (j,k) is of the second type and $k = p_0 + 2^s + 2^{\nu(n)-1}$, then $\binom{n-1-j-k}{j+k}$ is even due to the $2^{\nu(n)-1}$-position, since p_0, n, and j are all divisible by $2^{\nu(n)}$.

If $j = 2^{e-1} - 2^\ell - 2^h$ and $2^\ell \leqslant k < 2^\ell + 2^h$ with $k = p_0 + 2^s + 2^r$ with $r > s$, we claim that $\binom{n-1-j-k}{j+k}$ is even due to the 2^{e-2}-position. Indeed, $2^{e-1} - 2^h \leqslant j + k < 2^{e-1}$, so $j + k$ has a 1 in the 2^{e-2}-position, while

$$2^{e-1} \leqslant n - j - k - 1 < 2^{e-1} + 2^{t+1} - 2^h < 2^{e-1} + 2^{e-2}$$

since $t < e - 3$. If $j = 2^{e-1} - 2^{e-2} - 2^h$ and $2^{t+1} < k < 2^h$, one easily verifies that $\binom{n-1-j-k}{j+k}$ is even due to the 2^{e-3}-position. Finally, if $j = 2^{e-1} - 2^\ell - 2^h$ with $h < \ell < e - 2$ and $2^{t+1} < k < 2^h$, then $\binom{n-1-j-k}{j+k}$ is even due to the 2^{e-2}-position, as is easily proved. ■

Our final step in the proof of Theorem 1.7 is to prove strict inequality in (2.35) when $j > 2^{e-1}$. Proposition 2.1 implies the result if $k \geqslant 2^t$ or if $j > 2^e$. Thus, by Proposition 2.6, it suffices to prove (2.43) when $2^{e-1} < j \leqslant 2^e$ and $0 \leqslant k < 2^t$. Recall that q is as in (2.39). Because $k < 2^t$, it must be the case that if $\delta = 1$, then 2^{t+r} appears in $q - k$, and similarly 2^w if $\epsilon = 1$ and $w > t$. These will contribute to $\alpha(q-k)$. Thus the only ways to have $D_k := \alpha(q-k) - (\alpha(q) - \alpha(p_0)) \leqslant 0$ are (a) $k = p_0$ and $D_k = 0$; (b) $k = p_0 + 2^{\nu(n)-1}$ and $D_k = -1$; and (c) $k = p_0 + 2^{\nu(n)-1} - 2^v$ and $D_k = 0$.

Similarly to Lemma 2.40, we have for $2^{e-1} \leqslant j \leqslant 2^e$ and $0 \leqslant k < 2^{e-1}$, $\phi(j,k) \leqslant e$ with equality iff $j = 2^e - 2^h$ and $0 \leqslant k < 2^h$ for some $0 \leqslant h < e$, or $j = 2^e$. We will be done once we prove the following lemma. ■

Lemma 2.44. *If $p \in S_n$ and $2^{e-1} < j \leqslant 2^e$ and $0 \leqslant k < 2^t$, then*

1. *if $k = p_0$ or $p_0 + 2^{\nu(n)-1} - 2^v$ for some v, and $\phi(j,k) = e$, then $\binom{n-1-j-k}{j+k}$ is even.*

2. *if* $k = p_0 + 2^{\nu(n)-1}$ *and* $\phi(j,k) = e$, *then* $\widetilde{\nu}_2\binom{n-1-j-k}{j+k} = 2$.
3. *if* $\phi(j,k) = e-1$ *and* $k = p_0 + 2^{\nu(n)-1}$, *then* $\binom{n-1-j-k}{j+k}$ *is even.*

Proof. If $j = 2^e$, then $\widetilde{\nu}\binom{n-1-j-k}{j+k} = 0 = \nu\binom{n-1-j-k}{j+k}$, so now we may assume $j < 2^e$. If $k = p_0$ or $p_0 + 2^{\nu(n)-1}$ and $\phi(j,k) = e$, then $j+k > n-1-j-k$ (and hence $\binom{n-1-j-k}{j+k} = 0$) unless $h = e-2$ and $t = e-1$. But part of the definition of S_n said that if $t = e-1$, then $p_0 \geqslant \Delta$, and hence $j+k > n-1-j-k$ in this case, too. For part (2), we also need that $\binom{n-1-j-k}{j+k-2}$ is even, but it will also be 0, using that $2^{\nu(n)} - 2 \geqslant 0$.

