

## A GENERALIZED DIVISOR PROBLEM AND THE SUM OF CHOWLA AND WALUM II

XIAODONG CAO, JUN FURUYA, YOSHIO TANIGAWA, WENGUANG ZHAI

**Abstract:** In this paper, we study the relation between the discrete and the continuous mean values of  $\Delta_a^2(x)$ , where  $\Delta_a(x)$  ( $-1 < a < 1$ ) is the error term in the generalized divisor problem. We try to find the formula of the difference of these mean values in a sufficiently explicit form. As an application we give the asymptotic formula of the discrete mean square of  $\Delta_a(n)$  in the range  $-1 < a < 1, a \neq 0$ . We also study the integral containing the error term in the weighted two-dimensional divisor problem.

**Keywords:** a generalized divisor problem, mean values of the error term, sum of Chowla and Walum.

### 1. Introduction

Let  $\sigma_a(n)$  denote the sum of  $a$ -th powers of positive divisors of  $n$ . This function  $\sigma_a(n)$  is an important arithmetical function in the theory of zeta-functions. Let  $\Delta_a(x)$  be the error function defined by

$$\Delta_a(x) = \sum_{n \leq x} \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a}x^{1+a}, \quad a \neq 0. \quad (1.1)$$

For the mean value  $\sum_{n \leq x} \Delta_a(n)$ , many deep results were obtained. For instance, see Chowla and Pillai [7], Segal [19], MacLeod [16] and Ishibashi [13]. See also Pétermann [18] for  $\Omega$ -results of  $\Delta_a(x)$ .

The case  $a = 0$  is of special importance, in which case the error term  $\Delta_0(x)$  (we use the notation  $\Delta(x)$  instead of  $\Delta_0(x)$ ) should be defined by  $\Delta_0(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1)$ , where  $\gamma$  is the Euler constant. Hardy [12] studied the difference between the continuous and the discrete mean squares of  $\Delta(x)$  and its relevant

---

Xiaodong Cao and Wenguang Zhai are supported by National Natural Science Foundation of China (Grant No. 11171344) and Beijing Natural Science Foundation (Grant No. 1112010). Yoshio Tanigawa is supported by Grant-in-Aid for Scientific Research (No. 21540012) from JSPS.

**2010 Mathematics Subject Classification:** primary: 11N37

function, and consequently he derived the upper bound estimates of two kinds of mean squares. See also our forthcoming paper [5] for this topic.

Recently Furuya [10] proved that

$$\sum_{n \leq x} \Delta^2(n) = \int_1^x \Delta^2(t) dt + \frac{1}{6} x \log^2 x + \frac{8\gamma - 1}{12} x \log x + \frac{8\gamma^2 - 2\gamma + 1}{12} x + R(x), \quad (1.2)$$

where

$$R(x) = \begin{cases} O(x^{\frac{3}{4}} \log x), \\ \Omega_{\pm}(x^{\frac{3}{4}} \log x). \end{cases}$$

Cao and Zhai [2] obtained the precise expression of  $R(x)$ ,

$$R(x) = \frac{(\log x + 2\gamma)}{2\sqrt{2}\pi^2} x^{\frac{3}{4}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{\frac{5}{4}}} \sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + \left(\frac{1}{2} - \psi(x)\right) \Delta^2(x) + O(\sqrt{x} \log x).$$

In this paper, as a continuation of Furuya's result (1.2), we shall investigate the relation between the discrete mean value  $\sum_{n \leq x} \Delta_a^2(n)$  and the continuous mean value  $\int_1^x \Delta_a^2(t) dt$ . We shall try to express the formula of the difference between these two mean values in a sufficiently explicit form.

In our formulation, the sum of Chowla and Walum and its generalization play an important role. We shall prepare some notation. Let  $B_k(x)$  be the Bernoulli polynomial of degree  $k$ . We define the periodic Bernoulli function of degree  $k$  by

$$P_k(x) = B_k(x - [x]),$$

where  $[x]$  denotes the largest integer not exceeding  $x$ . It is well-known that, for instance,  $B_1(x) = x - \frac{1}{2}$ ,  $B_2(x) = x^2 - x + \frac{1}{6}$ ,  $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$  and  $B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$ . The Bernoulli number  $B_n$  of degree  $n$  is defined by  $B_n = B_n(0)$ . Especially we use the notation  $\psi(x) = x - [x] - \frac{1}{2}$  for the first periodic Bernoulli function. Let  $\mathbf{a} = (a_1, a_2)$  be a vector such that  $a_1$  and  $a_2$  are positive numbers. We define the generalization of the sum of Chowla and Walum by

$$G_{b,k}^{\mathbf{a}}(x) = \sum_{n \leq x^{\frac{1}{a_1 + a_2}}} n^b P_k\left(\left(\frac{x}{n^{a_2}}\right)^{1/a_1}\right). \quad (1.3)$$

When  $\mathbf{a} = (1, 1)$ , the above sum is so-called the sum of Chowla and Walum. In this special case, we follow the traditional notation  $G_{b,k}(x) = G_{b,k}^{(1,1)}(x)$ .

For later use, we shall recall some known facts on the sum of Chowla and Walum and the mean square estimate of  $\Delta_a(x)$ . In [8], Chowla and Walum considered the sum  $G_{b,k}(x)$  and posed a conjecture

$$G_{b,k}(x) \ll x^{\frac{b}{2} + \frac{1}{4} + \varepsilon}$$

for every non-negative integer  $b$ , where  $\varepsilon$  is any positive real number. Furthermore they proved that this conjecture is true for  $b = 1$  and  $k = 2$ . After that, Kanemitsu and Sita Rama Chandra Rao [14] proved that

$$G_{b,k}(x) \ll \begin{cases} x^{\frac{1}{2}} \log x & \text{if } b = \frac{1}{2}, \\ x^{\frac{b}{2} + \frac{1}{4}} & \text{if } b > \frac{1}{2}, \\ x^{\frac{3+4b}{10}} & \text{if } 0 \leq b < \frac{1}{2} \end{cases} \quad (1.4)$$

for  $k \geq 2$ . The bound in the last line of (1.4) was recently improved to

$$G_{b,k}(x) \ll x^{\frac{269+410b}{948}} \quad (1.5)$$

by [4, Theorem 2].

The mean square estimate of  $G_{b,k}(x)$  is also known, for instance, by Kanemitsu and Sita Rama Chandra Rao [14] for the case  $|b| \leq 1/2$  and  $k = 2$  and by Balakrishnan and Srinivasan [1] for the case  $b > -1/2$  and  $k \geq 2$ . In [3], by showing the analogue of the Voronoï formula for  $G_{b,k}(x)$ , we derived the asymptotic formula

$$\int_1^T G_{b,k}^2(t) dt = c T^{\frac{3}{2}+b} + O(T^{\frac{3}{2}+b-\delta(b,k)} \log^6 T) \quad (1.6)$$

for the case  $b > -1/2$  and  $k \geq 1$ , where  $c > 0$  is some constant and  $\delta(b, k) > 0$ . See [3] for the details. This means that the conjecture of Chowla and Walum is true in the sense of average. Especially the estimate in the middle line of (1.4) is best possible.

The investigation of the continuous mean value  $\int_1^x \Delta_a^2(t) dt$  also has a long and rich history. Cramér [9] proved that

$$\int_1^x \Delta_a^2(t) dt \sim C(a) x^{\frac{3}{2}+a} \quad \text{if } |a| < 1/2,$$

where  $C(a)$  is the function of  $a$  defined by (1.12) below. Walfisz [21] (see also Introduction in [6]) showed that

$$\begin{aligned} \int_1^x \Delta_1^2(t) dt &= \frac{36 + 5\pi^2}{432} x^3 + O(x^{\frac{5}{2}} \log x), \\ \int_1^x \Delta_{-1}^2(t) dt &= \left( \left( \frac{\gamma + \log 2\pi}{2} \right)^2 + \frac{5\pi^2}{144} \right) x + O(x^{\frac{1}{2}} \log x). \end{aligned}$$

Chowla [6] studied the mean square of  $\Delta_a(x)$  in detail for  $1/2 \leq |a| < 1$ . In fact he showed that

$$\int_1^x \Delta_a^2(t) dt = \begin{cases} \left( \frac{\zeta^2(-a)}{4} + \frac{\zeta(-2a)\zeta^2(1-a)}{12\zeta(2-2a)} \right) x \\ \quad + O(x^{\frac{3}{2}+a} \log x) & \text{if } -1 < a < -\frac{1}{2}, \\ O(x \log x) & \text{if } a = -\frac{1}{2}, \\ O(x^2 \log x) & \text{if } a = \frac{1}{2}, \\ \frac{\zeta(2a)\zeta^2(1+a)}{12(1+2a)\zeta(2+2a)} x^{1+2a} + o(x^{1+2a}), & \text{if } \frac{1}{2} < a < 1. \end{cases} \quad (1.7)$$

Sixty four years later, Meurman [17] showed that

$$\int_1^x \Delta_a^2(t) dt = \begin{cases} C(a)x^{\frac{3}{2}+a} + O(x) & \text{if } -\frac{1}{2} < a < 0, \\ \frac{\zeta^2(3/2)}{24\zeta(3)} x \log x + O(x) & \text{if } a = -\frac{1}{2}. \end{cases} \quad (1.8)$$

When  $0 < a < 1/2$ , Cao, Tanigawa and Zhai [3] proved that

$$\int_1^x \Delta_a^2(t) dt = C(a)x^{\frac{3}{2}+a} + O\left(x^{\frac{3a}{2}+\frac{5}{4}} \log^8 x\right). \quad (1.9)$$

We note that the definition of  $\Delta_a(x)$  in [17, 3] is slightly different from (1.1). However, by Lemmas 2.4 and Proposition 5.1 below we can modify their proofs to get (1.8) and (1.9).

As is stated before, we are interested in the detailed description of the difference between the discrete and the continuous mean squares of  $\Delta_a(x)$ . Our main results can be stated as follows.

**Theorem 1.** *Let  $-1 < a < 1$  and  $a \neq 0$  and let  $U_a(x)$  be the function defined by*

$$U_a(x) = \sum_{n \leq x} \Delta_a^2(n) - \int_1^x \Delta_a^2(t) dt - \left( \frac{1}{2} - \psi(x) \right) \Delta_a^2(x).$$

(i) For  $a \neq \pm 1/2$ , we have

$$\begin{aligned}
 U_a(x) = & C_1(a)x + C_2(a)x^{1+a} + C_3(a)x^{2a+1} \\
 & - \frac{\zeta^2(1+a)}{3}P_3(x)x^{2a} + \frac{a(a-1)}{12(2a-1)}B_4\zeta^2(1+a)x^{2a-1} \\
 & + \zeta(1+a)\left\{\frac{\zeta(-a)}{2}P_2(x) - \frac{2\zeta(1-a)}{3}P_3(x)\right\}x^a \\
 & - \left(\frac{1}{3}P_3(\sqrt{x}) + \psi(\sqrt{x})P_2(\sqrt{x})\right)\left(\zeta(1-a)x^{\frac{a+1}{2}} + \zeta(1+a)x^{\frac{3a+1}{2}}\right) \\
 & + \zeta(1-a)\left\{\frac{a-3}{12}P_4(\sqrt{x}) + \frac{a}{3}\psi(\sqrt{x})P_3(\sqrt{x})\right. \\
 & \left.+ \frac{a+1}{4}P_2^2(\sqrt{x}) + \left(P_2(x) - \frac{1}{6}\right)\left(2P_2(\sqrt{x}) + \frac{1}{12}\right)\right\}x^{\frac{a}{2}} \\
 & + \zeta(1+a)\left\{\frac{2a-3}{12}P_4(\sqrt{x}) + \frac{2a}{3}\psi(\sqrt{x})P_3(\sqrt{x})\right. \\
 & \left.+ \frac{2a+1}{4}P_2^2(\sqrt{x}) + \left(P_2(x) - \frac{1}{6}\right)\left(2P_2(\sqrt{x}) + \frac{1}{12}\right)\right\}x^{\frac{3a}{2}} \\
 & + \mathcal{E}_a(x) + \mathcal{C}(a) + O(x^{(3a-1)/2} + x^{(a-1)/2}), \tag{1.10}
 \end{aligned}$$

with

$$\begin{aligned}
 C_1(a) &= \frac{\zeta(1-a)}{2} \left( \frac{\zeta(1-a)}{3} - \zeta(-a) \right), \\
 C_2(a) &= \frac{\zeta(1+a)}{1+a} \left( \frac{\zeta(1-a)}{3} - \frac{\zeta(-a)}{2} \right), \\
 C_3(a) &= \frac{\zeta^2(1+a)}{6(1+2a)},
 \end{aligned}$$

where  $\mathcal{E}_a(x)$  and  $\mathcal{C}(a)$  are the functions given by (3.10) and (3.11) below.

