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CONCERNING DENSE SUBIDEALS IN COMMUTATIVE BANACH ALGEBRAS

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Abstract: In the paper [2] we have shown that in the case of a separable Banach algebra A the necessary and sufficient condition in order that a closed ideal $I \subset A$ has a dense subideal is that I is not finitely (algebraically) generated. We conjectured that this result is true in the general case. In this paper we give an example showing that this conjecture fails to be true.

Keywords: commutative Banach algebras, dense subideals, the first uncountable ordinal.

Let A be a commutative complex unital Banach algebra and I its proper closed ideal. We say that I is finitely generated, if there are elements $x_1, \dots, x_n$ in A such that

$$I = x_1A + \cdots + x_nA.$$

Clearly, the elements x_i must belong to I. Otherwise we say that I is infinitely generated.

In the paper [2] we conjectured that an ideal I has a proper dense subideal if and only if it is infinitely generated, and proved this conjecture in the case when the algebra A is separable. The term "subideal" of I means here any ideal of A contained in I.

This conjecture was suggested by a result of Grauert and Remmert, which says that a commutative Banach algebra is Noetherian if and only if it is finite dimensional (see Appendix to §5 in [1]). Their basic lemma ([1] Bemerkung 2, p. 54) says that for any commutative Banach algebra A, if the closure $\overline{I}$ of an ideal I in A is finitely generated, then $I = \overline{I}$. Thus a finitely generated closed ideal has no proper dense subideal. Our conjecture meant that the converse result also holds true. The aim of this note is to disprove this conjecture by exhibiting an example of a commutative Banach algebra A and its infinitely generated closed ideal which has no dense proper subideal.

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Denote by Ω the set of all ordinal numbers not greater than ω_1 – the first uncountable ordinal. It is well know that Ω provided with the interval topology (the basis for this topology is the family of all open intervals of ordinal numbers) is a compact space. Note that for each countable ordinal there is a greater (also countable) ordinal which is an isolated point in Ω.

Put $A = C(\Omega)$ – the algebra of all continuous complex-valued functions on Ω provided with the supremum norm. It is known that if a function $x(t)$ is in A, then there is a countable ordinal t_0 such that $x(t) = x(t_0)$ for all $t \geqslant t_0$. In this case $x(t_0) = x(\omega_1)$ and $F(x) = x(\omega_1)$ is a multiplicative linear functional on A. Denote by M its kernel, it is a maximal (closed) ideal in A.

Our claim follows from the following

Proposition. *The above described ideal M is infinitely generated and contains no proper dense subideal.*

Proof. Note first that if $x_1, \ldots, x_n$ are in M and if $t_1, \ldots, t_n$ are countable ordinals such that $x_i(t) = 0$ for all $t \in \Omega, t \geqslant t_i, i = 1, \ldots, n$, then taking an isolated point t_0 in Ω greater than all t_i's and taking a function y on Ω such that $y(t) = 1$ for $t \leqslant t_0$ and $y(t) = 0$ for $t > t_0$, we obtain an element of M which does not belong to the ideal of A generated by $x_1, \ldots, x_n$. Thus M is not finitely generated.

Let now I be a dense subideal of M. We shall be done, if we show that any element x of M must belong to I. So fix an element x_0 in M and let t_0 be an isolated point in Ω such that $x_0(t) = 0$ for all $t \geqslant t_0$. Put $S = \{t \in \Omega : 1 \leqslant t \leqslant t_0\}$, it is a closed (and so compact) subset of Ω. Consider the restriction map $h(x) = x|S$. It is easy to see that h maps A onto $C(S)$ and it is a continuous homomorphism. Thus the image $h(I)$ is an ideal in $C(S)$ which is proper or not. We show that $h(I)$ cannot be proper. Otherwise it would be contained in a maximal ideal of $C(S)$ which is the set of all restrictions $h(x)$ vanishing at some $t_1 \in S$. That means that all elements of I vanish at t_1 and the same would hold for all elements of M, since I is dense in M. That is impossible, since the functionals F and F_1 given by $F_1(x) = x(t_1)$ are different. Consequently $h(I) = C(S)$. Thus there is an element y in I such that

$$h(y) = h(x_0). \tag{1}$$

Since the point t_0 is isolated in Ω, the function $z(t)$ equal to one for t in S and equal to zero for t in $\Omega \setminus S$ belongs to M. Consequently the product zy is in I. By (1) we have $(zy)(t) = x_0(t)$ for t in S, and $(zy)(t) = x_0(t) = 0$ for t in $\Omega \setminus S$. Thus $x_0 = zy$ is in I and the conclusion follows. ■

The question when a closed ideal I of a commutative Banach algebra contains a dense subideal is still open. By the mentioned above result of Grauert and Remmert the necessary condition is that I is infinitely generated. We do not know, however, what is a reasonable sufficient condition.

References

[1] H. Grauert and R. Remmert, *Analytische Stellenalgebren*, Springer-Verlag, Berlin 1971.

[2] W. Żelazko, *When does a closed ideal of a commutative unital Banach algebra contain a dense subideal?*, Funct. Approx. Comment. Math. **44** (2011), 285–287.

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