If $k = p_0 + 2^{\nu(n)-1} - 2^v$, then $\binom{n-1-j-k}{j+k}$ is even due to the $2^{\min(v,\nu(n)-1)}$-position if $v \neq \nu(n)-1$, while if $v = \nu(n)-1$, we are in the case $k = p_0$ already handled. A similar argument works for part (3), using the $2^{\min(\nu(j),\nu(k))}$-position, provided $\nu(j) \neq \nu(k)$. However, equality of $\nu(j)$ and $\nu(k)$ will not occur, because one can easily prove by induction on j that if $2^{e-1} \leqslant j < 2^e$ and $0 \leqslant k < 2^{e-1}$ and $\nu(j) = \nu(k)$, then $\alpha(j+k) + \nu(j) - \alpha(k) < e-1$. ■

3. Proofs of results about $\nu\left(\sum \binom{n}{2i+1} i^k\right)$

In this section, we prove four propositions about $\nu\left(\sum \binom{n}{2i+1} i^k\right)$ which were stated and used in the previous section. The polynomials $q_m(x)$ which we introduce in Definition 3.1 might be of independent interest.

Our first proof utilizes an argument of Sun ([13]).

Proof of Proposition 2.3. We mimic the argument in the proof of [13, Thm 1.3]. Let $C_{m,\ell,b} = \sum_i \binom{m}{2i+b}\binom{i}{\ell}$. Using an identity which relates i^k to Stirling numbers, we obtain

$$\sum_i \binom{2n+\epsilon}{2i+b} i^k = \sum_i \binom{2n+\epsilon}{2i+b} \sum_\ell \binom{i}{\ell} \ell! S(k,\ell)$$
$$= \sum_\ell C_{2n+\epsilon,\ell,b}\, \ell! S(k,\ell).$$

Since $C_{2n+1,n,0} = 2n+1$, $C_{2n+1,n-1,0} = \frac{2}{3}(2n+1)(n+1)n$, $C_{2n,n,0} = 1$, $C_{2n,n-1,0} = 2n^2$, $C_{2n,n,1} = 0$, $C_{2n,n-1,1} = 2n$, $C_{2n+1,n,1} = 1$, and $C_{2n+1,n-1,1} = 2n(n+1)$, our result follows from

$$\nu(\ell! C_{2n+\epsilon,\ell,b}) \geqslant \nu((2n+\epsilon)!) - \ell = \nu(n!) + n - \ell,$$

where we have used [14, Thm 1.1] at the first step. ■

The remaining proofs utilize a new family of polynomials $q_m(x)$.

Definition 3.1. *For $m \geqslant 1$, we define polynomials $q_m(x)$ inductively by $q_1(x) = x-1$, and if*

$$(x+1)x(x-1)\cdots(x-m+2) = \sum_{j=1}^{m} b_{j,m}x^j, \tag{3.2}$$

then

$$(x+1)x(x-1)\cdots(x-m+2) = \sum_{j=1}^{m} 2^{m-j}b_{j,m}q_j(x). \tag{3.3}$$

For example, $q_2(x) = x^2 - x + 2$. The relevance of these polynomials is given by the following result.

Theorem 3.4. *For all integers $x \geqslant m$,*

$$\sum_i i^m \binom{x+1}{2i+1} = 2^{x-2m}q_m(x).$$

Proof. The proof is by induction on m. Validity when $m = 1$ follows from

$$2\sum i\binom{x+1}{2i+1} + 2^x = \sum(2i+1)\binom{x+1}{2i+1} = (x+1)\sum\binom{x}{2i} = (x+1)2^{x-1}.$$

We show that $2^{2m-x}\sum i^m\binom{x+1}{2i+1}$ satisfies the equation (3.3) which defines $q_m(x)$. We insert this expression for $q_j(x)$ into the RHS of (3.3) and obtain

$$\begin{aligned} 2^{m-x}\sum_i \binom{x+1}{2i+1}\sum_j (2i)^j b_{j,m} &= 2^{m-x}\sum\binom{x+1}{2i+1}(2i+1)\cdots(2i-m+2) \\ &= 2^{m-x}(x+1)\cdots(x-m+2)\sum\binom{x-m+1}{x-2i}, \end{aligned}$$

but $\sum\binom{x-m+1}{x-2i} = 2^{x-m}$, since it is the sum of all $\binom{x-m+1}{j}$ with j in a fixed parity. Thus we obtain $(x+1)\cdots(x-m+2)$, as desired. At the second step above, we have used (3.2) with $x = 2i$. ■

Proposition 2.4 is an immediate consequence of Theorems 3.4 and 3.5.