(ii) For  $a = \pm 1/2$ , we have

$$\begin{aligned}
 U_{-1/2}(x) = & C_1\left(-\frac{1}{2}\right)x + C_2\left(-\frac{1}{2}\right)x^{\frac{1}{2}} + q_1(x)x^{\frac{1}{4}} + p_1 \log x + p_2 \\
 & + q_2(x)x^{-\frac{1}{4}} + q_3(x)x^{-\frac{1}{2}} + \mathcal{E}_{-1/2}(x) + O(x^{-\frac{3}{4}}),
 \end{aligned}$$

and

$$\begin{aligned}
 U_{1/2}(x) = & C_3\left(\frac{1}{2}\right)x^2 + C_2\left(\frac{1}{2}\right)x^{\frac{3}{2}} + q_4(x)x^{\frac{5}{4}} + q_5(x)x + q_6(x)x^{\frac{3}{4}} \\
 & + q_7(x)x^{\frac{1}{2}} + \mathcal{E}_{1/2}(x) + O(x^{\frac{1}{4}}),
 \end{aligned}$$

where  $p_j$  are absolute constants and  $q_j(x)$  are bounded functions of  $x$ .

For a simpler form, we have the next corollary whose proof is immediately obtained by Theorem 1, (1.4) and (1.5).

**Corollary 1.** *Let  $-1 < a < 1$  and  $a \neq 0$ . Then we have*

$$U_a(x) = \begin{cases} C_1(a)x + C_2(a)x^{1+a} + O(x^{\frac{1+a}{2}}) & \text{if } -1 < a \leq -\frac{1}{3}, \\ C_1(a)x + C_2(a)x^{1+a} + C_3(a)x^{1+2a} + O(x^{\frac{1+a}{2}}) & \text{if } -\frac{1}{3} < a < 0, \\ C_3(a)x^{1+2a} + C_2(a)x^{1+a} + C_1(a)x + O(x^{\frac{1+3a}{2}}) & \text{if } 0 < a < \frac{1}{3}, \\ C_3(a)x^{1+2a} + C_2(a)x^{1+a} + O(x^{\frac{1+3a}{2}}) & \text{if } \frac{1}{3} \leq a < 1 \end{cases} \\ + \hat{\mathcal{E}}_a(x), \quad (1.11)$$

where the constants  $C_j(a)$  are the same as those in Theorem 1, and

$$\hat{\mathcal{E}}_a(x) = -\frac{1}{2} (\zeta(1-a) + \zeta(1+a)x^a) (G_{1+a,2}(x) + x^a G_{1-a,2}(x)).$$

In particular, for every  $\varepsilon > 0$ , we have

$$\hat{\mathcal{E}}_a(x) \ll \begin{cases} x^{\frac{679+410a}{948}+\varepsilon} & \text{if } -1 < a < -\frac{1}{2}, \\ x^{\frac{1}{2}} \log x & \text{if } a = -\frac{1}{2}, \\ x^{\frac{a}{2}+\frac{3}{4}} & \text{if } -\frac{1}{2} < a < 0, \\ x^{\frac{3a}{2}+\frac{3}{4}} & \text{if } 0 < a < \frac{1}{2}, \\ x^{\frac{3}{2}} \log x & \text{if } a = \frac{1}{2}, \\ x^{\frac{679+1486a}{948}+\varepsilon} & \text{if } \frac{1}{2} < a < 1. \end{cases}$$

For the mean square of  $\Delta_a(x)$ , we shall prove the following Theorem 2, which improves the last two formulas in (1.7) and the formula (1.9).

**Theorem 2.** *we have*

$$\int_1^x \Delta_a^2(t) dt = \begin{cases} \frac{\zeta^2(3/2)\zeta(a+3/2)\zeta(3/2-a)}{(4a+6)\pi^2\zeta(3)} x^{\frac{3}{2}+a} + O(x^{1+2a}) & \text{if } 0 < a < \frac{1}{2}, \\ \frac{\zeta^2(3/2)}{48\zeta(3)} x^2 \log x + O(x^2) & \text{if } a = \frac{1}{2}, \\ \frac{\zeta(2a)\zeta^2(1+a)}{12(1+2a)\zeta(2+2a)} x^{1+2a} + O(x^{\frac{3}{2}+a} \log x) & \text{if } \frac{1}{2} < a < 1. \end{cases}$$

As an arithmetic application of Theorem 1 and Theorem 2, one can study the discrete mean square of  $\Delta_a(n)$  for any  $-1 < a < 1, a \neq 0$ . The main result is the following

**Theorem 3.** *Let  $-1 < a < 1$  and  $a \neq 0$ . Then we have*

$$\sum_{n \leq x} \Delta_a^2(n) = \begin{cases} C(a)x + O(x^{\frac{3}{2}+a} \log x) & \text{if } -1 < a < -\frac{1}{2}, \\ C(a)x \log x + O(x) & \text{if } a = -\frac{1}{2}, \\ C(a)x^{\frac{3}{2}+a} + O(x) & \text{if } -\frac{1}{2} < a < 0, \\ C(a)x^{\frac{3}{2}+a} + O(x^{1+2a}) & \text{if } 0 < a < \frac{1}{2}, \\ C(a)x^2 \log x + O(x^2) & \text{if } a = \frac{1}{2}, \\ C(a)x^{1+2a} + O(x^{\frac{3}{2}+a} \log x) & \text{if } \frac{1}{2} < a < 1, \end{cases}$$

where

$$C(a) = \frac{1}{(4a+6)\pi^2} \sum_{n=1}^{\infty} \frac{\sigma_a^2(n)}{n^{a+3/2}} = \frac{\zeta^2(3/2)\zeta(a+3/2)\zeta(3/2-a)}{(4a+6)\pi^2\zeta(3)} \quad (1.12)$$

if  $|a| < 1/2$ ,

$$C(a) = \frac{\zeta(1-a)}{2} \left( \frac{\zeta(1-a)}{3} - \zeta(-a) \right) + \frac{1}{4}\zeta^2(-a) + \frac{\zeta(-2a)\zeta^2(1-a)}{12\zeta(2-2a)}$$

if  $-1 < a < -1/2$  and

$$C(a) = \begin{cases} \frac{\zeta^2(3/2)}{24\zeta(3)} & \text{if } a = -\frac{1}{2}, \\ \frac{\zeta^2(3/2)}{12} \left( \frac{1}{4\zeta(3)} + 1 \right) & \text{if } a = \frac{1}{2}, \\ \frac{\zeta^2(1+a)}{6(1+2a)} + \frac{\zeta(2a)\zeta^2(1+a)}{12(1+2a)\zeta(2+2a)} & \text{if } \frac{1}{2} < a < 1. \end{cases}$$

**Remark 1.1.** From Theorem 2, Theorem 3, (1.7) and (1.8), we see that

$$\begin{aligned} \sum_{n \leq x} \Delta_a^2(n) - \int_1^x \Delta_a^2(t) dt \\ = \begin{cases} o\left(\int_1^x \Delta_a^2(t) dt\right) & \text{if } -\frac{1}{2} \leq a < \frac{1}{2} \text{ and } a \neq 0 \\ (\hat{c}(a) + o(1)) \left(\int_1^x \Delta_a^2(t) dt\right) & \text{if } -1 < a < -\frac{1}{2} \text{ or } \frac{1}{2} \leq a < 1, \end{cases} \end{aligned}$$

where  $\hat{c}(a)$  is a positive constant.<sup>1</sup> This means that the leading terms of the asymptotic formulas of  $\sum_{n \leq x} \Delta_a^2(n)$  and  $\int_1^x \Delta_a^2(t) dt$  are the same for  $-1/2 \leq a < 1/2$  and  $a \neq 0$  and different for  $-1 < a < -1/2$  or  $1/2 \leq a < 1$ .

<sup>1</sup>It is clear that  $\hat{c}(a) > 0$  for  $1/2 < a < 1$ . While in the case  $-1 < a < -1/2$ , we have  $\hat{c}(a) = \zeta(1-a)(\zeta(1-a)/3 - \zeta(-a))/2$ , from which we can find that  $\hat{c}(a) > 0$  since  $\zeta(-a) < 0$  and  $\zeta(1-a) > 0$ .

This paper is organized as follows. In Section 2, in the category of a general weighted two-dimensional lattice points problem, we give a sharper form for its error term. Although the method used in this section is very elementary and classical, we would like to give a full proof for the sake of completeness. In Section 3, we shall give a proof of Theorem 1 based on Lemma 2.4. In the following section, we first use the idea of Chowla [6] to prove a key auxiliary lemma (i.e. Lemma 4.1 below), then give the proof of Theorem 2, and finally finish the proof of Theorem 3. In the last section, we further develop our method in the previous paper [4] and derive an asymptotic representation of the integral defined by (5.1) below. The formula of Proposition 5.1 is interesting in itself.

For confirmation, the authors have checked all formulas in Theorem 1 and Proposition 5.1 with the help of Mathematica 8.0.

## 2. A weighted two-dimensional lattice points problem

Let  $\mathbf{a} = (a_1, a_2)$  with positive numbers  $a_1$  and  $a_2$ , and let  $\mathbf{b} = (b_1, b_2)$  with  $b_1 \neq -1$  and  $b_2 \neq -1$ . We consider a weighted two-dimensional lattice points problem

$$D(\mathbf{a}, \mathbf{b}; x) = \sum_{n_1^{a_1} n_2^{a_2} \leq x} n_1^{b_1} n_2^{b_2}.$$

As usual, the error term of  $D(\mathbf{a}, \mathbf{b}; x)$  is defined by

$$\Delta(\mathbf{a}, \mathbf{b}; x) = D(\mathbf{a}, \mathbf{b}; x) - H(\mathbf{a}, \mathbf{b}; x),$$

where, in the case  $a_1(b_2 + 1) \neq a_2(b_1 + 1)$ , for instance,

$$H(\mathbf{a}, \mathbf{b}; x) = \frac{\zeta((b_1 + 1)\frac{a_2}{a_1} - b_2)}{b_1 + 1} x^{\frac{b_1+1}{a_1}} + \frac{\zeta((b_2 + 1)\frac{a_1}{a_2} - b_1)}{b_2 + 1} x^{\frac{b_2+1}{a_2}}$$

is the well-known main term. See (2.11) for details. If  $a_1$  and  $a_2$  are positive integers, this is a weighted two-dimensional divisor problem. For the history and the results of the two-dimensional divisor problems, see Vogts [20] and Chapter 5 in Krätzel [15].

The main purpose of this section is to give a sharper form for  $\Delta(\mathbf{a}, \mathbf{b}; x)$ . For this purpose, we prepare some lemmas.

