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# FORMAL PROOFS OF DEGREE 5 BINARY BBP-TYPE FORMULAS

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**Abstract:** We give formal proofs for the binary BBP-type formulas for $\zeta(5)$, $\pi^4 \log 2$, $\pi^2 \log^3 2$ and $\log^5 2$. These formulas were found experimentally by David H. Bailey, using his PSLQ algorithm. This present paper provides the first avenue where the hitherto unproved formulas are formally proved.

**Keywords:** BBP type formulas, digit extraction formulas, polylogarithm constants.

## 1. Introduction

BBP-type formulas are formulas of the form

$$\alpha = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{l} a_j/(kl+j)^s$$

where $s$, $b$, $l$ and $a_j$ are integers, and $\alpha$ is some constant. Formulas of this type were first introduced in a 1996 paper [1], where a formula of this type for $\pi$ was given. Such formulas allow digit extraction — the $i$-th digit of a mathematical constant $\alpha$ in base $b$ can be calculated directly, without needing to compute any of the previous $i-1$ digits, by means of simple algorithms that do not require multiple-precision arithmetic [2].

Apart from digit extraction, another reason the study of BBP-Type formulas has continued to attract attention is that BBP-Type constants are conjectured to be either rational or normal to base $b$ [3, 4, 5], that is their base-$b$ digits are randomly distributed.

BBP-Type formulas are usually discovered experimentally, through computer searches, by using Bailey and Ferguson's PSLQ (Partial Sum of Squares – Lower Quadrature) algorithm [6] or its variations. PSLQ and other integer relation finding schemes typically do not suggest proofs [4, 7]. Formal proofs must be developed

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after the formulas have been discovered. In reference [9] a wide range of general degree 1 BBP-Type formulas in arbitrary bases were derived. Many known BBP-Type formulas are seen to be particular cases of the formulas presented. In [10], two alternative approaches which lead directly – that is, without doing computer searches – to the discovery of digit extraction formulas were presented, thus removing the burden of finding the formulas first and finding the proofs later. In a recent paper [11] numerous ternary (base 3) BBP-Type formulas were discovered. A clear exposition on the non-computer-search-based discovery of binary BBP-Type formulas may be found in references [12] and [13].

Explicit binary BBP-type formulas exist for $\zeta(5)$, $\pi^4 \log 2$, $\pi^2 \log^3 2$ and $\log^5 2$ [2]. These formulas are however hitherto unproved. In this paper we give formal proofs of the formulas.

## 2. Generators of Degree 5 Binary BBP-type Formulas

The following two-variable degree 5 polylogarithm functional equation was derived by Broadhurst [8] (Eq. 63 pg. 5)