Theorem 3.5. *For all integers $x \geqslant m$,*

$$\nu(q_m(x)) \geqslant m - x + \nu((x+1)!) = m + 1 - \alpha(x+1).$$

Proof. The proof is by induction on m. When $m = 1$, it reduces to $\alpha(x+1) + \nu(x-1) \geqslant 2$.

For the LHS of (3.3), note that

$$\nu((x+1)\cdots(x-m+2)) \geqslant \nu((x+1)!) - (x-m),$$

using (2.2). For the j-term ($j < m$) in the sum in (3.3), by induction on m we have 2-exponent

$$\geqslant m - j + \nu(b_{j,m}) + j - x + \nu((x+1)!) \geqslant m - x + \nu((x+1)!).$$

Thus the inequality for $\nu(q_m(x))$ follows by induction. ■

The proof of Proposition 2.5 requires the following two lemmas, and the result follows easily from the second and Theorem 3.4.

Lemma 3.6. *If $b_{j,m}$ is as in Definition 3.1, then*

$$\nu(b_{j,m}) \geqslant \nu(m!) - \nu(j!) - (m-j)$$

with equality iff $\binom{j}{m-j}$ is odd.

Proof. We have

$$\begin{aligned}
\sum_{j\geqslant 0} x^j \sum_{m\geqslant j} b_{j,m} \tfrac{z^m}{m!} &= \sum_{m\geqslant 0} \tfrac{z^m}{m!} \sum_{j=0}^{m} b_{j,m} x^j \\
&= \sum_{m\geqslant 0} \tfrac{1}{m!}(x+1)_m z^m = \sum_{m\geqslant 0} \tbinom{x+1}{m} z^m \\
&= (1+z)^{x+1} = e^{(x+1)\log(1+z)} \\
&= \sum_{k\geqslant 0} \tfrac{1}{k!}(\log(1+z))^k (1+x)^k \\
&= \sum_{k\geqslant 0} \tfrac{1}{k!}(\log(1+z))^k \sum_{i=0}^{k} \tbinom{k}{i} x^i.
\end{aligned}$$

Here we have introduced the notation $(x+1)_m = (x+1)x\cdots(x-m+2)$. Equate coefficients of $x^j z^m$, and get

$$\tfrac{1}{m!} b_{j,m} = \tfrac{1}{j!} \sum_k \tfrac{1}{(k-j)!}([z^m](\log(1+z))^k).$$

Here $[z^m]p(z)$ denotes the coefficient of z^m in $p(z)$. Let $\ell(z) = \log(1+z)/z$. The claim of the lemma reduces to

$$\nu\left(\sum_k \tfrac{1}{(k-j)!}([z^{m-k}]\ell(z)^k)\right) \geqslant -(m-j),$$

or equivalently

$$\nu\left(\sum_k \tfrac{2^k}{(k-j)!}([z^{m-k}]\ell(2z)^k)\right) \geqslant j \text{ with equality iff } \tbinom{j}{m-j} \text{ is odd.}$$

Since $\ell(2z) \equiv 1+z \bmod 2$, and $\nu((k-j)!) \leqslant k-j$ with equality iff $k=j$, all terms in the sum have $\nu(-) \geqslant j$ with equality iff $k=j$ and $\binom{j}{m-j}$ is odd. ■

Lemma 3.7. *Let $q_m(-)$ be as in Definition 3.1, and let x be any integer. Then $\nu(q_m(x)) \geqslant \nu(m!)$ with equality iff $\binom{x-m}{m}$ is odd.*