**Lemma 2.1.** *Let  $z$  be any complex number and let*

$$W_z(x) = \int_1^x t^z \psi(t) dt.$$

*Then we have for  $x \geq 1$  and a positive integer  $N$ ,*

$$W_z(x) = \beta(z) + \sum_{j=2}^N \beta_j(z) P_j(x) x^{z+2-j} + O(x^{\Re z+1-N}), \quad (2.1)$$



where

$$\beta(z) = \begin{cases} \frac{1}{2} - \gamma & \text{if } z = -2, \\ \frac{1}{2} \log 2\pi - 1 & \text{if } z = -1, \\ -\frac{z}{2(z+1)(z+2)} + \frac{\zeta(-z-1)}{z+1} & \text{otherwise,} \end{cases} \quad (2.2)$$

and

$$\beta_2(z) = \frac{1}{2} \quad \text{and} \quad \beta_j(z) = (-1)^j \frac{z(z-1)\cdots(z-j+3)}{j!} \quad \text{for } j \geq 3.$$

**Proof.** First assume that  $\Re z < 0$ . It is well-known that  $\beta(z)$  of (2.2) is equal to

$$\int_1^\infty t^z \psi(t) dt.$$

Splitting the line of integration at  $x$  and applying integration by parts we get

$$\beta(z) = W_z(x) - \frac{1}{2} P_2(x) x^z - \frac{z}{2} \int_x^\infty t^{z-1} P_2(t) dt.$$

The integral in the right hand side converges in the region  $\Re z < 1$ . Repeating this process, we get

$$W_z(x) = \beta(z) + \sum_{j=2}^M \beta_j(z) P_j(x) x^{z+2-j} + O(x^{\Re z+1-M}) \quad (2.3)$$

for any positive integer  $M \geq 2$  with  $\Re z + 1 < M$ .

Let  $N$  be given. If  $\Re z < N$ , we can take  $M = N + 1$ . If  $\Re z \geq N$ , we take  $M > \Re z + 1$  in (2.3) and evaluate the terms for  $j \geq N + 1$  as an error term. This completes the proof of (2.1).  $\blacksquare$

**Lemma 2.2.**

(i) For  $z \neq -1$ , we have for  $x > 1$

$$\sum_{n \leq x} n^z = \frac{x^{z+1}}{z+1} - \psi(x) x^z + \frac{z-1}{2(z+1)} + z W_{z-1}(x). \quad (2.4)$$

(ii) Let  $z$  be a complex number with  $z \neq -1$ . Then we have for  $x > 1$

$$\sum_{n \leq x} n^z = \zeta(-z) + \frac{x^{z+1}}{z+1} + R_z(x) + O(x^{\Re z-4}), \quad (2.5)$$

and

$$\sum_{n \leq x} n^z \log n = -\zeta'(-z) + \frac{x^{z+1} \log x}{z+1} - \frac{x^{z+1}}{(z+1)^2} \quad (2.6)$$

$$+ R_z(x) \log x + Q_z(x) + O(x^{\Re z - 4} \log x),$$

where we put

$$R_z(x) = -\psi(x)x^z + \langle z \rangle_2 P_2(x)x^{z-1} - \langle z \rangle_3 P_3(x)x^{z-2} + \langle z \rangle_4 P_4(x)x^{z-3} \quad (2.7)$$

with

$$\langle z \rangle_j = \frac{z(z-1) \cdots (z-j+2)}{j!} \quad (j \geq 2)$$

and

$$Q_z(x) = \frac{1}{2} P_2(x)x^{z-1} - \frac{2z-1}{6} P_3(x)x^{z-2} + \frac{3z^2-6z+2}{24} P_4(x)x^{z-3}.$$

**Proof.** The assertion (2.4) is Lemma 1 in [11]. The formula (2.5) follows immediately from Lemma 2.1 and (2.4) by taking  $N = 4$ . For (2.6), we note that the error term of (2.5) can be expanded into an asymptotic series with the error term as

$$f(z) \int_x^\infty t^{z-N} P_N(t) dt$$

for large  $N > \Re z + 1$ , where  $f(z)$  is a polynomial of  $z$  of degree  $N$ . Since this integral converges uniformly with respect to  $z$  with  $\Re z < N$ , we can differentiate this formula under the integral sign with respect to  $z$ , hence we get the error term described above. Noting that  $\frac{\partial}{\partial z} R_z(x) = R_z(x) \log x + Q_z(x)$ , we get the formula (2.6).  $\blacksquare$

**Remark 2.1.** When  $z = -1$ , the formulas (2.5) and (2.6) hold true in the forms:

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + R_{-1}(x) + O(x^{-5}),$$

and

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2} \log^2 x - \gamma_1 + R_{-1}(x) \log x + Q_{-1}(x) + O(x^{-5} \log x),$$

where  $\gamma_1$  is the generalized Euler constant of order 1 defined by  $\zeta(s) = 1/(s-1) + \gamma + \gamma_1(s-1) + O((s-1)^2)$  for  $s$  near to 1.

**Remark 2.2.** The asymptotic expression (2.6) may be obtained by (2.5) with an error term as an integral form and partial summation. But by the above method, we can determine the constant  $-\zeta'(-z)$  very easily.

Now we shall state the main result of this section. In this lemma and in the sequel of this paper, we write  $\tilde{\mathbf{u}} = (u_2, u_1)$  for the vector  $\mathbf{u} = (u_1, u_2)$ .

**Lemma 2.3.** *Let  $\mathbf{a} = (a_1, a_2)$  with positive numbers  $a_1$  and  $a_2$ , and let  $\mathbf{b} = (b_1, b_2)$  with  $b_1 \neq -1$  and  $b_2 \neq -1$ . Define  $\rho$ ,  $\tilde{\rho}$ ,  $C_{3,1}(\mathbf{a}, \mathbf{b})$ ,  $C_{4,1}(\mathbf{a}, \mathbf{b})$  and  $C_{4,2}(\mathbf{b})$  by*

$$\rho = \rho(\mathbf{a}, \mathbf{b}) = b_2 - \frac{a_2(b_1 + 1)}{a_1}, \quad (2.8)$$

$$\tilde{\rho} = \rho(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) = b_1 - \frac{a_1(b_2 + 1)}{a_2}, \quad (2.9)$$

$$C_{3,1}(\mathbf{a}, \mathbf{b}) = - \left( \frac{\langle \rho \rangle_3 - \langle b_2 \rangle_3}{b_1 + 1} + \frac{\langle \tilde{\rho} \rangle_3 - \langle b_1 \rangle_3}{b_2 + 1} \right),$$

$$C_{4,1}(\mathbf{a}, \mathbf{b}) = \frac{\langle \rho \rangle_4 - \langle b_2 \rangle_4}{b_1 + 1} + \frac{\langle \tilde{\rho} \rangle_4 - \langle b_1 \rangle_4}{b_2 + 1}$$

and

$$C_{4,2}(\mathbf{b}) = -(\langle b_1 \rangle_3 + \langle b_2 \rangle_3)$$

Let  $x > 1$  and let  $y = x^{\frac{1}{a_1 + a_2}}$ . Then we have

$$D(\mathbf{a}, \mathbf{b}; x) = H(\mathbf{a}, \mathbf{b}; x) + \Delta(\mathbf{a}, \mathbf{b}; x), \quad (2.10)$$

where

$$H(\mathbf{a}, \mathbf{b}; x) \quad (2.11)$$

$$= \begin{cases} \frac{\zeta(-\rho)}{b_1 + 1} x^{\frac{b_1 + 1}{a_1}} + \frac{\zeta(-\tilde{\rho})}{b_2 + 1} x^{\frac{b_2 + 1}{a_2}} & \text{if } a_1(b_2 + 1) \neq a_2(b_1 + 1), \\ \left( \frac{2 + b_1 + b_2}{(b_1 + 1)(b_2 + 1)} \left( \frac{\log x}{a_1 + a_2} + \gamma \right) - \frac{1}{(b_1 + 1)(b_2 + 1)} \right) x^{\frac{b_1 + 1}{a_1}} & \text{otherwise,} \end{cases}$$

and

$$\Delta(\mathbf{a}, \mathbf{b}; x) = \sum_{j=1}^4 E_j(\mathbf{a}, \mathbf{b}; x)$$

with

$$E_1(\mathbf{a}, \mathbf{b}; x) = -x^{\frac{b_1}{a_1}} G_{\rho + \frac{a_2}{a_1}, 1}^{\mathbf{a}}(x) - x^{\frac{b_2}{a_2}} G_{\tilde{\rho} + \frac{a_1}{a_2}, 1}^{\tilde{\mathbf{a}}}(x),$$

$$E_2(\mathbf{a}, \mathbf{b}; x) = \sum_{j=2}^4 (-1)^j \left( \langle b_1 \rangle_j x^{\frac{b_1-(j-1)}{a_1}} G_{\rho+\frac{a_2 j}{a_1}, j}^{\mathbf{a}}(x) + \langle b_2 \rangle_j x^{\frac{b_2-(j-1)}{a_2}} G_{\tilde{\rho}+\frac{a_1 j}{a_2}, j}^{\tilde{\mathbf{a}}}(x) \right),$$

$$\begin{aligned} E_3(\mathbf{a}, \mathbf{b}; x) &= \zeta(-b_1)\zeta(-b_2) - \left\{ \frac{1}{2} \left( \frac{a_2}{a_1} + \frac{a_1}{a_2} \right) P_2(y) + \psi^2(y) \right\} y^{b_1+b_2} \\ &\quad + \left( C_{3,1}(\mathbf{a}, \mathbf{b}) P_3(y) + \frac{b_1+b_2}{2} \psi(y) P_2(y) \right) y^{b_1+b_2-1} \\ &\quad + \left( C_{4,1}(\mathbf{a}, \mathbf{b}) P_4(y) + C_{4,2}(\mathbf{b}) \psi(y) P_3(y) - \frac{b_1 b_2}{4} P_2^2(y) \right) y^{b_1+b_2-2} \end{aligned}$$

and

$$E_4(\mathbf{a}, \mathbf{b}; x) = O \left( \left( x^{\frac{b_1-4}{a_1}} + x^{\frac{b_2-4}{a_2}} \right) \log x + y^{b_1-4} + y^{b_2-4} + y^{b_1+b_2-3} \right). \quad (2.12)$$

Here  $G_{b,k}^{\mathbf{a}}(x)$  is the generalized sum of Chowla and Walum defined by (1.3).

**Remark 2.3.** In fact, the coefficients of  $y^{b_1+b_2-j}$  ( $j = 0, 1, 2$ ) in  $E_3(\mathbf{a}, \mathbf{b}; x)$  are polynomials of degree  $j+2$  in  $\psi(y)$ .

**Remark 2.4.** For the case  $\mathbf{a} = (a, b)$  and  $\mathbf{b} = (0, 0)$ , Lemma 2.3 is superior to Lemma 3.1 in Cao and Zhai [2].

**Remark 2.5.** The coefficient  $C_{3,1}(\mathbf{a}, \mathbf{b})$  is explicitly given by

$$C_{3,1}(\mathbf{a}, \mathbf{b}) = \frac{a_2(2b_2-1)}{6a_1} + \frac{a_1(2b_1-1)}{6a_2} - \frac{a_2^2(b_1+1)}{6a_1^2} - \frac{a_1^2(b_2+1)}{6a_2^2}.$$

**Remark 2.6.** If we assume

$$-1 < b_1 < 4 + \frac{a_1(b_2+1)}{a_2} \quad \text{and} \quad -1 < b_2 < 4 + \frac{a_2(b_1+1)}{a_1},$$

the error estimate (2.12) becomes

$$E_4(\mathbf{a}, \mathbf{b}; x) \ll y^{b_1+b_2-3}.$$

**Remark 2.7.** The assertion of Lemma 2.3 holds for any fixed complex numbers  $b_j \neq -1$  ( $j = 1, 2$ ). But  $b_j$  in the error term  $E_4(\mathbf{a}, \mathbf{b}; x)$  should be replaced by  $\Re b_j$ .