$$
\begin{aligned}
&\mathrm{Li}_5\left[\frac{x\alpha}{y\beta}\right] + \mathrm{Li}_5\left[x\alpha\, y\eta\right] + \mathrm{Li}_5\left[\frac{x\alpha\,\beta}{\eta}\right] + \mathrm{Li}_5\left[x\xi\, y\beta\right] + \mathrm{Li}_5\left[\frac{x\xi}{y\eta}\right] \\
&+ \mathrm{Li}_5\left[\frac{x\xi\,\eta}{\beta}\right] + \mathrm{Li}_5\left[\frac{\alpha\, y\beta}{\xi}\right] + \mathrm{Li}_5\left[\frac{\alpha}{\xi\, y\eta}\right] + \mathrm{Li}_5\left[\frac{\alpha\,\eta}{\xi\,\beta}\right] \\
&- 9\,\mathrm{Li}_5\left[xy\right] - 9\,\mathrm{Li}_5\left[x\beta\right] - 9\,\mathrm{Li}_5\left[x\eta\right] - 9\,\mathrm{Li}_5\left[\frac{x}{y}\right] - 9\,\mathrm{Li}_5\left[\frac{x}{\beta}\right] \\
&- 9\,\mathrm{Li}_5\left[\frac{x}{\eta}\right] - 9\,\mathrm{Li}_5\left[\alpha\, y\right] - 9\,\mathrm{Li}_5\left[\alpha\,\beta\right] - 9\,\mathrm{Li}_5\left[\alpha\,\eta\right] \\
&- 9\,\mathrm{Li}_5\left[\frac{\alpha}{y}\right] - 9\,\mathrm{Li}_5\left[\frac{\alpha}{\beta}\right] - 9\,\mathrm{Li}_5\left[\frac{\alpha}{\eta}\right] - 9\,\mathrm{Li}_5\left[\xi\, y\right] - 9\,\mathrm{Li}_5\left[\xi\,\beta\right] \\
&- 9\,\mathrm{Li}_5\left[\xi\,\eta\right] - 9\,\mathrm{Li}_5\left[\frac{y}{\xi}\right] - 9\,\mathrm{Li}_5\left[\frac{\beta}{\xi}\right] - 9\,\mathrm{Li}_5\left[\frac{\eta}{\xi}\right] \\
&+ 18\,\mathrm{Li}_5\left[x\right] + 18\,\mathrm{Li}_5\left[\alpha\right] + 18\,\mathrm{Li}_5\left[\xi\right] + 18\,\mathrm{Li}_5\left[y\right] + 18\,\mathrm{Li}_5\left[\beta\right] \\
&+ 18\,\mathrm{Li}_5\left[\eta\right] - 18\,\zeta(5) = 3/10\ (\log\xi)^5 + 3/4\ (\log y - \log x)\,(\log\xi)^4 \\
&+ 3/2\ (3\ \log y - \log\eta)\,(\log\eta)^2\,(\log\xi)^2 \\
&+ 1/2\,\pi^2\,(\log\xi - 3\ \log\eta)\,(\log\xi)^2 + 1/5\,\pi^4\log\xi\,.
