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OPTIMAL GROUPS FOR THE r–RANK ARTIN CONJECTURE

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Abstract: For any finitely generated subgroup Γ of $\mathbb{Q}^*$, Pappalardi and the first–named author [1] found a formula to compute the density of the primes ℓ for which the reduction modulo ℓ of Γ contains a primitive root modulo ℓ. They conjectured a characterization of optimal groups, free or torsion, i.e. subgroups with maximal density. In this paper we prove their conjecture and give a similar characterization for optimal positive groups.

Keywords: Artin primitive root conjecture, finitely generated subgroups of $\mathbb{Q}^*$.

1. Introduction and main results

Let Γ be a finitely generated subgroup of $\mathbb{Q}^*$, the multiplicative group of non–zero rational numbers. We denote the rank of Γ by r, and we assume $r \geqslant 1$. We define $\mathrm{Supp}(\Gamma)$ as the (finite) set of primes ℓ such that $\nu_\ell(a) \neq 0$ for some $a \in \Gamma$. Hereafter, ℓ will always denote a prime number. For any $\ell \notin \mathrm{Supp}(\Gamma)$, we set $\Gamma \bmod \ell = \{a \bmod \ell : a \in \Gamma\}$, which is a subgroup of the multiplicative group $\mathbb{F}_\ell^*$. For any positive real number x, let $N_\Gamma(x) = \#\{\ell \leqslant x : \ell \notin \mathrm{Supp}(\Gamma) \text{ and } \Gamma \bmod \ell = \mathbb{F}_\ell^*\}$.

The Artin Conjecture for primitive roots states that $N_\Gamma(x) \to \infty$ for $x \to \infty$, when Γ is generated by an integer a which is different from -1 and is not a perfect square. Under the Generalized Riemann Hypothesis for some number fields, Hooley [2] proved that $N_\Gamma(x) \sim \delta_\Gamma \frac{x}{\log x}$, when $\Gamma = \langle a \rangle$ with a as above, giving an explicit formula to compute the density. In the general case of groups Γ of any rank, Pappalardi [6] proved the same asympototic formula for $N_\Gamma(x)$, and Pappalardi and the first–named author [1] gave a complicated formula to compute δ_Γ. Indeed, they proved that $\delta_\Gamma = A_r b_\Gamma c_\Gamma$, where

$$A_r = \prod_{\ell > 2} \left(1 - \frac{1}{\ell^r(\ell - 1)}\right) \tag{1}$$

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is the r–rank Artin constant,

$$b_\Gamma = \prod_{\ell > 2} \left(1 - \frac{\ell^{r - r_\ell} - 1}{\ell^r(\ell - 1) - 1} \right) \tag{2}$$

and

$$c_\Gamma = 1 - \frac{1}{2^{r_2}} \sum_{\xi \in \widetilde{\Gamma}} \mu(|s(\xi)|) \prod_{\ell | s(\xi)} \frac{1}{\ell^{r_\ell}(\ell - 1) - 1}. \tag{3}$$

Here, $r_\ell = \dim_{\mathbb{F}_\ell}(\Gamma \mathbb{Q}^{*\ell}/\mathbb{Q}^{*\ell})$, where $\mathbb{Q}^{*\ell} = \{a^\ell : a \in \mathbb{Q}^*\}$. Furthermore, for any $\xi \in \mathbb{Q}^*/\mathbb{Q}^{*2}$, we let $s(\xi)$ denote the unique square–free integer in the equivalence class ξ. Then, $\widetilde{\Gamma} = \{\xi \in \Gamma\mathbb{Q}^{*2}/\mathbb{Q}^{*2} : s(\xi) \equiv 1 \pmod 4\}$. Since $r_\ell < r$ only for finitely many primes ℓ (see Section 2), then the product defining b_Γ is finite, so that b_Γ is a positive rational number. We also note that c_Γ is rational, since $\widetilde{\Gamma}$ is finite.

We refer the reader to the paper by Moree [4] for a comprehensive survey on Artin's primitive root conjecture, written both for a general audience and for specialists, including some historical remarks, a complete bibliography, open problems and outlines to many variations of the conjecture. With regard to generalizations to the higher rank case, we point out the papers by Pappalardi and Susa [8], Pappalardi [7], and Menici and Pehlivan [3]. It is also interesting to note that Moree and Stevenhagen [5] recovered the above formula for δ_Γ using a unified general approach for the computation of Artin primitive root densities.