**Proof.** We first consider the case  $a_1(b_2+1) \neq a_2(b_1+1)$ . Applying the Dirichlet hyperbola method, we have

$$\begin{aligned} D(\mathbf{a}, \mathbf{b}; x) &= \sum_{n_1 \leq y} n_1^{b_1} \sum_{n_2^{a_2} \leq x/n_1^{a_1}} n_2^{b_2} + \sum_{n_2 \leq y} n_2^{b_2} \sum_{n_1^{a_1} \leq x/n_2^{a_2}} n_1^{b_1} \\ &\quad - \sum_{n_1 \leq y} n_1^{b_1} \sum_{n_2 \leq y} n_2^{b_2} \\ &= \Sigma_1 + \Sigma_2 - \Sigma_3, \end{aligned} \quad (2.13)$$

say.

Applying (2.5) in Lemma 2.2, we get

$$\begin{aligned} \Sigma_1 = & \zeta(-b_2) \sum_{n_1 \leq y} n_1^{b_1} + \frac{x^{\frac{b_2+1}{a_2}}}{b_2+1} \sum_{n_1 \leq y} n_1^{\tilde{\rho}} + \sum_{n_1 \leq y} n_1^{b_1} R_{b_2} \left( \left( \frac{x}{n_1^{a_1}} \right)^{1/a_2} \right) \\ & + O \left( x^{\frac{b_2-4}{a_2}} \sum_{n_1 \leq y} n_1^{b_1 - \frac{a_1(b_2-4)}{a_2}} \right). \end{aligned} \quad (2.14)$$

The last error term in the right hand side of (2.14) is dominated from above by  $x^{(b_2-4)/a_2} \log x + y^{b_1+b_2-3}$ . Applying (2.5) again to the first two sums of the right hand side of (2.14) and rearranging the terms, we have

$$\begin{aligned} \Sigma_1 = & \zeta(-b_1)\zeta(-b_2) + \frac{\zeta(-b_2)}{b_1+1} y^{b_1+1} + \zeta(-b_2) R_{b_1}(y) \\ & + \frac{\zeta(-\tilde{\rho})}{b_2+1} x^{\frac{b_2+1}{a_2}} + \frac{1}{(b_2+1)(\tilde{\rho}+1)} y^{b_1+b_2+2} + \frac{1}{b_2+1} x^{\frac{b_2+1}{a_2}} R_{\tilde{\rho}}(y) \\ & + \sum_{n_1 \leq y} n_1^{b_1} R_{b_2} \left( \left( \frac{x}{n_1^{a_1}} \right)^{1/a_2} \right) + O \left( x^{\frac{b_2-4}{a_2}} \log x + y^{b_1-4} + y^{b_1+b_2-3} \right). \end{aligned} \quad (2.15)$$

Similarly to  $\Sigma_1$ , we have

$$\begin{aligned} \Sigma_2 = & \zeta(-b_1)\zeta(-b_2) + \frac{\zeta(-b_1)}{b_2+1} y^{b_2+1} + \zeta(-b_1) R_{b_2}(y) \\ & + \frac{\zeta(-\rho)}{b_1+1} x^{\frac{b_1+1}{a_1}} + \frac{1}{(b_1+1)(\rho+1)} y^{b_1+b_2+2} + \frac{1}{b_1+1} x^{\frac{b_1+1}{a_1}} R_{\rho}(y) \\ & + \sum_{n_2 \leq y} n_2^{b_2} R_{b_1} \left( \left( \frac{x}{n_2^{a_2}} \right)^{1/a_1} \right) + O \left( x^{\frac{b_1-4}{a_1}} \log x + y^{b_2-4} + y^{b_1+b_2-3} \right). \end{aligned} \quad (2.16)$$

As for  $\Sigma_3$ , we use (2.5) again to get

$$\begin{aligned} \Sigma_3 = & \left( \zeta(-b_1) + \frac{1}{b_1+1} y^{b_1+1} + R_{b_1}(y) + O(y^{b_1-4}) \right) \\ & \times \left( \zeta(-b_2) + \frac{1}{b_2+1} y^{b_2+1} + R_{b_2}(y) + O(y^{b_2-4}) \right). \end{aligned} \quad (2.17)$$

Substituting (2.15), (2.16) and (2.17) into (2.13), and noting that

$$\frac{1}{(b_2+1)(\tilde{\rho}+1)} + \frac{1}{(b_1+1)(\rho+1)} = \frac{1}{(b_1+1)(b_2+1)},$$

we easily get

$$\begin{aligned}
D(\mathbf{a}, \mathbf{b}; x) &= \zeta(-b_1)\zeta(-b_2) + \frac{\zeta(-\rho)}{b_1+1}x^{\frac{b_1+1}{a_1}} + \frac{\zeta(-\tilde{\rho})}{b_2+1}x^{\frac{b_2+1}{a_2}} \\
&+ \sum_{n_1 \leq y} n_1^{b_1} R_{b_2} \left( \left( \frac{x}{n_1^{a_1}} \right)^{1/a_2} \right) + \sum_{n_2 \leq y} n_2^{b_2} R_{b_1} \left( \left( \frac{x}{n_2^{a_2}} \right)^{1/a_1} \right) \\
&+ \frac{1}{b_1+1} \left( x^{\frac{b_1+1}{a_1}} R_{\rho}(y) - y^{b_1+1} R_{b_2}(y) \right) \\
&+ \frac{1}{b_2+1} \left( x^{\frac{b_2+1}{a_2}} R_{\tilde{\rho}}(y) - y^{b_2+1} R_{b_1}(y) \right) - R_{b_1}(y) R_{b_2}(y) \\
&+ O \left( \left( x^{\frac{b_1-4}{a_1}} + x^{\frac{b_2-4}{a_2}} \right) \log x + y^{b_1-4} + y^{b_2-4} + y^{b_1+b_2-3} \right). \quad (2.18)
\end{aligned}$$

The second and the third terms in the right hand side constitute the main term  $H(\mathbf{a}, \mathbf{b}; x)$ , and we can easily see that the fourth and the fifth terms become  $E_1(\mathbf{a}, \mathbf{b}; x)$  and  $E_2(\mathbf{a}, \mathbf{b}; x)$  by the definition (2.7) of  $R_z(x)$ . For the other terms, substitute the definition (2.7) of  $R_z(x)$  again, and we get

$$\begin{aligned}
&x^{\frac{b_1+1}{a_1}} R_{\rho}(y) - y^{b_1+1} R_{b_2}(y) \\
&= y^{b_1+b_2+1} \left\{ -\frac{a_2(b_1+1)}{2a_1} P_2(y) y^{-1} - (\langle \rho \rangle_3 - \langle b_2 \rangle_3) P_3(y) y^{-2} \right. \\
&\quad \left. + (\langle \rho \rangle_4 - \langle b_2 \rangle_4) P_4(y) y^{-3} \right\},
\end{aligned}$$

$$\begin{aligned}
&x^{\frac{b_2+1}{a_2}} R_{\tilde{\rho}}(y) - y^{b_2+1} R_{b_1}(y) \\
&= y^{b_1+b_2+1} \left\{ -\frac{a_1(b_2+1)}{2a_2} P_2(y) y^{-1} - (\langle \tilde{\rho} \rangle_3 - \langle b_1 \rangle_3) P_3(y) y^{-2} \right. \\
&\quad \left. + (\langle \tilde{\rho} \rangle_4 - \langle b_1 \rangle_4) P_4(y) y^{-3} \right\}
\end{aligned}$$

and

$$\begin{aligned}
R_{b_1}(y) R_{b_2}(y) &= \psi^2(y) y^{b_1+b_2} - \frac{b_1+b_2}{2} \psi(y) P_2(y) y^{b_1+b_2-1} \\
&+ \left( (\langle b_1 \rangle_3 + \langle b_2 \rangle_3) \psi(y) P_3(y) + \frac{b_1 b_2}{4} P_2^2(y) \right) y^{b_1+b_2-2} \\
&+ O(y^{b_1+b_2-3}).
\end{aligned}$$

Substituting these formulas into (2.18) we get

$$\begin{aligned}
D(\mathbf{a}, \mathbf{b}; x) &= H(\mathbf{a}, \mathbf{b}; x) + E_1(\mathbf{a}, \mathbf{b}; x) + E_2(\mathbf{a}, \mathbf{b}; x) + \zeta(-b_1)\zeta(-b_2) \\
&\quad - \left\{ \frac{1}{2} \left( \frac{a_2}{a_1} + \frac{a_1}{a_2} \right) P_2(y) + \psi^2(y) \right\} y^{b_1+b_2} \\
&\quad + \left\{ - \left( \frac{\langle \rho \rangle_3 - \langle b_2 \rangle_3}{b_1 + 1} + \frac{\langle \tilde{\rho} \rangle_3 - \langle b_1 \rangle_3}{b_2 + 1} \right) P_3(y) \right. \\
&\quad \left. + \frac{b_1 + b_2}{2} \psi(y) P_2(y) \right\} y^{b_1+b_2-1} \\
&\quad + \left\{ \left( \frac{\langle \rho \rangle_4 - \langle b_2 \rangle_4}{b_1 + 1} + \frac{\langle \tilde{\rho} \rangle_4 - \langle b_1 \rangle_4}{b_2 + 1} \right) P_4(y) \right. \\
&\quad \left. - (\langle b_1 \rangle_3 + \langle b_2 \rangle_3) \psi(y) P_3(y) - \frac{b_1 b_2}{4} P_2^2(y) \right\} y^{b_1+b_2-2} \\
&\quad + O \left( \left( x^{\frac{b_1-4}{a_1}} + x^{\frac{b_2-4}{a_2}} \right) \log x + y^{b_1-4} + y^{b_2-4} + y^{b_1+b_2-3} \right).
\end{aligned}$$

This proves the assertion of Lemma 2.3 in the case  $a_1(b_2 + 1) \neq a_2(b_1 + 1)$ .