\end{aligned}
\tag{1}
$$

In the above formula $\xi = 1 - x$, $\eta = 1 - y$, $\alpha = -x/\xi$ and $\beta = -y/\eta$. Evaluating Eq. (1) at coordinates $(1/2, 1/2)$ gives

$$
\begin{aligned}
&\frac{403}{4}\,\zeta(5) - \frac{2}{3}\,\pi^4\log 2 + \pi^2\log^3 2 - \frac{3}{2}\,\log^5 2 \\
&\qquad = 144\,\mathrm{Li}_5\left[\frac{1}{2}\right] - \frac{81}{2}\,\mathrm{Li}_5\left[\frac{1}{4}\right] + 4\,\mathrm{Li}_5\left[-\frac{1}{8}\right]\,.
\end{aligned}
\tag{2}
$$

Evaluating at $(-i, i)$ and taking the real part gives

$$\begin{aligned}\frac{4371}{128}\,\zeta(5) - \frac{349}{3072}\,\pi^4\log 2 + \frac{7}{128}\,\pi^2\log^3 2 - \frac{3}{64}\,\log^5 2 \\ = 36\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] \\ + 4\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 18\,\mathrm{Li}_5\left[\frac{1}{2}\right] - \frac{9}{16}\mathrm{Li}_5\left[-\frac{1}{4}\right].\end{aligned} \tag{3}$$

Evaluating the identity at $(-1, i)$ and taking the real part gives

$$\begin{aligned}\frac{279}{8}\,\zeta(5) - \frac{977}{6144}\,\pi^4\log 2 + \frac{97}{768}\,\pi^2\log^3 2 - \frac{15}{128}\,\log^5 2 \\ = 2\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ - 32\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + 37\,\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{7}{8}\,\mathrm{Li}_5\left[-\frac{1}{4}\right].\end{aligned} \tag{4}$$

Broadhurst proved (Eq. 68 of [8] written out) that

$$\begin{aligned}\frac{31}{32}\,\zeta(5) - \frac{343}{99360}\,\pi^4\log 2 + \frac{5}{2484}\,\pi^2\log^3 2 - \frac{2}{1035}\,\log^5 2 \\ = \frac{128}{69}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\,\exp\left(\frac{i\pi}{4}\right)\right] - \frac{20}{69}\,\mathrm{Li}_5\left[\frac{1}{2}\right].\end{aligned} \tag{5}$$