For any r, it is easy to find a subgroup Γ with rank r such that δ_Γ is arbitrarily small. In contrast, it is not evident that for any r there is a maximum value of δ_Γ, varying Γ among all the subgroups of $\mathbb{Q}^*$ with rank r. In [1], the authors conjecture that, for any given rank r, there exists a free group of rank r having maximal density, and the same is stated for torsion groups of rank r. Moreover, they propose a characterization of free groups, and of torsion groups, having maximal density, which they call *optimal*. In the present paper we prove their claims, with just a minor correction, and complete the picture, taking into account also the positive groups, that is the subgroups of $\mathbb{Q}^+$.

We say that a free (or torsion) subgroup of $\mathbb{Q}^*$ with rank r is an *optimal group* when its density is maximal in the set of the densities of all free (or torsion, respectively) subgroups of $\mathbb{Q}^*$ with rank r.

Let $(p_i)_{i \geqslant 1}$ be the increasing sequence of all the odd primes.

Theorem 1. *The free group $\langle (-1/p_i)p_i : i = 1, \dots, r \rangle$ is optimal, and its density is*

$$A_r \left(1 - \frac{1}{2^r} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^r(p_i - 1) - 1} \right) \right).$$

Moreover, a free subgroup Γ of $\mathbb{Q}^*$ with rank r is optimal if and only if $\widetilde{\Gamma} = \langle (-1/p_i)p_i\mathbb{Q}^{*2} : i = 1, \dots, r \rangle$, and $r_\ell = r$ for every ℓ when $r \geqslant 2$, while $r_\ell = 1$ for every $\ell \neq 3$ when $r = 1$.

Theorem 2. *The torsion group $\big\langle -1, p_i : i = 1, \dots, r\big\rangle$ is optimal, and its density is*

$$A_r \left(1 - \frac{1}{2^{r+1}} \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^r(p_i - 1) - 1}\right)\right).$$

Moreover, a torsion subgroup Γ of $\mathbb{Q}^*$ with rank r is optimal if and only if $\widetilde{\Gamma} = \big\langle (-1/p_i)p_i\mathbb{Q}^{*2} : i = 1, \dots, r\big\rangle$, and $r_\ell = r$ for every $\ell > 2$.

By Theorem 1 no positive group is optimal, as a free group. However, we can say that a positive subgroup of $\mathbb{Q}^*$ with rank r is an *optimal group* when its density is maximal in the set of the densities of all positive subgroups of $\mathbb{Q}^*$ with rank r. Let $(q_i)_{i\geqslant 1}$ be the increasing sequence of all the primes q satisfying $q \equiv 1 \pmod 4$.

Theorem 3. *The positive group $\big\langle q_i : i = 1, \dots, r\big\rangle$ is optimal, and its density is*

$$A_r \left(1 - \frac{1}{2^{r}} \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^r(q_i - 1) - 1}\right)\right).$$

Moreover, a positive subgroup Γ of $\mathbb{Q}^*$ with rank r is optimal if and only if $\widetilde{\Gamma} = \big\langle q_i\mathbb{Q}^{*2} : i = 1, \dots, r\big\rangle$, and $r_\ell = r$ for every ℓ when $r \geqslant 2$, while $r_\ell = 1$ for every $\ell \neq 5$ when $r = 1$.

In Section 2, we sketch the proof of our results and give some remarks related to finitely generated subgroups of $\mathbb{Q}^*$. In Section 3, we prove a basic technical lemma about certain sums over subgroups of $\mathbb{Q}^*/\mathbb{Q}^{*2}$, which will be the the main tool in the proof of our theorems. In Sections 4, 5 and 6, we prove Theorems 1, 2 and 3, respectively.

2. Outline of the proof and preliminary remarks

The idea behind the proof of the characterization of optimal groups is the following. Given a non–optimal group Γ, we look for some other group (with the same rank) having density greater than that of Γ. This is attained by recursively removing, adding, or substituting primes in $\mathrm{Supp}(\Gamma)$. Since, for any fixed r, $\delta_\Gamma = A_r b_\Gamma c_\Gamma$ and A_r is constant, we have to maximize the product $b_\Gamma c_\Gamma$. The term we mainly have to control is c_Γ, while b_Γ is dealt with easily in a second phase, when some compensation may occur. Hence we are led to study the sum in the formula of c_Γ, and this can be more easily undertaken in a more general set, considering similar sums over subgroups of $\mathbb{Q}^*/\mathbb{Q}^{*2}$. When dealing with the product $b_\Gamma c_\Gamma$ in Sections 4, 5 and 6, we shall need some general remarks that we list below.