If  $a_1(b_2 + 1) = a_2(b_1 + 1)$ , we can prove the assertion of the lemma similarly. ■

In the application to the generalized divisor problem, we shall consider the case  $\mathbf{a} = (1, 1)$  in Lemma 2.3. It is appropriate here to present the explicit formula in this case. To avoid the complicated notation, we shall use the simplified one like

$$D_{\mathbf{b}}(x) = D((1, 1), \mathbf{b}; x).$$

**Lemma 2.4.** *Let  $\mathbf{b} = (b_1, b_2)$  such that  $b_j \neq -1$  and  $|b_1 - b_2| < 5$ . Then we have*

$$D_{\mathbf{b}}(x) = H_{\mathbf{b}}(x) + \Delta_{\mathbf{b}}(x),$$

where

$$H_{\mathbf{b}}(x) = \begin{cases} \frac{\zeta(b_1 - b_2 + 1)}{b_1 + 1} x^{b_1+1} + \frac{\zeta(b_2 - b_1 + 1)}{b_2 + 1} x^{b_2+1} & \text{if } b_1 \neq b_2, \\ \frac{\log x + 2\gamma - 1/(b_1 + 1)}{b_1 + 1} x^{b_1+1} & \text{if } b_1 = b_2, \end{cases}$$

and

$$\Delta_{\mathbf{b}}(x) = E_{1,\mathbf{b}}(x) + E_{2,\mathbf{b}}(x) + E_{3,\mathbf{b}}(x)$$

with

$$\begin{aligned}
E_{1,\mathbf{b}}(x) &= -x^{b_1}G_{b_2-b_1,1}(x) - x^{b_2}G_{b_1-b_2,1}(x), \\
E_{2,\mathbf{b}}(x) &= \frac{b_1}{2}x^{b_1-1}G_{b_2-b_1+1,2}(x) + \frac{b_2}{2}x^{b_2-1}G_{b_1-b_2+1,2}(x) \\
&\quad - \frac{b_1(b_1-1)}{6}x^{b_1-2}G_{b_2-b_1+2,3}(x) - \frac{b_2(b_2-1)}{6}x^{b_2-2}G_{b_1-b_2+2,3}(x) \\
&\quad + \frac{b_1(b_1-1)(b_1-2)}{24}x^{b_1-3}G_{b_2-b_1+3,4}(x) \\
&\quad + \frac{b_2(b_2-1)(b_2-2)}{24}x^{b_2-3}G_{b_1-b_2+3,4}(x)
\end{aligned}$$

and

$$\begin{aligned}
E_{3,\mathbf{b}}(x) &= \zeta(-b_1)\zeta(-b_2) - (P_2(\sqrt{x}) + \psi^2(\sqrt{x})x^{\frac{b_1+b_2}{2}} \\
&\quad + \left(\frac{b_1+b_2-4}{6}P_3(\sqrt{x}) + \frac{b_1+b_2}{2}\psi(\sqrt{x})P_2(\sqrt{x})\right)x^{\frac{b_1+b_2-1}{2}} \\
&\quad - \left(\frac{2b_1^2+2b_2^2-3b_1b_2-2b_1-2b_2+6}{12}P_4(\sqrt{x}) \right. \\
&\quad \left. + \frac{b_1^2+b_2^2-b_1-b_2}{6}\psi(\sqrt{x})P_3(\sqrt{x}) + \frac{b_1b_2}{4}P_2^2(\sqrt{x})\right)x^{\frac{b_1+b_2-2}{2}} \\
&\quad + O\left(x^{\frac{b_1}{2}-2} + x^{\frac{b_2}{2}-2} + x^{\frac{b_1+b_2-3}{2}}\right).
\end{aligned}$$

**Remark 2.8.** Lemma 2.4 is a generalization and an improvement of Lemma 15 in Chowla [6].

### 3. The Proof of Theorem 1

**Lemma 3.1.** Let  $f(n)$  be an arithmetic function and  $E(x)$  be the error term defined by

$$E(x) = \sum_{n \leq x} f(n) - g(x).$$

Suppose that  $g(x)$  is continuously differentiable. For every fixed positive integer  $k$ , we have

$$\begin{aligned}
\sum_{n \leq x} E^k(n) &= \left(\frac{1}{2} - \psi(x)\right) E^k(x) + \int_1^x E^k(u) du \\
&\quad + k \int_1^x \left(\frac{1}{2} - \psi(u)\right) g'(u) E^{k-1}(u) du.
\end{aligned} \tag{3.1}$$



**Proof.** This is Lemma 1 of Furuya [10]. ■

**Proof of Theorem 1.** As before, we shall use the abbreviated notation

$$D_{b_1, b_2}(x) = D((1, 1), (b_1, b_2); x)$$

in the sequel. Let  $-1 < a < 1$  and  $a \neq 0$ . We take  $f(n) = \sigma_a(n)$ ,  $g(x) = \zeta(1-a)x + \frac{\zeta(1+a)}{1+a}x^{1+a}$  in Lemma 3.1. By (3.1) with  $k = 2$ , we have

$$\begin{aligned} \sum_{n \leq x} \Delta_a^2(n) &= \left(\frac{1}{2} - \psi(x)\right) \Delta_a^2(x) + \int_1^x \Delta_a^2(t) dt \\ &\quad + 2 \int_1^x \left(\frac{1}{2} - \psi(t)\right) (\zeta(1-a) + \zeta(1+a)t^a) \Delta_a(t) dt \\ &= \left(\frac{1}{2} - \psi(x)\right) \Delta_a^2(x) + \int_1^x \Delta_a^2(t) dt + T_1 - 2T_2, \end{aligned} \quad (3.2)$$

where we put

$$\begin{aligned} T_1 &= \int_1^x (\zeta(1-a) + \zeta(1+a)t^a) \Delta_a(t) dt, \\ T_2 &= \int_1^x (\zeta(1-a) + \zeta(1+a)t^a) \psi(t) \Delta_a(t) dt. \end{aligned}$$

In order to derive the asymptotic behaviour of  $T_1$  and  $T_2$ , we have to consider the integrals of the types

$$J_1(x, \delta) = \int_1^x t^\delta \Delta_a(t) dt \quad \text{and} \quad J_2(x, \delta) = \int_1^x t^\delta \psi(t) \Delta_a(t) dt$$

for  $\delta = 0$  and  $a$ . However, we treat more general  $\delta$  in the following lemma.

**Lemma 3.2.** *Let  $-1 < a < 1$ ,  $a \neq 0$  and suppose  $\delta$  is a real number such that  $\delta \neq -1, -2, -(2+a)$  and  $\delta < \min(3-a, 3)$ . Then we have*

$$\begin{aligned} J_1(x, \delta) &= \frac{x^{1+\delta}}{1+\delta} D_{a,0}(x) - \frac{1}{1+\delta} D_{a+1+\delta, 1+\delta}(x) - \frac{\zeta(1-a)}{2+\delta} (x^{2+\delta} - 1) \\ &\quad - \frac{\zeta(1+a)}{(1+a)(2+a+\delta)} (x^{2+a+\delta} - 1) \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} J_2(x, \delta) &= (W_\delta(x) - \beta(\delta)) D_{a,0}(x) - \frac{1}{12} D_{a+\delta, \delta}(x) - \beta_4(\delta) B_4 D_{a+\delta-2, \delta-2}(x) \\ &\quad + 6\beta_6(\delta) A(\delta) - \zeta(1-a) W_{1+\delta}(x) - \frac{\zeta(1+a)}{1+a} W_{a+1+\delta}(x) \\ &\quad + O(x^{\delta-3}(x^a + 1)), \end{aligned} \quad (3.4)$$

where  $A(\delta)$  is a certain constant depending only on  $a$  and  $\delta$ .

**Proof.** We follow the method used in [11]. First we treat  $J_1(x, \delta)$ . Substituting the definition (1.1) of  $\Delta_a(x)$  into the integral and interchanging the order of summation and integration, we get

$$J_1(x, \delta) = \frac{x^{1+\delta}}{1+\delta} \sum_{n \leq x} \sigma_a(n) - \frac{1}{1+\delta} \sum_{n \leq x} \sigma_a(n) n^{1+\delta} - \frac{\zeta(1-a)}{2+\delta} (x^{2+\delta} - 1) \\ - \frac{\zeta(1+a)}{(1+a)(2+a+\delta)} (x^{2+a+\delta} - 1).$$

By noting that

$$\sum_{n \leq x} \sigma_a(n) n^\mu = D_{a+\mu, \mu}(x) \quad (3.5)$$

for any  $\mu$ , we get the assertion (3.3).

Next we consider the integral  $J_2(x, \delta)$ . We again substitute the definition (1.1) of  $\Delta_a(x)$ . In this case we obtain that

$$J_2(x, \delta) = \sum_{n \leq x} \sigma_a(n) (W_\delta(x) - W_\delta(n)) - \zeta(1-a) W_{1+\delta}(x) \\ - \frac{\zeta(1+a)}{1+a} W_{1+a+\delta}(x). \quad (3.6)$$

In view of Lemma 2.1, it remains to consider the sum  $\sum_{n \leq x} \sigma_a(n) W_\delta(n)$ . In order to get the precise evaluation of this sum with  $x = n$  and  $N = 5$  in (2.1), we need the integral expression of the error term. In fact, we have

$$W_\delta(n) = \beta(\delta) + \frac{1}{2} B_2 n^\delta + \beta_4(\delta) B_4 n^{\delta-2} - 6\beta_6(\delta) \int_n^\infty t^{\delta-4} P_5(t) dt.$$

Hence, by (3.5),

$$\sum_{n \leq x} \sigma_a(n) W_\delta(n) = \beta(\delta) D_{a,0}(x) + \frac{1}{12} D_{a+\delta, \delta}(x) \\ + \beta_4(\delta) B_4 D_{a+\delta-2, \delta-2}(x) - 6\beta_6(\delta) R \quad (3.7)$$

with

$$R = \sum_{n \leq x} \sigma_a(n) \int_n^\infty t^{\delta-4} P_5(t) dt.$$

Now  $R$  is transformed as

$$R = \sum_{n=1}^\infty \sigma_a(n) \int_n^\infty t^{\delta-4} P_5(t) dt - \sum_{n > x} \sigma_a(n) \int_n^\infty t^{\delta-4} P_5(t) dt.$$

Since  $\int_n^\infty t^{\delta-4} P_5(t) st \ll n^{\delta-4}$  and  $\sum_{n \leq y} \sigma_a(n) \ll y(1+y^a)$ , we find that the first sum in the right hand side, which we denote by  $A(\delta)$ , converges absolutely, while

the second one is estimated by  $x^{\delta-3}(x^a+1)$  from the assumption  $\delta < \min(3-a, 3)$ . Hence

$$R = A(\delta) + O(x^{\delta-3}(x^a+1)). \quad (3.8)$$

Combining (3.6), (3.7) and (3.8), we get the formula (3.4). ■

We note that for  $\delta = 0$  we have the more precise form of  $J_2(x, 0)$  than that in Lemma 3.2 as

$$J_2(x, 0) = W_0(x)D_{a,0}(x) - \zeta(1-a)W_1(x) - \frac{\zeta(1+a)}{1+a}W_{1+a}(x).$$

Now we go back to the expression (3.2). Since

$$T_j = \zeta(1-a)J_j(x, 0) + \zeta(1+a)J_j(x, a) \quad \text{for } j = 1, 2,$$

we have

$$\begin{aligned} T_1 - 2T_2 &= \zeta(1-a)\{xD_{a,0}(x) - D_{a+1,1}(x) - 2W_0(x)D_{a,0}(x)\} \\ &\quad + \zeta^2(1-a)\left\{-\frac{1}{2}(x^2-1) + 2W_1(x)\right\} \\ &\quad + \zeta(1+a)\left\{\frac{x^{1+a}}{1+a}D_{a,0}(x) - \frac{1}{1+a}D_{2a+1,a+1}(x) - 2(W_a(x) - \beta(a))D_{a,0}(x) \right. \\ &\quad \left. + \frac{1}{6}D_{2a,a}(x) + \frac{a(a-1)}{12}B_4D_{2a-2,a-2}(x) - 12\beta_6(a)A(a)\right\} \\ &\quad + \zeta^2(1+a)\left\{-\frac{1}{2(1+a)^2}(x^{2a+2}-1) + \frac{2}{1+a}W_{2a+1}(x)\right\} \\ &\quad + \zeta(1-a)\zeta(1+a)\left\{-\frac{1}{1+a}(x^{2+a}-1) + 2W_{a+1}(x) + \frac{2}{a+1}W_{a+1}(x)\right\} \\ &\quad + O(x^{a-3}(x^a+1)). \end{aligned} \quad (3.9)$$