Solving Eqs. (2), (3), (4) and (5) simultaneously for $\zeta(5)$, $\pi^4\log 2$, $\pi^2\log^3 2$ and $\log^5 2$ we find

$$\begin{aligned}\zeta(5) = {} & \frac{1317888}{1457}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{377856}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ & + \frac{56097792}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + \frac{1240064}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\ & - \frac{929664}{62651}\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{644112}{62651}\mathrm{Li}_5\left[\frac{1}{4}\right] - \frac{63616}{62651}\mathrm{Li}_5\left[-\frac{1}{8}\right] + \frac{616752}{62651}\,\mathrm{Li}_5\left[-\frac{1}{4}\right],\end{aligned} \tag{6}$$

$$\begin{aligned}\pi^4\log 2 &= \frac{18593280}{47}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{5294592}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad+\frac{794230272}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]\\&\quad+\frac{16467456}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad-\frac{11408256}{2021}\operatorname{Li}_5\left[\frac{1}{2}\right]+\frac{9121248}{2021}\operatorname{Li}_5\left[\frac{1}{4}\right]\\&\quad-\frac{900864}{2021}\operatorname{Li}_5\left[-\frac{1}{8}\right]+\frac{8769816}{2021}\operatorname{Li}_5\left[-\frac{1}{4}\right],\end{aligned}\tag{7}$$

$$\begin{aligned}\pi^2\log^3 2 &= \frac{17440704}{47}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{4835904}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad+\frac{747218112}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]\\&\quad+\frac{12619200}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad-\frac{7432176}{2021}\operatorname{Li}_5\left[\frac{1}{2}\right]+\frac{8731962}{2021}\operatorname{Li}_5\left[\frac{1}{4}\right]\\&\quad-\frac{862416}{2021}\operatorname{Li}_5\left[-\frac{1}{8}\right]+\frac{8350599}{2021}\operatorname{Li}_5\left[-\frac{1}{4}\right]\end{aligned}\tag{8}$$

and

$$\begin{aligned}\log^5 2 &= \frac{6218880}{47}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]-\frac{1689472}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad+\frac{266699392}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]+\frac{3780736}{2021}\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]\\&\quad-\frac{2092736}{2021}\operatorname{Li}_5\left[\frac{1}{2}\right]+\frac{3217563}{2021}\operatorname{Li}_5\left[\frac{1}{4}\right]\\&\quad-\frac{317784}{2021}\operatorname{Li}_5\left[-\frac{1}{8}\right]+\frac{3005666}{2021}\operatorname{Li}_5\left[-\frac{1}{4}\right].\end{aligned}\tag{9}$$

We are now ready to derive formulas for $\zeta(5)$, $\pi^4\log 2$, $\pi^2\log^3 2$ and $\log^5 2$. In order to save space, we will give the formulas using the compact P-notation [2]:

$$P(s,b,l,A)\equiv\sum_{k=0}^{\infty}\frac{1}{b^k}\sum_{j=1}^{l}\frac{a_j}{(kl+j)^s},$$

where $s$, $b$ and $l$ are integers, and $A=(a_1,a_2,\dots,a_n)$ is a vector of integers.

## 3. Binary BBP-type Formula for $\zeta(5)$

Using the identity

$$\mathrm{Re}\,\mathrm{Li}_5\left[pe^{iq}\right] = \sum_{k=1}^{\infty} \frac{p^k \cos(kq)}{k^5} \qquad p \in [0,1], q \in [0, 2\pi]\,,$$

to write each polylogarithm term in Eq. (6) as a BBP-type formula of length 120 and base $2^{60}$ and forming the indicated linear combination we obtain the degree 5 binary BBP-type formulas for $\zeta(5)$:

$$\begin{aligned}
\zeta(5) = \frac{1}{62651 \cdot 2^{49}} P(5, 2^{60}, 120, (&279 \cdot 2^{59}, -7263 \cdot 2^{60}, 293715 \cdot 2^{57},\\
&-13977 \cdot 2^{60}, -1153683 \cdot 2^{56}, 28377 \cdot 2^{60}, 279 \cdot 2^{56}, 83871 \cdot 2^{59},\\
&-293715 \cdot 2^{54}, -7263 \cdot 2^{56}, -279 \cdot 2^{54}, -889173 \cdot 2^{53}, -279 \cdot 2^{53},\\
&-7263 \cdot 2^{54}, 429705 \cdot 2^{52}, 83871 \cdot 2^{55}, 279 \cdot 2^{51}, 28377 \cdot 2^{54},\\
&-279 \cdot 2^{50}, 1041309 \cdot 2^{49}, 293715 \cdot 2^{48}, -7263 \cdot 2^{50}, 279 \cdot 2^{48},\\
&1153125 \cdot 2^{47}, 1153683 \cdot 2^{46}, -7263 \cdot 2^{48}, 293715 \cdot 2^{45}, -13977 \cdot 2^{48},\\
&-279 \cdot 2^{45}, 28377 \cdot 2^{48}, 279 \cdot 2^{44}, 83871 \cdot 2^{47}, -293715 \cdot 2^{42},\\
&-7263 \cdot 2^{44}, -1153683 \cdot 2^{41}, -889173 \cdot 2^{41}, -279 \cdot 2^{41}, -7263 \cdot 2^{42},\\
&-293715 \cdot 2^{39}, 188811 \cdot 2^{39}, 279 \cdot 2^{39}, 28377 \cdot 2^{42}, -279 \cdot 2^{38},\\
&-13977 \cdot 2^{40}, -429705 \cdot 2^{37}, -7263 \cdot 2^{38}, 279 \cdot 2^{36}, 1153125 \cdot 2^{35},\\
&279 \cdot 2^{35}, -7263 \cdot 2^{36}, 293715 \cdot 2^{33}, -13977 \cdot 2^{36}, -279 \cdot 2^{33},\\
&28377 \cdot 2^{36}, 1153683 \cdot 2^{31}, 83871 \cdot 2^{35}, -293715 \cdot 2^{30}, -7263 \cdot 2^{32},\\
&-279 \cdot 2^{30}, 16497 \cdot 2^{33}, -279 \cdot 2^{29}, -7263 \cdot 2^{30}, -293715 \cdot 2^{27},\\
&83871 \cdot 2^{31}, 1153683 \cdot 2^{26}, 28377 \cdot 2^{30}, -279 \cdot 2^{26}, -13977 \cdot 2^{28},\\
&293715 \cdot 2^{24}, -7263 \cdot 2^{26}, 279 \cdot 2^{24}, 1153125 \cdot 2^{23}, 279 \cdot 2^{23},\\
&-7263 \cdot 2^{24}, -429705 \cdot 2^{22}, -13977 \cdot 2^{24}, -279 \cdot 2^{21}, 28377 \cdot 2^{24},\\
&279 \cdot 2^{20}, 188811 \cdot 2^{19}, -293715 \cdot 2^{18}, -7263 \cdot 2^{20}, -279 \cdot 2^{18},\\
&-889173 \cdot 2^{17}, -1153683 \cdot 2^{16}, -7263 \cdot 2^{18}, -293715 \cdot 2^{15}, 83871 \cdot 2^{19},\\
&279 \cdot 2^{15}, 28377 \cdot 2^{18}, -279 \cdot 2^{14}, -13977 \cdot 2^{16}, 293715 \cdot 2^{12},\\
&-7263 \cdot 2^{14}, 1153683 \cdot 2^{11}, 1153125 \cdot 2^{11}, 279 \cdot 2^{11}, -7263 \cdot 2^{12},\\
&293715 \cdot 2^{9}, 1041309 \cdot 2^{9}, -279 \cdot 2^{9}, 28377 \cdot 2^{12}, 279 \cdot 2^{8},\\
&83871 \cdot 2^{11}, 429705 \cdot 2^{7}, -7263 \cdot 2^{8}, -279 \cdot 2^{6}, -889173 \cdot 2^{5},\\
&-279 \cdot 2^{5}, -7263 \cdot 2^{6}, -293715 \cdot 2^{3}, 83871 \cdot 2^{7}, 279 \cdot 2^{3},\\
&28377 \cdot 2^{6}, -2307366, -13977 \cdot 2^{4}, 293715, -29052, 279, 0))\,. \qquad (10)
\end{aligned}$$