Let Γ be a finitely generated subgroup of $\mathbb{Q}^*$ with rank r. Then, Γ is free if and only if $-1 \notin \Gamma$, and is torsion otherwise. In both cases, there exist $a_i \in \mathbb{Q}^*$, for $i = 1, \dots, r$, such that $a_1, \dots, a_r$ are multiplicatively independent, and $\Gamma = \langle a_1, \dots, a_r\rangle$ when Γ is free, while $\Gamma = \langle -1, a_1, \dots, a_r\rangle$ when Γ is torsion; in the latter case, we may assume that $a_i > 0$, for $i = 1, \dots, r$.

We recall that $r_\ell = \dim_{\mathbb{F}_\ell}(\Gamma\mathbb{Q}^{*\ell}/\mathbb{Q}^{*\ell})$. In other words, r_ℓ is the maximal number of elements in Γ that are multiplicatively independent modulo ℓ–powers. Therefore, $0 \leqslant r_\ell \leqslant r$ for every odd prime ℓ, and $0 \leqslant r_2 \leqslant r$ when Γ is free, while $1 \leqslant r_2 \leqslant r+1$ when Γ is torsion.

We note that $\widetilde{\Gamma}$ is a subgroup of $\Gamma\mathbb{Q}^{*2}/\mathbb{Q}^{*2}$ and $-\mathbb{Q}^{*2} \notin \widetilde{\Gamma}$. If we let $t = \dim_{\mathbb{F}_2}(\widetilde{\Gamma})$, then $0 \leqslant t \leqslant \min\{r_2, r\}$. It easily follows that c_Γ is positive when r_2 is positive, while $c_\Gamma = 0$ when $r_2 = 0$. We also note that if $\widetilde{\Gamma} = \big\langle(-1/\ell_i)\ell_i : i = 1, \dots, r\big\rangle$ for some primes ℓ_i, then $r_2 = r$ when Γ is free, and $r_2 = r+1$ when Γ is torsion.

If $\mathrm{Supp}(\Gamma) = \{\ell_1, \dots, \ell_s\}$ then $s \geqslant r$, and there exists a matrix $M = (m_{ij})$ of size $r \times s$, with integer entries, such that $|a_i| = \prod_{j=1}^{s} \ell_j^{m_{ij}}$. It is shown in [1] that $r_\ell = \mathrm{rank}(M \bmod \ell)$ for every odd prime ℓ, and $r_2 = \mathrm{rank}(M \bmod 2)$ when $-1 \notin \Gamma\mathbb{Q}^{*2}$, while $r_2 = \mathrm{rank}(M \bmod 2) + 1$ when $-1 \in \Gamma\mathbb{Q}^{*2}$ (which is the case when Γ is torsion). Moreover, for every odd prime ℓ, we have $r_\ell = r$ if and only if $\ell \nmid \Delta(M)$, where $\Delta(M)$ is the greatest common divisor of the minors of maximum size (i.e. r) of M. Hence, $r_\ell < r$ only for finitely many primes ℓ. In addition, $r_\ell = r$ for all ℓ if and only if $\Delta(M) = 1$, while $r_\ell = r$ for all $\ell \neq 3$ (or $\ell \neq 5$) if and only if $\Delta(M) = 3^n$ (or 5^n, respectively) for some integer $n \geqslant 0$. This shows that the condition on the r_ℓ's in Theorems 1, 2 and 3 can be reformulated in terms of $\Delta(M)$.

3. Sums over subgroups of $\mathbb{Q}^*/\mathbb{Q}^{*2}$

Let G be a finite subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$. Each element of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ can be uniquely written as $m\mathbb{Q}^{*2}$, where m is a square–free integer. Hence, hereafter m will denote a square–free integer, and we shall write an element of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ as $m\mathbb{Q}^{*2}$. According to the notation in Section 1, for $\xi = m\mathbb{Q}^{*2}$ we have $m = s(\xi)$. We suppose that $-\mathbb{Q}^{*2} \notin G$; this implies that, for all $m \in \mathbb{Z}$, if $m\mathbb{Q}^{*2} \in G$ then $-m\mathbb{Q}^{*2} \notin G$.