We suppose further that  $a \neq \pm 1/2$ . Then we can apply Lemma 2.4 to each of the function  $D_{b_1, b_2}(x)$  in the right hand side of (3.9). After some simplifications,

we obtain the assertion (1.10) of Theorem 1 with

$$\begin{aligned}
\mathcal{E}_a(x) = & \left( P_2(x) - \frac{1}{6} \right) (\zeta(1-a) + \zeta(1+a)x^a) (x^a G_{-a,1}(x) + G_{a,1}(x)) \quad (3.10) \\
& - \frac{1}{2} (\zeta(1-a) + \zeta(1+a)x^a) (x^a G_{1-a,2}(x) + G_{1+a,2}(x)) \\
& - \frac{a}{2} \left( P_2(x) - \frac{1}{6} \right) (\zeta(1-a) + x^a \zeta(1+a)) G_{1-a,2}(x) x^{a-1} \\
& + \frac{a}{12} \zeta(1+a) (x^a G_{1-a,2}(x) + G_{1+a,2}(x)) x^{a-1} \\
& + \frac{a}{6} \zeta(1+a) (x^a G_{2-a,3}(x) + G_{2+a,3}(x)) x^{a-1} \\
& + \frac{a}{3} (\zeta(1-a) + x^a \zeta(1+a)) G_{2-a,3}(x) x^{a-1} \\
& - \frac{a(a-1)}{8} \zeta(1-a) G_{3-a,4}(x) x^{a-2} \\
& - \frac{a}{24} \zeta(1+a) ((7a-4)x^{2a-2} G_{3-a,4}(x) + (a-1)x^{a-2} G_{3+a,4}(x)) \\
& + \frac{\zeta(1-a)\zeta(-a)}{2} \left( P_2(x) - \frac{1}{6} \right) + \frac{\zeta^2(1-a)}{3} (1 - P_3(x))
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}(a) = & -\frac{\zeta(1+a)\zeta(-1-2a)\zeta(-1-a)}{a+1} + \frac{\zeta(1+a)\zeta(-2a)\zeta(-a)}{6} \quad (3.11) \\
& + \frac{1}{12} \zeta(1-a)\zeta(-1-a) + \frac{a(a-1)}{12} B_4 \zeta(1+a)\zeta(2-2a)\zeta(2-a) \\
& + \frac{\zeta^2(1+a)}{(1+a)^2} \left( \frac{1}{2a+3} + \zeta(-2-2a) \right) \\
& + \frac{2\zeta(1-a)\zeta(1+a)}{1+a} \left( \frac{1}{a+3} + \zeta(-2-a) \right) - 12\zeta(1+a)\beta_6(a)A(a).
\end{aligned}$$

We shall consider the case (ii) of Theorem 1. We note that  $U_{\pm 1/2}(x) = \lim_{a \rightarrow \pm 1/2} U_a(x)$  and the error term (1.10) is bounded near the point  $a = \pm 1/2$ . Hence we can take the limits as  $a \rightarrow \pm 1/2$  in (1.10). In fact, if  $a = -1/2$ , the divergent terms in (1.10) and (3.11) are  $\frac{\zeta^2(1+a)}{6(2a+1)} x^{2a+1}$  and  $\frac{\zeta(1+a)\zeta(-2a)\zeta(-a)}{6}$  and their residues at  $a = -1/2$  are  $\frac{1}{12}\zeta^2(1/2)$  and  $-\frac{1}{12}\zeta^2(1/2)$ , respectively. Hence the sum of these two terms are holomorphic at  $a = -1/2$ . Furthermore we can see easily that

$$\begin{aligned}
\lim_{a \rightarrow -\frac{1}{2}} \left\{ \frac{\zeta^2(1+a)}{6(2a+1)} x^{2a+1} + \frac{\zeta(1+a)\zeta(-2a)\zeta(-a)}{6} \right\} \\
= \frac{1}{6} \zeta \left( \frac{1}{2} \right) \left( \zeta \left( \frac{1}{2} \right) \log x + \gamma \zeta \left( \frac{1}{2} \right) + \zeta' \left( \frac{1}{2} \right) \right).
\end{aligned}$$

The other terms have their values at  $a = -1/2$ . Therefore we get the assertion for  $U_{-1/2}(x)$  in (ii) of Theorem 1. The assertion for  $U_{1/2}(x)$  follows similarly, but we note that the log-term is contained in the error term in this case.

This completes the proof of Theorem 1. ■

#### 4. The proof of Theorem 2 and Theorem 3

To prove Theorem 2, we shall need the following relations, which improves Lemma 27 of Chowla [6].

**Lemma 4.1 (Reciprocal relation).** *Let  $-1 < a < 1$  and  $a \neq 0$ . Then we have*

$$\begin{aligned} \Delta_a(x) = & x^a \Delta_{-a}(x) + \frac{\zeta(a)}{2} x^a - \frac{\zeta(-a)}{2} + \frac{a}{2} x^{-1} (x^a G_{1-a,2}(x) + G_{1+a,2}(x)) \\ & - \frac{a}{6} x^{-2} \{(a-1)x^a G_{2-a,3}(x) - (a+1)G_{2+a,3}(x)\} \\ & + \frac{a}{24} x^{-3} \{(a-1)(a-2)x^a G_{3-a,4}(x) + (a+1)(a+2)G_{3+a,4}(x)\} \\ & + a \left( \psi(\sqrt{x}) P_2(\sqrt{x}) + \frac{1}{3} P_3(\sqrt{x}) \right) x^{\frac{a-1}{2}} \\ & + \frac{a}{3} (P_4(\sqrt{x}) + \psi(\sqrt{x}) P_3(\sqrt{x})) x^{\frac{a}{2}-1} + O(x^{\frac{a-3}{2}}). \end{aligned} \quad (4.1)$$

In particular, we have

$$\begin{aligned} \Delta_a(x) = & x^a \Delta_{-a}(x) + \frac{\zeta(a)}{2} x^a - \frac{\zeta(-a)}{2} + \frac{a}{2} x^{-1} (x^a G_{1-a,2}(x) + G_{1+a,2}(x)) \\ & + O(x^{\frac{a-1}{2}}). \end{aligned} \quad (4.2)$$

**Proof.** The assertion (4.1) is obtained from Lemma 2.4. We obtain (4.2) from (4.1) easily. ■

Now we begin to prove Theorem 2. For convenience, we let

$$\hat{C}(a) = \begin{cases} \frac{1}{4} \zeta^2(-a) + \frac{\zeta(-2a) \zeta^2(1-a)}{12 \zeta(2-2a)} & \text{if } -1 < a < -\frac{1}{2}, \\ \frac{\zeta^2(3/2)}{24 \zeta(3)} & \text{if } a = -\frac{1}{2}, \\ \frac{\zeta^2(3/2) \zeta(a+3/2) \zeta(3/2-a)}{(4a+6) \pi^2 \zeta(3)} & \text{if } |a| < \frac{1}{2}. \end{cases}$$

In the latter discussion in this section, we may assume that  $0 < a < 1$ .

Squaring (4.2) we have

$$\begin{aligned} \Delta_a^2(t) = & t^{2a} \Delta_{-a}^2(t) + \frac{1}{4} \zeta^2(a) t^{2a} + \zeta(a) t^{2a} \Delta_{-a}(t) - \zeta(-a) t^a \Delta_{-a}(t) \\ & + a t^{a-1} \Delta_{-a}(t) \{t^a G_{1-a,2}(t) + G_{1+a,2}(t)\} \\ & + O(t^{\frac{3a-1}{2}} |\Delta_{-a}(t)|) + O(t^{\frac{3a}{2}}). \end{aligned} \quad (4.3)$$

For the integral of the first term of the right hand side of (4.3) for  $1/2 < a < 1$ , we apply integration by parts and use the estimate in the first line of (1.7). Thus we get for  $1/2 < a < 1$

$$\begin{aligned} \int_1^x t^{2a} \Delta_{-a}^2(t) dt &= x^{2a} \int_1^x \Delta_{-a}^2(t) dt - 2a \int_1^x t^{2a-1} \left( \int_1^t \Delta_{-a}^2(u) du \right) dt \\ &= \frac{1}{1+2a} \hat{C}(-a) x^{1+2a} + O(x^{\frac{3}{2}+a} \log x). \end{aligned} \quad (4.4)$$

In the same way, by the two estimates in (1.8), one has for  $0 < a \leq \frac{1}{2}$ ,

$$\int_1^x t^{2a} \Delta_{-a}^2(t) dt = \begin{cases} \frac{3-2a}{3+2a} \hat{C}(-a) x^{\frac{3}{2}+a} + O(x^{1+2a}), & \text{if } 0 < a < \frac{1}{2}, \\ \frac{1}{2} \hat{C}(-\frac{1}{2}) x^2 \log x + O(x^2), & \text{if } a = \frac{1}{2}. \end{cases} \quad (4.5)$$

Here note that  $\hat{C}(a) = \frac{3-2a}{3+2a} \hat{C}(-a) = C(a)$  for  $0 < a < \frac{1}{2}$ .

For the integral of the third and the fourth terms of the right hand side of (4.3), we use (3.3) of Lemma 3.2 and Lemma 2.4, and get

$$\begin{aligned} \int_1^x t^{2a} \Delta_{-a}(t) dt &= \frac{x^{1+2a}}{1+2a} D_{-a,0}(x) - \frac{1}{1+2a} D_{1+a,1+2a}(x) \\ &\quad - \frac{\zeta(1+a)}{2+2a} (x^{2+2a} - 1) - \frac{\zeta(1-a)}{(1-a)(2+a)} (x^{2+a} - 1) \\ &= -\frac{\zeta(a)}{2(1+2a)} x^{1+2a} - \frac{1}{2} x^a G_{1+a,2}(x) - \frac{1}{2} x^{2a} G_{1-a,2}(x) \\ &\quad + O(x^{\frac{3}{2}a+\frac{1}{2}}) \\ &= -\frac{\zeta(a)}{2(1+2a)} x^{1+2a} + O(x^{\frac{3}{2}a+1}). \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} \int_1^x t^a \Delta_{-a}(t) dt &= \frac{x^{1+a}}{1+a} D_{-a,0}(x) - \frac{1}{1+a} D_{1,1+a}(x) \\ &\quad - \frac{\zeta(1+a)}{2+a} (x^{2+a} - 1) - \frac{\zeta(1-a)}{2(1-a)} (x^2 - 1) \\ &= -\frac{\zeta(a)}{2(1+a)} x^{1+a} - \frac{1}{2} G_{1+a,2}(x) \\ &\quad - \frac{1}{2} x^a G_{1-a,2}(x) + O(x^{\frac{a+1}{2}}) \\ &\ll x^{1+a}. \end{aligned} \quad (4.7)$$

Furthermore, by Cauchy's inequality, integration by parts and (1.6), we have

$$\begin{aligned}
 & \int_1^x t^{a-1} \Delta_{-a}(t) (t^a G_{1-a,2}(t) + G_{1+a,2}(t)) dt \\
 & \ll \left( \int_1^x \Delta_{-a}^2(t) dt \right)^{\frac{1}{2}} \left\{ \left( \int_1^x (t^{2a-1} G_{1-a,2}(t))^2 dt \right)^{\frac{1}{2}} \right. \\
 & \quad \left. + \left( \int_1^x (t^{a-1} G_{1+a,2}(t))^2 dt \right)^{\frac{1}{2}} \right\} \\
 & \ll x^{\frac{1}{4} + \frac{3}{2}a} \left( \int_1^x \Delta_{-a}^2(t) dt \right)^{\frac{1}{2}}. \tag{4.8}
 \end{aligned}$$

Hence the left hand side of (4.8) is estimated as

$$\ll \begin{cases} x^{\frac{3}{2}a + \frac{3}{4}} & \text{if } \frac{1}{2} < a < 1, \\ x^{\frac{3}{2}} \log x & \text{if } a = \frac{1}{2}, \\ x^{a+1} & \text{if } 0 < a < \frac{1}{2}. \end{cases} \tag{4.9}$$

By Cauchy's inequality again,

$$\int_1^x t^{\frac{3a-1}{2}} |\Delta_{-a}(t)| dt \ll \begin{cases} x^{\frac{3}{2}a + \frac{1}{2}} & \text{if } \frac{1}{2} < a < 1, \\ x^{\frac{5}{4}} \log x & \text{if } a = \frac{1}{2}, \\ x^{a + \frac{3}{4}} & \text{if } 0 < a < \frac{1}{2}. \end{cases} \tag{4.10}$$

We also note that

$$\int_1^x t^{2a} dt = \frac{1}{1+2a} (x^{1+2a} - 1). \tag{4.11}$$

Now, by combining (4.3)–(4.11), the assertion of Theorem 2 is obtained. This finishes the proof of Theorem 2.  $\blacksquare$

Finally Theorem 3 follows immediately from Corollary 1, (1.11), (1.7), (1.8), Theorem 2 and the upper bound estimates of  $\Delta_a(x)$ .  $\blacksquare$

**Remark 4.1.** If we use the Cauchy inequality in (4.7), we get the estimate  $x^{a/2+5/4}$  for  $0 < a < 1/2$ . This estimate is worse than  $x^{1+a}$ , hence (4.7) can be said to be non-trivial.