Proof. We have

$$q_m(x) = (x+1)_m - \sum_{j=1}^{m-1} 2^{m-j} b_{j,m} q_j(x).$$

Note that $\nu((x+1)_m) \geqslant \nu(m!)$ with equality iff $\binom{x+1}{m}$ is odd. By induction, the j-term T_j in the sum satisfies

$$\nu(T_j) \geqslant m - j + \nu(m!) - \nu(j!) - (m-j) + \nu(j!) = \nu(m!)$$

with equality iff $\binom{j}{m-j}$ is odd and $\binom{x-j}{j}$ is odd. This implies the inequality. Equality occurs iff

$$\binom{x+1}{m} + \sum_{j=0}^{m-1} \binom{j}{m-j}\binom{x-j}{j} \tag{3.8}$$

is odd. By Lemma 3.10, $\sum\limits_{j=0}^{m} \binom{j}{m-j}\binom{x-j}{j} \equiv \binom{x+1}{m}$ mod 2. Thus the expression in (3.8) is congruent to $\binom{x-m}{m}$, establishing the claim. ■

Proposition 2.6 follows immediately from Theorem 3.4 and the following result, which is a refinement of Lemma 3.7.

Theorem 3.9. *If m is a positive integer and x is any integer, then, mod 4,*

$$q_m(x)/m! \equiv \binom{x-m}{m} + \begin{cases} 2\binom{x-m}{m-2} & \textit{if } x \textit{ and } m \textit{ are even} \\ 0 & \textit{otherwise.} \end{cases}$$

The proof of Theorem 3.9 requires several subsidiary results.

Lemma 3.10. *If m and x are integers with $m \geqslant 0$, then*

$$\sum_{j=0}^{m} \binom{j}{m-j}\binom{x-j}{j} \equiv \binom{x+1}{m} + 2\binom{x+1}{m-1} \pmod 4.$$

Proof. This follows easily from Jensen's Formula (see e.g., [10]), which says that if A, B, and D are integers with $D \geqslant 0$, then

$$\sum_{j=0}^{D} \binom{j+B}{D-j}\binom{A-j}{j} = \sum_{j=0}^{D} (-1)^j \binom{A+B-j}{D-j}.$$

This implies that the sum in our lemma equals $\sum_{j=0}^{m}(-1)^j\binom{x-j}{m-j}$. We prove that this is congruent, mod 4, to the RHS of our lemma when $x \geqslant 0$ by induction on x. The formula is easily seen to be true if $x = 0$ (note that when $x = 0$ and $m = 1$

the LHS equals -1 while the RHS equals 3), and the induction step is by Pascal's formula. For $x<0$, let $y=-x$ with $y>0$. The equation to be proved becomes

$$\sum_{j=0}^{m}\binom{m+y-1}{m-j}\equiv\binom{y+m-2}{m}-2\binom{y+m-3}{m-1}\pmod 4.$$

When $y=1$, both sides equal $\delta_{m,0}+2\delta_{m,1}$ and the result follows by induction on y using Pascal's formula. ■

The next result refines Lemma 3.6.

Lemma 3.11. *If $b_{j,m}$ is as in Theorem 3.1, then, mod 4,*

$$2^{m-j}j!b_{j,m}/m!\equiv\binom{j}{m-j}+2c_{j,m},\ \textit{where } c_{j,m}=\begin{cases}\binom{j}{m-j-1} & \textit{if } j \textit{ is even}\\ \binom{j}{m-j-2} & \textit{if } j \textit{ is odd.}\end{cases}$$

Proof. As in the proof of 3.6, we have

$$2^{m-j}j!b_{j,m}/m!=\sum_{k\geqslant j}\tfrac{2^{k-j}}{(k-j)!}([z^{m-k}]\ell(2z)^k). \tag{3.12}$$

Since, mod 4, $\ell(2z)\equiv 1-z-2z^3$, and $2^{k-j}/(k-j)!\equiv 0$ unless $k-j$ equals 0 or a 2-power, (3.12) equals

$$[z^{m-j}](1-z-2z^3)^j+2\sum_{e\geqslant 0}[z^{m-j-2^e}](1-z-2z^3)^{j+2^e}$$
$$\equiv[z^{m-j}](1-z-2z^3)^j+2\sum_{e\geqslant 0}\binom{j+2^e}{m-j-2^e}.$$

Replace $m-j$ by ℓ. We must prove, mod 4,

$$A_{j,\ell}+2B_{j,\ell}\equiv C_{j,\ell}+2D_{j,\ell}, \tag{3.13}$$

where

$$A_{j,\ell}=\binom{j}{\ell},\qquad C_{j,\ell}=[z^\ell](1-z-2z^3)^j,\qquad D_{j,\ell}=\sum_{e\geqslant 0}\binom{j+2^e}{\ell-2^e},$$

and

$$B_{j,\ell}=\begin{cases}\binom{j}{\ell-1} & j \text{ even}\\ \binom{j}{\ell-2} & j \text{ odd.}\end{cases}$$

If $j=0$, both sides of (3.13) are congruent to $\delta_{\ell,0}+2\delta_{\ell,1}$. For the RHS, note that if $\ell=2^f$ with $f\geqslant 1$, then $2D_{0,\ell}\equiv 0$ as it obtains a 2 from $e=f$ and from $e=f-1$.