## 4. Binary BBP-type Formula for $\pi^4 \log 2$

Using Eq. (7) and proceeding exactly as in the previous section, we obtain the binary BBP-type formula for $\pi^4 \log 2$ to be

$$
\begin{aligned}
\pi^4 \log 2 = \frac{1}{2021 \cdot 2^{50}} P(5, 2^{60}, 120, (&5157 \cdot 2^{59}, -89127 \cdot 2^{61}, 7805295 \cdot 2^{57},\\
&- 195183 \cdot 2^{61}, -32325939 \cdot 2^{56}, 1621107 \cdot 2^{59}, 5157 \cdot 2^{56},\\
&37287 \cdot 2^{65}, -7805295 \cdot 2^{54}, -89127 \cdot 2^{57}, -5157 \cdot 2^{54},\\
&- 24620409 \cdot 2^{53}, -5157 \cdot 2^{53}, -89127 \cdot 2^{55}, 12255165 \cdot 2^{52},\\
&37287 \cdot 2^{61}, 5157 \cdot 2^{51}, 1621107 \cdot 2^{53}, -5157 \cdot 2^{50},\\
&29192697 \cdot 2^{49}, 7805295 \cdot 2^{48}, -89127 \cdot 2^{51}, 5157 \cdot 2^{48},\\
&32315625 \cdot 2^{47}, 32325939 \cdot 2^{46}, -89127 \cdot 2^{49}, 7805295 \cdot 2^{45},\\
&- 195183 \cdot 2^{49}, -5157 \cdot 2^{45}, 1621107 \cdot 2^{47}, 5157 \cdot 2^{44},\\
&37287 \cdot 2^{53}, -7805295 \cdot 2^{42}, -89127 \cdot 2^{45}, -32325939 \cdot 2^{41},\\
&- 24620409 \cdot 2^{41}, -5157 \cdot 2^{41}, -89127 \cdot 2^{43}, -7805295 \cdot 2^{39},\\
&5866263 \cdot 2^{39}, 5157 \cdot 2^{39}, 1621107 \cdot 2^{41}, -5157 \cdot 2^{38},\\
&- 195183 \cdot 2^{41}, -12255165 \cdot 2^{37}, -89127 \cdot 2^{39}, 5157 \cdot 2^{36},\\
&32315625 \cdot 2^{35}, 5157 \cdot 2^{35}, -89127 \cdot 2^{37}, 7805295 \cdot 2^{33},\\
&- 195183 \cdot 2^{37}, -5157 \cdot 2^{33}, 1621107 \cdot 2^{35}, 32325939 \cdot 2^{31},\\
&37287 \cdot 2^{41}, -7805295 \cdot 2^{30}, -89127 \cdot 2^{33}, -5157 \cdot 2^{30},\\
&480951 \cdot 2^{33}, -5157 \cdot 2^{29}, -89127 \cdot 2^{31}, -7805295 \cdot 2^{27},\\
&37287 \cdot 2^{37}, 32325939 \cdot 2^{26}, 1621107 \cdot 2^{29}, -5157 \cdot 2^{26},\\
&- 195183 \cdot 2^{29}, 7805295 \cdot 2^{24}, -89127 \cdot 2^{27}, 5157 \cdot 2^{24},\\
&32315625 \cdot 2^{23}, 5157 \cdot 2^{23}, -89127 \cdot 2^{25}, -12255165 \cdot 2^{22},\\
&- 195183 \cdot 2^{25}, -5157 \cdot 2^{21}, 1621107 \cdot 2^{23}, 5157 \cdot 2^{20},\\
&5866263 \cdot 2^{19}, -7805295 \cdot 2^{18}, -89127 \cdot 2^{21}, -5157 \cdot 2^{18},\\
&- 24620409 \cdot 2^{17}, -32325939 \cdot 2^{16}, -89127 \cdot 2^{19}, -7805295 \cdot 2^{15},\\
&37287 \cdot 2^{25}, 5157 \cdot 2^{15}, 1621107 \cdot 2^{17}, -5157 \cdot 2^{14}, -195183 \cdot 2^{17},\\
&7805295 \cdot 2^{12}, -89127 \cdot 2^{15}, 32325939 \cdot 2^{11}, 32315625 \cdot 2^{11}, 5157 \cdot 2^{11},\\
&- 89127 \cdot 2^{13}, 7805295 \cdot 2^{9}, 29192697 \cdot 2^{9}, -5157 \cdot 2^{9}, 1621107 \cdot 2^{11},\\
&5157 \cdot 2^{8}, 37287 \cdot 2^{17}, 12255165 \cdot 2^{7}, -89127 \cdot 2^{9}, -5157 \cdot 2^{6},\\
&- 24620409 \cdot 2^{5}, -5157 \cdot 2^{5}, -89127 \cdot 2^{7}, -7805295 \cdot 2^{3}, 37287 \cdot 2^{13},\\
&5157 \cdot 2^{3}, 1621107 \cdot 2^{5}, -64651878, -195183 \cdot 2^{5}, 7805295,\\
&- 89127 \cdot 2^{3}, 5157, 0))\,. \qquad (11)
\end{aligned}
$$