## 5. An asymptotic representation of the integral $I_{a,b}(\theta; x)$

In the previous sections, we saw that the integral of the form  $\int_1^x t^\delta \Delta_a(t) dt$  plays an important role. We treated this kind of integral in our previous papers [4] and [11], where we used the notation  $-\theta$  instead of  $\delta$  under the assumption  $\Re \theta \geq 0$ . It is interesting to consider the generalization of this integral in the frame of the weighted two-dimensional divisor problem. In this section we are concerned with such a problem.

Let  $x \geq 1$  and  $\theta$  be a complex number. We define

$$I_{\mathbf{a}, \mathbf{b}}(\theta; x) = \int_1^x t^{-\theta} \Delta(\mathbf{a}, \mathbf{b}; t) dt, \quad (5.1)$$

here  $\Delta(\mathbf{a}, \mathbf{b}; t)$  is the error term defined by (2.10).

To state the result we prepare the following notation. For  $\mathbf{a} = (a_1, a_2)$  and  $\mathbf{b} = (b_1, b_2)$ , we define

$$l_1 = b_1 + (1 - \theta)a_1 \quad \text{and} \quad l_2 = b_2 + (1 - \theta)a_2.$$

Note that  $\rho(\mathbf{a}, (l_1, l_2)) = \rho(\mathbf{a}, \mathbf{b})$  and  $\tilde{\rho}(\mathbf{a}, (l_1, l_2)) = \tilde{\rho}(\mathbf{a}, \mathbf{b})$ .

Now we shall show the following (more general) asymptotic representation under the assumption that  $a_1$  and  $a_2$  are positive integers.

**Proposition 5.1.** *Let  $\mathbf{a} = (a_1, a_2)$  with positive integers  $a_1$  and  $a_2$  and let  $\mathbf{b} = (b_1, b_2)$  with  $-1 < b_1 < 4 + \frac{a_1(b_2+1)}{a_2}$  and  $-1 < b_2 < 4 + \frac{a_2(b_1+1)}{a_1}$ . Let  $y = x^{\frac{1}{a_1+a_2}}$ , and let  $\rho$  and  $\tilde{\rho}$  be the numbers defined by (2.8) and (2.9), respectively. Suppose that  $\theta$  is a complex number such that*

$$\Re \theta < \min \left\{ 1 + \frac{b_1+1}{a_1}, 1 + \frac{b_2+1}{a_2} \right\}. \quad (5.2)$$

Then we have

$$\begin{aligned} I_{\mathbf{a}, \mathbf{b}}(\theta; x) &= A_{\mathbf{a}, \mathbf{b}}(\theta; x) + B_{\mathbf{a}, \mathbf{b}}(\theta) \\ &+ \sum_{j=2}^4 \left\{ \lambda_j x^{\frac{b_1+1-j}{a_1}+1-\theta} G_{\rho+\frac{a_2 j}{a_1}, j}^{\mathbf{a}}(x) + \tilde{\lambda}_j x^{\frac{b_2+1-j}{a_2}+1-\theta} G_{\tilde{\rho}+\frac{a_1 j}{a_2}, j}^{\tilde{\mathbf{a}}}(x) \right\} \\ &- \left( \delta_{3,1} P_3(y) + \delta_{3,2} \psi(y) P_2(y) \right) x^{\frac{b_1+b_2-1}{a_1+a_2}+1-\theta} \\ &+ \left( \delta_{4,1} P_4(y) + \delta_{4,2} \psi(y) P_3(y) + \delta_{4,3} P_2(y)^2 \right) x^{\frac{b_1+b_2-2}{a_1+a_2}+1-\theta} \\ &+ O \left( x^{\frac{b_1+b_2-3}{a_1+a_2}+1-\Re \theta} \log^\epsilon x \right), \end{aligned} \quad (5.3)$$

where

$$A_{\mathbf{a}, \mathbf{b}}(\theta; x) = \begin{cases} \frac{\zeta(-b_1)\zeta(-b_2)}{1-\theta} x^{1-\theta} - \frac{\zeta(-l_1)\zeta(-l_2)}{1-\theta} & \text{if } \theta \neq 1, \\ \zeta(-b_1)\zeta(-b_2) \log x + a_1 \zeta(-b_2) \zeta'(-b_1) + a_2 \zeta(-b_1) \zeta'(-b_2) & \text{if } \theta = 1, \end{cases}$$

and

$$B_{\mathbf{a}, \mathbf{b}}(\theta) = \begin{cases} \frac{\zeta(-\rho)a_1}{(b_1+1)(l_1+1)} + \frac{\zeta(-\tilde{\rho})a_2}{(b_2+1)(l_2+1)} & \text{if } a_1(b_2+1) \neq a_2(b_1+1), \\ \frac{\gamma a_1}{(b_1+1)(l_1+1)} \left( 1 + \frac{a_1}{a_2} \right) - \frac{a_1^2(b_1+l_1+2)}{a_2(b_1+1)^2(l_1+1)^2} & \text{otherwise.} \end{cases}$$



The coefficients  $\lambda_j, \tilde{\lambda}_j$  and  $\delta_{i,j}$  are given by

$$\begin{aligned}\lambda_2 &= -\frac{a_1}{2}, & \tilde{\lambda}_2 &= -\frac{a_2}{2}, \\ \lambda_3 &= \frac{a_1}{6} (2b_1 - 1 + a_1(1 - \theta)), & \tilde{\lambda}_3 &= \frac{a_2}{6} (2b_2 - 1 + a_2(1 - \theta)), \\ \lambda_4 &= -\frac{1}{24} \{ (3b_1^2 - 6b_1 + 2)a_1 + 3(b_1 - 1)(1 - \theta)a_1^2 + (1 - \theta)^2 a_1^3 \}, \\ \tilde{\lambda}_4 &= -\frac{1}{24} \{ (3b_2^2 - 6b_2 + 2)a_2 + 3(b_2 - 1)(1 - \theta)a_2^2 + (1 - \theta)^2 a_2^3 \}, \\ \delta_{3,1} &= \frac{1}{6} \left( \frac{a_2^2}{a_1} + \frac{a_1^2}{a_2} \right), & \delta_{3,2} &= \frac{a_1 + a_2}{2}, \\ \delta_{4,1} &= \frac{1}{24} \left\{ \frac{3(b_2 - 1)a_2^2}{a_1} + \frac{3(b_1 - 1)a_1^2}{a_2} - \frac{(b_1 + 1)a_2^3}{a_1^2} - \frac{(b_2 + 1)a_1^3}{a_2^2} \right. \\ &\quad \left. + (1 - \theta) \left( \frac{a_2^3}{a_1} + \frac{a_1^3}{a_2} \right) \right\}, \\ \delta_{4,2} &= \frac{1}{6} \{ (2b_1 - 1)a_1 + (2b_2 - 1)a_2 + (1 - \theta)(a_1^2 + a_2^2) \}, \\ \delta_{4,3} &= \frac{1}{4} \{ b_1 a_2 + a_1 b_2 + (1 - \theta)a_1 a_2 \}\end{aligned}$$

and

$$\epsilon = \begin{cases} 1 & \text{if } \theta = 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Suppose that  $\theta \neq 1$ . We first assume that  $a_1(b_2 + 1) \neq a_2(b_1 + 1)$ . Since we assume that  $a_1$  and  $a_2$  are positive integers, we have

$$D(\mathbf{a}, \mathbf{b}; x) = \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n),$$

where

$$d(\mathbf{a}, \mathbf{b}; n) = \sum_{m_1^{a_1} m_2^{a_2} = n} m_1^{b_1} m_2^{b_2}.$$

We substitute the definition (2.10) of  $\Delta(\mathbf{a}, \mathbf{b}; t)$  into the integral  $I_{\mathbf{a}, \mathbf{b}}(\theta; x)$  and get

$$\begin{aligned}I_{\mathbf{a}, \mathbf{b}}(\theta; x) &= \int_1^x t^{-\theta} \sum_{n \leq t} d(\mathbf{a}, \mathbf{b}; n) dt - \int_1^x t^{-\theta} H(\mathbf{a}, \mathbf{b}; t) dt \\ &= \frac{x^{1-\theta}}{1-\theta} \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) - \frac{1}{1-\theta} \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) n^{1-\theta} \\ &\quad - \frac{\zeta(-\rho) \left( x^{1-\theta + \frac{b_1+1}{a_1}} - 1 \right)}{(b_1 + 1) \left( 1 - \theta + \frac{b_1+1}{a_1} \right)} - \frac{\zeta(-\tilde{\rho}) \left( x^{1-\theta + \frac{b_2+1}{a_2}} - 1 \right)}{(b_2 + 1) \left( 1 - \theta + \frac{b_2+1}{a_2} \right)}. \quad (5.4)\end{aligned}$$

We note that

$$\sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) n^{1-\theta} = D(\mathbf{a}, (l_1, l_2); x).$$

Clearly  $a_1(b_2 + 1) \neq a_2(b_1 + 1)$  is equivalent to  $a_1(l_2 + 1) \neq a_2(l_1 + 1)$ . Furthermore it is easily seen that

$$-1 < \Re l_1 < 4 + \frac{a_1(\Re l_2 + 1)}{a_2}, \quad -1 < \Re l_2 < 4 + \frac{a_2(\Re l_1 + 1)}{a_1}$$

by (5.2) and the assumption on  $b_j$  ( $j = 1, 2$ ). Hence  $l_1$  and  $l_2$  satisfy the condition in Remark 2.6. Therefore we can apply Lemma 2.3 to the first and the second terms in the right hand side of (5.4) with the simplest error term. After some simplifications, we get the assertion of Proposition 5.1 in this case. We should note that the conditions on  $a_j, b_j$  ( $j = 1, 2$ ) and  $\theta$  in the proposition implies that the error terms are included in  $O(x^{(b_1+b_2-3)/(a_1+a_2)+1-\Re\theta})$ .

If  $a_1(b_2 + 1) = a_2(b_1 + 1)$ , the difference from the above is only the main term  $H(\mathbf{a}, \mathbf{b}; x)$  of  $D(\mathbf{a}, \mathbf{b}; x)$ . So using the lower formula of (2.11) we can obtain the corresponding assertion similarly.