Let $\chi : G \to \{\pm 1\}$ be a homomorphism of multiplicative groups. Let $f(\ell)$ be a real function defined over the set of primes, with values in the open unit interval $(0, 1)$. For G, χ and f as above, let

$$S(G, \chi, f) = \sum_{m\mathbb{Q}^{*2} \in G} \chi(m\mathbb{Q}^{*2}) \prod_{\ell \mid m} f(\ell).$$

If $G = \{\mathbb{Q}^{*2}\}$, the above sum equals 1. Furthermore, if χ_1 is the trivial homomorphism (that is the one with constant value 1), then $S(G, \chi_1, f) \geqslant 1$ for any G and any f, where the equality holds if and only if $G = \{\mathbb{Q}^{*2}\}$.

Let $\mathrm{Supp}(G)$ be the (finite) set of primes ℓ dividing m for some integer m with $m\mathbb{Q}^{*2} \in G$. For $\ell \in \mathrm{Supp}(G)$, let G_ℓ be the subgroup of G of the elements $m\mathbb{Q}^{*2} \in G$ such that $\ell \nmid m$. Clearly, $\ell \notin \mathrm{Supp}(G_\ell)$.

Lemma 4. *For all G, χ and f, we have*

$$S(G, \chi, f) > 0,$$

and for each $\ell \in \operatorname{Supp}(G)$

$$S(G, \chi, f) \geqslant \big(1 - f(\ell)\big) S(G_\ell, \chi, f),$$

where the equality holds if and only if $\pm\ell\mathbb{Q}^{*2} \in G$ *and* $\chi(\pm\ell\mathbb{Q}^{*2}) = -1$.

Proof. We argue by induction on $h = \big|\operatorname{Supp}(G)\big|$. If $h = 0$, then $G = \{\mathbb{Q}^{*2}\}$, thus $S(G, \chi, f) = 1$. If $h \geqslant 1$, in order to fix the ideas, let $\operatorname{Supp}(G) = \{\ell_1, \dots, \ell_h\}$ and $\ell = \ell_1$. Even if not required, we prove directly also the case $h = 1$: now $G = \{\mathbb{Q}^{*2}, \ell\mathbb{Q}^{*2}\}$ or $G = \{\mathbb{Q}^{*2}, -\ell\mathbb{Q}^{*2}\}$, so that

$$S(G, \chi, f) = 1 + \chi(\pm\ell\mathbb{Q}^{*2}) f(\ell).$$

Since $G_\ell = \{\mathbb{Q}^{*2}\}$, we have $S(G_\ell, \chi, f) = 1$, and the result follows from this and $0 < f(\ell) < 1$.

Let $h > 1$. Since $\operatorname{Supp}(G_\ell) \subseteq \{\ell_2, \dots, \ell_h\}$, by the inductive hypothesis we have

$$S(G_\ell, \chi, f) > 0. \tag{4}$$

We distinguish two cases.

First case. Suppose that $\pm\ell\mathbb{Q}^{*2} \in G$, that is $\ell\mathbb{Q}^{*2} \in G$ or $-\ell\mathbb{Q}^{*2} \in G$ (but not both of them). Since G_ℓ is a subgroup of G with index 2, we have:

$$\text{if } m\mathbb{Q}^{*2} \in G \setminus G_\ell, \text{ then } \ell \mid m \text{ and } \pm\frac{m}{\ell}\mathbb{Q}^{*2} \in G_\ell,$$

and

$$\text{if } m\mathbb{Q}^{*2} \in G_\ell, \text{ then } \ell \nmid m \text{ and } \pm\ell m\mathbb{Q}^{*2} \in G \setminus G_\ell.$$

Hence

$$S(G, \chi, f) - S(G_\ell, \chi, f) = \chi(\pm\ell\mathbb{Q}^{*2}) f(\ell) S(G_\ell, \chi, f). \tag{5}$$

Since $0 < f(\ell) < 1$, by (4) and (5) we obtain the result.

Second case. Suppose now that $\ell\mathbb{Q}^{*2} \notin G$ and $-\ell\mathbb{Q}^{*2} \notin G$. Let H be the subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ generated by the elements of G and by $\ell\mathbb{Q}^{*2}$. We lift χ to a homomorphism on H, which we still call χ, by putting $\chi(\ell\mathbb{Q}^{*2}) = 1$. We consider H_ℓ and note that $\operatorname{Supp}(H_\ell) = \{\ell_2, \dots, \ell_h\}$. Hence, besides (4), we have

$$S(H_\ell, \chi, f) > 0. \tag{6}$$

Moreover, G and H_ℓ are subgroups of H with index 2, and $G_\ell = G \cap H_\ell$. As a result, we have:

$$\text{if } m\mathbb{Q}^{*2} \in G \setminus G_\ell, \text{ then } \ell \mid m \text{ and } \frac{m}{\ell}\mathbb{Q}^{*2} \in H_\ell \setminus G_\ell,$$

and

$$\text{if } m\mathbb{Q}^{*2} \in H_\ell \setminus G_\ell, \text{ then } \ell \nmid m \text{ and } \ell m\mathbb{Q}^{*2} \in G \setminus G_\ell.$$