Having proved the validity of (3.13) when $j=0$, we proceed by induction on j. If j is even, then, mod 4,

$$\begin{aligned}&A_{j+1,\ell}+2B_{j+1,\ell}-C_{j+1,\ell}-2D_{j+1,\ell}\\ &\qquad= A_{j,\ell}+A_{j,\ell-1}+2(B_{j,\ell-1}+B_{j,\ell-2})-(C_{j,\ell}-C_{j,\ell-1}-2C_{j,\ell-3})\\ &\qquad\quad -2(D_{j,\ell}+D_{j,\ell-1})\\ &\qquad\equiv (A_{j,\ell}+2B_{j,\ell}-C_{j,\ell}-2D_{j,\ell})+(A_{j,\ell-1}+2B_{j,\ell-1}-C_{j,\ell-1}-2D_{j-1,\ell})\\ &\qquad\quad -2B_{j,\ell}+2B_{j,\ell-2}+2C_{j,\ell-1}+2C_{j,\ell-3}\\ &\qquad\equiv -2\tbinom{j}{\ell-1}+2\tbinom{j}{\ell-3}+2\tbinom{j}{\ell-1}+2\tbinom{j}{\ell-3}\equiv 0,\end{aligned}$$

and a similar argument works when j is odd. ■

The following result relates the even parts in 3.9 and 3.11.

Lemma 3.14. *Let*

$$p_j(x)=\begin{cases}\binom{x-j}{j-2}, & x \text{ and } j \text{ even}\\ 0, & \text{otherwise}\end{cases}\qquad \text{and}\qquad c_{j,m}=\begin{cases}\binom{j}{m-j-1}, & j \text{ even}\\ \binom{j}{m-j-2}, & j \text{ odd.}\end{cases}$$

Then, mod 2, if x and m are integers with $m\geqslant 0$,

$$\binom{x+1}{m-1}\equiv\sum_{j=1}^{m}\Big(\binom{j}{m-j}p_j(x)+c_{j,m}\binom{x-j}{j}\Big). \tag{3.15}$$

Proof. First let x be odd. By Lemma 3.10, mod 2,

$$\binom{x+1}{m-1}\equiv\sum_j\binom{j}{m-j-1}\binom{x-j}{j}.$$

Since $p_j(x)=0$ and $\binom{x-j}{j}\equiv 0$ for odd j, this is equivalent to (3.15) in this case.

Now suppose x is even and m odd. We must prove, mod 2,

$$\binom{x+1}{m-1}\equiv\sum_{j\text{ odd}}\binom{j}{m-j-2}\binom{x-j}{j}+\sum_{j\text{ even}}\Big(\binom{j}{m-j}\binom{x-j}{j-2}+\binom{j}{m-j-1}\binom{x-j}{j}\Big).$$

By 3.10, the LHS is congruent to $\sum\binom{j}{m-j-1}\binom{x-j}{j}$. If j is odd, $\binom{j}{m-j-1}\equiv\binom{j}{m-j-2}$, and if j is even, $\binom{j}{m-j}\equiv 0$. The desired result is now immediate.

Finally suppose x and m are both even. Again using 3.10, we must show

$$\sum_{j\text{ odd}}\binom{j}{m-j-1}\binom{x-j}{j}\equiv\sum_{j\text{ even}}\binom{j}{m-j}\binom{x-j}{j-2}+\sum_{j\text{ odd}}\binom{j}{m-j-2}\binom{x-j}{j}$$

since $\binom{j}{m-j-1}\equiv 0$ if j is even. The terms on the LHS combine with the j-odd terms on the RHS to yield $\sum_{j\text{ odd}}\binom{j+1}{m-j-1}\binom{x-j}{j}$. Letting $k=j+1$, this becomes $\sum_{k\text{ even}}\binom{k}{m-k}\binom{x-k+1}{k-1}$. Since x and k are even, $\binom{x-k+1}{k-1}\equiv\binom{x-k}{k-2}$, and so all terms cancel. ■

Now we easily prove Theorem 3.9.