Next we treat the case  $\theta = 1$ . As in the case above, we assume  $a_1(b_2 + 1) \neq a_2(b_1 + 1)$  first. Then we have

$$\begin{aligned} I_{\mathbf{a}, \mathbf{b}}(\theta; x) &= \int_1^x t^{-1} \sum_{n \leq t} d(\mathbf{a}, \mathbf{b}; n) dt - \int_1^x t^{-1} H(\mathbf{a}, \mathbf{b}; t) dt \\ &= \log x \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) - \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) \log n \\ &\quad - \frac{a_1 \zeta(-\rho)}{(b_1 + 1)^2} \left( x^{\frac{b_1+1}{a_1}} - 1 \right) - \frac{a_2 \zeta(-\tilde{\rho})}{(b_2 + 1)^2} \left( x^{\frac{b_2+1}{a_2}} - 1 \right). \end{aligned} \quad (5.5)$$

The first sum in the right hand side of (5.5) has already been studied in Lemma 2.3. Hence it remains to consider the second sum. It is transformed as

$$\begin{aligned} \sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) \log n &= \sum_{n \leq x} \left( \sum_{m_1^{a_1} m_2^{a_2} = n} m_1^{b_1} m_2^{b_2} \right) \log n \\ &= a_1 \sum_{m_1^{a_1} m_2^{a_2} \leq x} m_1^{b_1} m_2^{b_2} \log m_1 + a_2 \sum_{m_1^{a_1} m_2^{a_2} \leq x} m_1^{b_1} m_2^{b_2} \log m_2, \end{aligned}$$

which we shall denote the sums in the right hand side by  $S_1$  and  $S_2$  in this order. Similarly to the proof of Lemma 2.3, we estimate  $S_j$  ( $j = 1, 2$ ) by Dirichlet's

hyperbola method, namely we have

$$S_1 = a_1 \left\{ \sum_{m_1 \leq y} m_1^{b_1} \log m_1 \sum_{m_2 \leq \left(\frac{x}{m_1^{a_1}}\right)^{1/a_2}} m_2^{b_2} \right. \\ \left. + \sum_{m_2 \leq y} m_2^{b_2} \sum_{m_1 \leq \left(\frac{x}{m_2^{a_2}}\right)^{1/a_1}} m_1^{b_1} \log m_1 - \left( \sum_{m_1 \leq y} m_1^{b_1} \log m_1 \right) \left( \sum_{m_2 \leq y} m_2^{b_2} \right) \right\}$$

and

$$S_2 = a_2 \left\{ \sum_{m_1 \leq y} m_1^{b_1} \sum_{m_2 \leq \left(\frac{x}{m_1^{a_1}}\right)^{1/a_2}} m_2^{b_2} \log m_2 \right. \\ \left. + \sum_{m_2 \leq y} m_2^{b_2} \log m_2 \sum_{m_1 \leq \left(\frac{x}{m_2^{a_2}}\right)^{1/a_1}} m_1^{b_1} - \left( \sum_{m_1 \leq y} m_1^{b_1} \right) \left( \sum_{m_2 \leq y} m_2^{b_2} \log m_2 \right) \right\}.$$

By repeated use of (2.5) and (2.6) in Lemma 2.2 in the right hand side of  $S_1$  and  $S_2$ , we have finally that

$$\sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) \log n = K_1(x) \log x + K_2(x) + O(y^{b_1+b_2-3} \log x),$$

where

$$K_1(x) = \frac{\zeta(-\rho)}{b_1+1} x^{\frac{b_1+1}{a_1}} + \frac{\zeta(-\tilde{\rho})}{b_2+1} x^{\frac{b_2+1}{a_2}} + \frac{1}{b_1+1} x^{\frac{b_1+1}{a_1}} R_\rho(y) + \frac{1}{b_2+1} x^{\frac{b_2+1}{a_2}} R_{\tilde{\rho}}(y) \\ - \frac{1}{b_1+1} y^{b_1+1} R_{b_2}(y) - \frac{1}{b_2+1} y^{b_2+1} R_{b_1}(y) - R_{b_1}(y) R_{b_2}(y) \\ + \sum_{m_1 \leq y} R_{b_2} \left( \left( \frac{x}{m_1^{a_1}} \right)^{1/a_2} \right) m_1^{b_1} + \sum_{m_2 \leq y} R_{b_1} \left( \left( \frac{x}{m_2^{a_2}} \right)^{1/a_1} \right) m_2^{b_2} \quad (5.6)$$

and

$$\begin{aligned}
K_2(x) = & -a_1\zeta(-b_2)\zeta'(-b_1) - a_2\zeta(-b_1)\zeta'(-b_2) \\
& + a_1 \sum_{m_2 \leq y} Q_{b_1} \left( \left( \frac{x}{m_2^{a_2}} \right)^{1/a_1} \right) m_2^{b_2} + a_2 \sum_{m_1 \leq y} Q_{b_2} \left( \left( \frac{x}{m_1^{a_1}} \right)^{1/a_2} \right) m_1^{b_1} \\
& - \frac{a_1}{(b_1+1)^2} x^{\frac{b_1+1}{a_1}} \{ \zeta(-\rho) + R_\rho(y) \} - \frac{a_2}{(b_2+1)^2} x^{\frac{b_2+1}{a_2}} \{ \zeta(-\tilde{\rho}) + R_{\tilde{\rho}}(y) \} \\
& + \frac{a_1}{(b_1+1)^2} y^{b_1+1} R_{b_2}(y) + \frac{a_2}{(b_2+1)^2} y^{b_2+1} R_{b_1}(y) \\
& - \frac{a_2}{b_1+1} y^{b_1+1} Q_{b_2}(y) - \frac{a_1}{b_2+1} y^{b_2+1} Q_{b_1}(y) \\
& - a_1 Q_{b_1}(y) R_{b_2}(y) - a_2 Q_{b_2}(y) R_{b_1}(y). \tag{5.7}
\end{aligned}$$

Using the formula (2.18) for  $D(\mathbf{a}, \mathbf{b}; x)$  and (5.6), we find that

$$\sum_{n \leq x} d(\mathbf{a}, \mathbf{b}; n) - K_1(x) = \zeta(-b_1)\zeta(-b_2) + O(y^{b_1+b_2-3}). \tag{5.8}$$

Combining the formulas (5.5), (5.7) and (5.8), we have

$$\begin{aligned}
I_{\mathbf{a}, \mathbf{b}}(\theta; x) = & \zeta(-b_1)\zeta(-b_2) \log x + a_1\zeta(-b_2)\zeta'(-b_1) \\
& + a_2\zeta(-b_1)\zeta'(-b_2) + \frac{a_1\zeta(-\rho)}{(b_1+1)^2} + \frac{a_2\zeta(-\tilde{\rho})}{(b_2+1)^2} \\
& - a_1 \sum_{m_2 \leq y} Q_{b_1} \left( \left( \frac{x}{m_2^{a_2}} \right)^{1/a_1} \right) m_2^{b_2} - a_2 \sum_{m_1 \leq y} Q_{b_2} \left( \left( \frac{x}{m_1^{a_1}} \right)^{1/a_2} \right) m_1^{b_1} \\
& + \frac{a_1}{(b_1+1)^2} \left\{ x^{\frac{b_1+1}{a_1}} R_\rho(y) - y^{b_1+1} R_{b_2}(y) \right\} \\
& + \frac{a_2}{(b_2+1)^2} \left\{ x^{\frac{b_2+1}{a_2}} R_{\tilde{\rho}}(y) - y^{b_2+1} R_{b_1}(y) \right\} \\
& + \frac{a_2}{b_1+1} y^{b_1+1} Q_{b_2}(y) + \frac{a_1}{b_2+1} y^{b_2+1} Q_{b_1}(y) \\
& + a_1 Q_{b_1}(y) R_{b_2}(y) + a_2 Q_{b_2}(y) R_{b_1}(y) + O(y^{b_1+b_2-3} \log x).
\end{aligned}$$

Substituting the definitions of  $R_z(x)$  and  $Q_z(x)$  into the above formula, we find that the assertion (5.3) also holds in the case  $\theta = 1$ .

The case  $a_1(b_2+1) = a_2(b_1+1)$  can be treated in a similar way.

This completes the proof of Proposition 5.1. ■

## References

- [1] U. Balakrishnan and S. Srinivasan, *A footnote to a conjecture in number theory II*, Rend. Accad. Naz. Sci. XL Mem. Mat. **10** (1986), 43–44.
- [2] X. Cao and W. Zhai, *On the mean square of the error term for the asymmetric two-dimensional divisor problems (II)*, Monatsh. Math. **160** (2010), 115–142.
- [3] X. Cao, Y. Tanigawa and W. Zhai, *On a conjecture of Chowla and Walum*, Sci. China Math. **53** (2010), 2755–2771.
- [4] X. Cao, J. Furuya, Y. Tanigawa and W. Zhai, *A generalized divisor problem and the sum of Chowla and Walum*, J. Math. Anal. Appl. **400** (2013), 15–21.
- [5] X. Cao, J. Furuya, Y. Tanigawa and W. Zhai, *On the differences between two kinds of mean value formulas of number-theoretic error terms*, preprint.
- [6] S. Chowla, *Contributions to the analytic theory of numbers*, Math. Z. **35** (1932), 279–299.
- [7] S. Chowla and S. Sivasankaranarayana Pillai, *On the error terms in some asymptotic formulae in the theory of numbers (II)*, J. Indian Math. Soc. **18** (1930), 181–184.
- [8] S. Chowla and H. Walum, *On the divisor problem*, Norske Vid. Selsk. Forh. (Trondheim) **36** (1963), 127–134 (Proc. Sympos. Pure Math. Vol. **8** (1965), 138–143).
- [9] H. Cramér, *Contributions to the analytic theory of numbers*, Proc. 5th Scand. Math. Congress, Helsingfors 1922, 266–272.
- [10] J. Furuya, *On the average orders of the error term in the Dirichlet divisor problem*, J. Number Theory **115**(1) (2005), 1–26.
- [11] J. Furuya and Y. Tanigawa, *Explicit representations of the integral containing the error term in the divisor problem*, Acta Math. Hungar. **129**(1-2) (2010), 24–46.
- [12] G.H. Hardy, *The average order of the arithmetical functions  $P(x)$  and  $\Delta(x)$* , Proc. London Math. Soc. **15** (2) (1916), 192–213.
- [13] M. Ishibashi, *Average order of the divisor functions with negative power weight*, Tsukuba J. Math. **17**(2) (1993), 513–535.
- [14] S. Kanemitsu and R. Sita Rama Chandra Rao, *On a conjecture of S. Chowla and of S. Chowla and H. Walum I*, J. Number Theory **20** (1985), 255–261; II, *ibid.* **20** (1985), 103–120.
- [15] E. Krätzel, *Lattice points*, Kluwer, Dordrecht-Boston-London, 1988.
- [16] R.A. MacLeod, *Fractional part sums and divisor functions*, J. Number Theory **14**(2) (1982), 185–227.
- [17] T. Meurman, *The mean square of the error term in a generalization of Dirichlet’s divisor problem*, Acta Arith. **74** (1996), 351–364.
- [18] Y.-F.S. Pétermann, *About a theorem of Paolo Codecà’s and omega estimates for arithmetical convolutions. II*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), no. 3, 343–353.
- [19] S.L. Segal, *A note on the average order of number-theoretic error terms*, Duke Math. J. **32** (1965), 279–284, (errata: **32** (1965), 765 and **33** (1966), 821).

- [20] M. Vogts, *Many-dimensional generalized divisor problems*, Math. Nachr. **124** (1985), 103–121.
- [21] A. Walfisz, *Teilerprobleme*, Zweite Abhandlung, Math. Z. **34** (1931), 448–472.

**Addresses:** Xiaodong Cao: Department of Mathematics and Physics, Beijing Institute of Petro-Chemical Technology, Beijing, 102617, P.R. China;

Jun Furuya: Department of Integrated Arts and Science, Okinawa National College of Technology, Nago, Okinawa, 905-2192, Japan;

Yoshio Tanigawa: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan;

Wenguang Zhai: Department of Mathematics, China University of Mining and Technology, Beijing 100083, P.R. China.

**E-mail:** caoxiaodong@bipt.edu.cn, jfuruya@okinawa-ct.ac.jp, tanigawa@math.nagoya-u.ac.jp, zhaiwg@hotmail.com

**Received:** 10 July 2012