## 5. Binary BBP-type Formula for $\pi^2 \log^3 2$

Writing each polylogarithm term in Eq. (8) as a base $2^{60}$, length 120 BBP-type formula and forming the indicated linear combination we obtain:

$$
\begin{aligned}
\pi^2 \log^3 2 = \frac{1}{2021 \cdot 2^{53}} P(5, 2^{60}, 120, (&21345 \cdot 2^{59}, -464511 \cdot 2^{61}, 47870835 \cdot 2^{57}, \\
&- 1312971 \cdot 2^{61}, -236170815 \cdot 2^{56}, 1579179 \cdot 2^{62}, 21345 \cdot 2^{56}, \\
&286131 \cdot 2^{65}, -47870835 \cdot 2^{54}, -464511 \cdot 2^{57}, -21345 \cdot 2^{54}, \\
&- 173704605 \cdot 2^{53}, -21345 \cdot 2^{53}, -464511 \cdot 2^{55}, 94128645 \cdot 2^{52}, \\
&286131 \cdot 2^{61}, 21345 \cdot 2^{51}, 1579179 \cdot 2^{56}, -21345 \cdot 2^{50}, \\
&215120589 \cdot 2^{49}, 47870835 \cdot 2^{48}, -464511 \cdot 2^{51}, 21345 \cdot 2^{48}, \\
&236128125 \cdot 2^{47}, 236170815 \cdot 2^{46}, -464511 \cdot 2^{49}, 47870835 \cdot 2^{45}, \\
&- 1312971 \cdot 2^{49}, -21345 \cdot 2^{45}, 1579179 \cdot 2^{50}, 21345 \cdot 2^{44}, \\
&286131 \cdot 2^{53}, -47870835 \cdot 2^{42}, -464511 \cdot 2^{45}, -236170815 \cdot 2^{41}, \\
&- 173704605 \cdot 2^{41}, -21345 \cdot 2^{41}, -464511 \cdot 2^{43}, -47870835 \cdot 2^{39}, \\
&56870019 \cdot 2^{39}, 21345 \cdot 2^{39}, 1579179 \cdot 2^{44}, -21345 \cdot 2^{38}, \\
&- 1312971 \cdot 2^{41}, -94128645 \cdot 2^{37}, -464511 \cdot 2^{39}, 21345 \cdot 2^{36}, \\
&236128125 \cdot 2^{35}, 21345 \cdot 2^{35}, -464511 \cdot 2^{37}, 47870835 \cdot 2^{33}, \\
&- 1312971 \cdot 2^{37}, -21345 \cdot 2^{33}, 1579179 \cdot 2^{38}, 236170815 \cdot 2^{31}, \\
&286131 \cdot 2^{41}, -47870835 \cdot 2^{30}, -464511 \cdot 2^{33}, -21345 \cdot 2^{30}, \\
&1950735 \cdot 2^{34}, -21345 \cdot 2^{29}, -464511 \cdot 2^{31}, -47870835 \cdot 2^{27}, \\
&286131 \cdot 2^{37}, 236170815 \cdot 2^{26}, 1579179 \cdot 2^{32}, -21345 \cdot 2^{26}, \\
&- 1312971 \cdot 2^{29}, 47870835 \cdot 2^{24}, -464511 \cdot 2^{27}, 21345 \cdot 2^{24}, \\
&236128125 \cdot 2^{23}, 21345 \cdot 2^{23}, -464511 \cdot 2^{25}, -94128645 \cdot 2^{22}, \\
&- 1312971 \cdot 2^{25}, -21345 \cdot 2^{21}, 1579179 \cdot 2^{26}, 21345 \cdot 2^{20}, \\
&56870019 \cdot 2^{19}, -47870835 \cdot 2^{18}, -464511 \cdot 2^{21}, -21345 \cdot 2^{18}, \\
&- 173704605 \cdot 2^{17}, -236170815 \cdot 2^{16}, -464511 \cdot 2^{19}, -47870835 \cdot 2^{15}, \\
&286131 \cdot 2^{25}, 21345 \cdot 2^{15}, 1579179 \cdot 2^{20}, -21345 \cdot 2^{14}, -1312971 \cdot 2^{17}, \\
&47870835 \cdot 2^{12}, -464511 \cdot 2^{15}, 236170815 \cdot 2^{11}, 236128125 \cdot 2^{11}, \\
&21345 \cdot 2^{11}, -464511 \cdot 2^{13}, 47870835 \cdot 2^{9}, 215120589 \cdot 2^{9}, -21345 \cdot 2^{9}, \\
&1579179 \cdot 2^{14}, 21345 \cdot 2^{8}, 286131 \cdot 2^{17}, 94128645 \cdot 2^{7}, -464511 \cdot 2^{9}, \\
&- 21345 \cdot 2^{6}, -173704605 \cdot 2^{5}, -21345 \cdot 2^{5}, -464511 \cdot 2^{7}, \\
&- 47870835 \cdot 2^{3}, 286131 \cdot 2^{13}, 21345 \cdot 2^{3}, 1579179 \cdot 2^{8}, -472341630, \\
&- 1312971 \cdot 2^{5}, 47870835, -464511 \cdot 2^{3}, 21345, 0)) \,.
\end{aligned}
\tag{12}
$$

## 6. Binary BBP-type Formula for $\log^5 2$

Finally, following the same procedure as in the previous sections, Eq. (9) leads to the binary BBP-type formula:

$$
\begin{aligned}
\log^5 2 = \frac{1}{2021\cdot 2^{52}} P(5, 2^{60}, 120, (&2783\cdot 2^{59}, -32699\cdot 2^{62}, 7171925\cdot 2^{57},\\
&-187547\cdot 2^{61}, -41252441\cdot 2^{56}, 9391097\cdot 2^{57}, 2783\cdot 2^{56},\\
&52183\cdot 2^{65}, -7171925\cdot 2^{54}, -32699\cdot 2^{58}, -2783\cdot 2^{54},\\
&-29483621\cdot 2^{53}, -2783\cdot 2^{53}, -32699\cdot 2^{56}, 17037475\cdot 2^{52},\\
&52183\cdot 2^{61}, 2783\cdot 2^{51}, 9391097\cdot 2^{51}, -2783\cdot 2^{50},\\
&38246123\cdot 2^{49}, 7171925\cdot 2^{48}, -32699\cdot 2^{52}, 2783\cdot 2^{48},\\
&41307505\cdot 2^{47}, 41252441\cdot 2^{46}, -32699\cdot 2^{50}, 7171925\cdot 2^{45},\\
&-187547\cdot 2^{49}, -2783\cdot 2^{45}, 9391097\cdot 2^{45}, 2783\cdot 2^{44},\\
&52183\cdot 2^{53}, -7171925\cdot 2^{42}, -32699\cdot 2^{46}, -41252441\cdot 2^{41},\\
&-29483621\cdot 2^{41}, -2783\cdot 2^{41}, -32699\cdot 2^{44}, -7171925\cdot 2^{39},\\
&12188517\cdot 2^{39}, 2783\cdot 2^{39}, 9391097\cdot 2^{39}, -2783\cdot 2^{38},\\
&-187547\cdot 2^{41}, -17037475\cdot 2^{37}, -32699\cdot 2^{40}, 2783\cdot 2^{36},\\
&41307505\cdot 2^{35}, 2783\cdot 2^{35}, -32699\cdot 2^{38}, 7171925\cdot 2^{33},\\
&-187547\cdot 2^{37}, -2783\cdot 2^{33}, 9391097\cdot 2^{33}, 41252441\cdot 2^{31},\\
&52183\cdot 2^{41}, -7171925\cdot 2^{30}, -32699\cdot 2^{34}, -2783\cdot 2^{30},\\
&5881627\cdot 2^{30}, -2783\cdot 2^{29}, -32699\cdot 2^{32}, -7171925\cdot 2^{27},\\
&52183\cdot 2^{37}, 41252441\cdot 2^{26}, 9391097\cdot 2^{27}, -2783\cdot 2^{26},\\
&-187547\cdot 2^{29}, 7171925\cdot 2^{24}, -32699\cdot 2^{28}, 2783\cdot 2^{24},\\
&41307505\cdot 2^{23}, 2783\cdot 2^{23}, -32699\cdot 2^{26}, -17037475\cdot 2^{22},\\
&-187547\cdot 2^{25}, -2783\cdot 2^{21}, 9391097\cdot 2^{21}, 2783\cdot 2^{20},\\
&12188517\cdot 2^{19}, -7171925\cdot 2^{18}, -32699\cdot 2^{22}, -2783\cdot 2^{18},\\
&-29483621\cdot 2^{17}, -41252441\cdot 2^{16}, -32699\cdot 2^{20}, -7171925\cdot 2^{15},\\
&52183\cdot 2^{25}, 2783\cdot 2^{15}, 9391097\cdot 2^{15}, -2783\cdot 2^{14}, -187547\cdot 2^{17},\\
&7171925\cdot 2^{12}, -32699\cdot 2^{16}, 41252441\cdot 2^{11}, 41307505\cdot 2^{11}, 2783\cdot 2^{11},\\
&-32699\cdot 2^{14}, 7171925\cdot 2^{9}, 38246123\cdot 2^{9}, -2783\cdot 2^{9}, 9391097\cdot 2^{9},\\
&2783\cdot 2^{8}, 52183\cdot 2^{17}, 17037475\cdot 2^{7}, -32699\cdot 2^{10}, -2783\cdot 2^{6},\\
&-29483621\cdot 2^{5}, -2783\cdot 2^{5}, -32699\cdot 2^{8}, -7171925\cdot 2^{3}, 52183\cdot 2^{13},\\
&2783\cdot 2^{3}, 9391097\cdot 2^{3}, -82504882, -187547\cdot 2^{5}, 7171925,\\
&-32699\cdot 2^{4}, 2783, 30315))\,. \qquad (13)
\end{aligned}
$$

## 7. Conclusion

Explicit binary BBP-type formulas for the $SC^*$ constants $\zeta(5)$, $\pi^4 \log 2$, $\pi^2 \log^3 2$ and $\log^5 2$ which were first obtained experimentally are now proved.

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