Proof of Theorem 3.9. The proof is by induction on m, with the case $m = 1$ immediate. Using notation of 3.14, equation (3.3) yields, mod 4,

$$\begin{aligned}
q_m(x)/m! &= \tbinom{x+1}{m} - \sum_{j=1}^{m-1} \frac{j!2^{m-j}b_{j,m}}{m!}\frac{q_j(x)}{j!} \\
&\equiv -2\tbinom{x+1}{m-1} + \sum_{j=0}^{m} \tbinom{j}{m-j}\tbinom{x-j}{j} - \sum_{j=1}^{m-1} \big(\tbinom{j}{m-j} + 2c_{j,m}\big)\big(\tbinom{x-j}{j} + 2p_j(x)\big) \\
&\equiv \tbinom{x-m}{m} - 2\big(\tbinom{x+1}{m-1} - \sum_{j=1}^{m-1} \big(\tbinom{j}{m-j}p_j(x) + c_{j,m}\tbinom{x-j}{j}\big)\big) \\
&\equiv \tbinom{x-m}{m} + 2p_m(x),
\end{aligned}$$

as desired. Here we have used 3.10 and 3.11 at the second step and 3.14 at the last step. ■

4. Relationship with Hensel's Lemma

In [5], the author introduced Lemma 2.8 and applied it to study $\nu(T_5(-))$ and $\nu(T_6(-))$ similarly to what we do here for all $T_n(-)$. Clarke was quick to observe in [3] that if $T_n(-)$ is considered as a function on $\mathbf{Z}_2$, then our conclusion that $\nu(T_n(x)) = \nu(x - x_0) + c_0$ when x is restricted to a congruence class C can be interpreted as saying that $T_n(x_0) = 0$. He showed that if $T_n(x_0) = 0$ and $|T_n'(x_0)| \neq 0$, then

$$|T_n(x)| = |x - x_0||T_n'(x_0)|$$

on a neighborhood of x_0, which corresponds to our congruence class C. Here again $|x| = 1/2^{\nu(x)}$ on $\mathbf{Z}_2$, and $d(x, y) = |x - y|$ defines the metric. Also, T_n' denotes the derivative. Moreover, Clarke noted that the iteration toward the root x_0 in our theorems is a disguised form of Hensel's Lemma for convergence toward a root of the function T_n.

We illustrate by considering the root of T_{13} of the form $4x_0 + 1$. See Theorem 1.2 and Table 1.3. For our iteration toward x_0, let

$$g(x) = \nu(T_{13}(4x + 1)) - 10. \tag{4.1}$$

Then $g(0) = 1$, $g(2^1) = 5$, $g(2^1 + 2^5) = 6$, etc. Thus our early approximation to $4x_0 + 1$ is

$$1 + 4(2^1 + 2^5 + 2^6), \tag{4.2}$$

and, continuing, we obtain that the last 18 digits in the binary expansion of $4x_0+1$ are

$$111001001110001001. \tag{4.3}$$

Note that each 1 in the binary expansion requires a separate calculation.

Now we describe the Hensel point of view, following Clarke ([3]). He showed that

$$T_n'(k) = \sum \binom{n}{2i+1}(2i+1)^k L(2i+1),$$

where

$$L(2i+1) = \sum_{j=1}^{\infty} (-1)^{j-1}(2i)^j/j$$

is the 2-adic logarithm. Hensel's Lemma applied to an analytic function f involves the iteration $k_{n+1} = k_n - \frac{f(k_n)}{f'(k_n)}$, which, under favorable hypotheses, converges to a root of f. We have $f = T_{13}$. Using `Maple`, we find

$$\nu(T_{13}'(k)) = \begin{cases} 8, & k \equiv 1, 2\ (4) \\ 9, & k \equiv 0\ (4) \\ \geqslant 11, & k \equiv 3\ (4). \end{cases} \tag{4.4}$$

To prove this, which involves an infinite sum (for L) and infinitely many values of k, first note that our only claim is about the mod 2^{11} value of T_{13}', and so the sum for L may be stopped after $j = 12$. Since $L(2i+1) \equiv 0 \bmod 4$, we are only concerned with $(2i+1)^k \bmod 2^9$. Since $(2i+1)^k \bmod 2^9$ has period 2^8 in k, performing the computation for 256 values of k would suffice.

Let $k_0 = 1$. Then `Maple` computes that $k_1 = 1 - \frac{T_{13}(1)}{T_{13}'(1)}$ has binary expansion ending 1001001, and so agrees with (4.3) mod 64. Next $k_2 = k_1 - \frac{T_{13}(k_1)}{T_{13}'(k_1)}$ has binary expansion ending 0001110001001, agreeing with (4.3) mod 2^{12}. Finally $k_3 = k_2 - \frac{T_{13}(k_2)}{T_{13}'(k_2)}$ agrees with (4.3), and hence is correct at least mod 2^{18}. This is much faster convergence than ours.

Let $\theta(x) = T_{13}(4x+1)$. Our algorithm essentially applies Hensel's Lemma to $\theta(x)$, but just takes the leading term each time. For all x, $\nu(\theta'(x)) = \nu(4T_{13}'(4x+1)) = 10$, and so our $g(x)$ equals $\nu(\theta(x)/\theta'(x))$. Thus when we let $x_{i+1} = x_i + 2^{g(x_i)}$, we are adding the leading term of $\theta(x_i)/\theta'(x_i)$. Once the limiting value, which we denote by x_0, is found, the root of T_{13} is $4x_0 + 1$.

In [3], Clarke defines, for an analytic function f,

$$\mathbf{g}(x, h) = \frac{f(x+h) - f(x) - hf'(x)}{h^2}$$

and shows that if $f(x_0) = 0$ and $|\mathbf{g}(x,h)| \leqslant 2^r$ for all relevant x and h, then the desired formula

$$|f(x)| = |x - x_0||f'(x_0)|$$

holds for all x satisfying

$$|x - x_0| < |f'(x_0)|/2^r. \tag{4.5}$$

He also notes that our $T_n(-)$ are analytic when restricted to all 2-adic integers of either parity.

For $f = T_{13}$, `Maple` suggests that $\nu(\mathbf{g}(x,h)) \geqslant 7$ for h even, with equality iff $x+h \equiv 0,3 \bmod 4$. This can be easily proved using `Maple` calculations and some elementary arguments. Hence $r=-7$. Using (4.4), we obtain the results of Table 1.3 for $n=13$ which are listed there as $C=[1,2\ (4)]$ and $[0,4\ (8)]$, since

$$\frac{|T'_{13}(x_0)|}{2^{-7}} = \begin{cases} 2^{-1}, & x_0 \equiv 1,2\ (4) \\ 2^{-2}, & x_0 \equiv 0\ (4). \end{cases}$$

Being less than this requires $|x-x_0| \leqslant 2^{-2}$ or 2^{-3} in (4.5), whose reciprocals are the moduli of the congruence classes in Table 1.3.

Another result of [3] gives a condition,

$$|f(x)| < \min\left(\tfrac{|f'(x)|}{2^{k-1}}, \tfrac{|f'(x)|^2}{2^r}\right) \tag{4.6}$$

(where $|\mathbf{g}(x,h)| \leqslant 2^r$ and f is analytic on $c+2^k\mathbf{Z}_2$), which guarantees that iteration of Hensel from x converges to a root of f. For $f=T_{13}$ and $x \equiv 3\ (4)$, $|T'_{13}(x)|^2/2^r \leqslant (2^{-11})^2/2^{-7} = 2^{-15}$ by (4.4), while by Table 1.3 $|T_{13}(x)|$ takes on values 2^{-11}, 2^{-13}, and 2^{-15}. Thus the condition (4.6) does not hold, consistent with our finding in Table 1.3 that $|T_{13}(x)|$ is constant on balls about 7, 3, and 11, so there is no root in these neighborhoods.

Clarke's approach is a very attractive alternative to ours. It converges faster, and it is more closely associated with analytic methods, such as the Hensel/Newton convergence algorithm. On the other hand, there is a certain combinatorial simplicity to our approach, especially Lemma 2.8 and its reduction to consideration of expressions such as (2.11) and (2.35), and subsequently to (2.38). We find it very attractive that for each $f=T_n$, it seems likely that $\mathbf{Z}_2$ can be partitioned into finitely many balls $B(x_0,\epsilon)$ on each of which $|f(x)|$ is linear in $|x-x_0|$ (including the possibility that it is constant). It is not clear which approach will be the better way to establish this.

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