Therefore

$$S(G, \chi, f) - S(G_\ell, \chi, f) = f(\ell)\big(S(H_\ell, \chi, f) - S(G_\ell, \chi, f)\big). \tag{7}$$

Recalling that $0 < f(\ell) < 1$, by (6) and (7) we get

$$S(G,\chi,f) > \big(1 - f(\ell)\big) S(G_\ell,\chi,f), \tag{8}$$

which is positive by (4). ■

Remark. In the second case of the above proof, besides G and H_ℓ, there exists a third subgroup of H containing G_ℓ, namely the group K generated by the elements of G_ℓ and by $\ell\mathbb{Q}^{*2}$. Then we may lift χ to a homomorphism χ_- on H by putting $\chi_-(\ell\mathbb{Q}^{*2}) = -1$, this time. We have $K_\ell = G_\ell$ and $S(K,\chi_-,f) = \big(1 - f(\ell)\big) S(G_\ell,\chi,f)$. Hence, the inequality (8) can be read as

$$S(G,\chi,f) > S(K,\chi_-,f),$$

thus relating Lemma 4 to the outline of the proof given at the beginning of Section 2.

We point out that $\widetilde{\Gamma}$ is a subgroup of $\mathbb{Q}^*/\mathbb{Q}^{*2}$ and that $-\mathbb{Q}^{*2} \notin \widetilde{\Gamma}$. Hence we are going to apply Lemma 4 to $\widetilde{\Gamma}$, with the homomorphism $\mu_+ : \mathbb{Q}^*/\mathbb{Q}^{*2} \to \{\pm 1\}$ defined by

$$\mu_+(m\mathbb{Q}^{*2}) = \mu(|m|).$$

4. Optimal free groups

We note that $2 \notin \mathrm{Supp}(\widetilde{\Gamma})$. For any odd prime ℓ, we let

$$f(\ell) = \frac{1}{\ell^{r_\ell}(\ell - 1) - 1},$$

so that $0 < f(\ell) < 1$.

We know that $\widetilde{\Gamma}$ has 2^t elements, for some integer t such that $0 \leqslant t \leqslant r_2 \leqslant r$. As a consequence, $\mathrm{Supp}(\widetilde{\Gamma})$ has at least t elements. By Lemma 4, using induction on t, there exist t primes $\ell_1, \ldots, \ell_t \in \mathrm{Supp}(\widetilde{\Gamma})$ such that

$$S(\widetilde{\Gamma},\mu_+,f) \geqslant \prod_{i=1}^{t} \big(1 - f(\ell_i)\big),$$

and the equality holds if and only if $\widetilde{\Gamma} = \big\langle (-1/\ell_1)\ell_1\mathbb{Q}^{*2}, \ldots, (-1/\ell_t)\ell_t\mathbb{Q}^{*2} \big\rangle$. It follows that there always exist r (instead of t) odd primes $\ell_1, \ldots, \ell_r$ (not necessarily in $\mathrm{Supp}(\widetilde{\Gamma})$) such that

$$S(\widetilde{\Gamma},\mu_+,f) \geqslant \prod_{i=1}^{r} \big(1 - f(\ell_i)\big),$$

and the equality holds if and only if $t = r$ and

$$\widetilde{\Gamma} = \big\langle (-1/\ell_1)\ell_1\mathbb{Q}^{*2}, \ldots, (-1/\ell_r)\ell_r\mathbb{Q}^{*2} \big\rangle. \tag{9}$$

Since $r_2 \leqslant r$, we have by (3)

$$c_\Gamma = 1 - \frac{1}{2^{r_2}} S(\widetilde{\Gamma}, \mu_+, f) \leqslant 1 - \frac{1}{2^{r_2}} \prod_{i=1}^{r} \big(1 - f(\ell_i)\big) \leqslant 1 - \frac{1}{2^{r}} \prod_{i=1}^{r} \big(1 - f(\ell_i)\big),$$

and the two equalities hold if and only if (9) holds and $r_2 = r$, respectively. Here we recall that for free groups (9) implies $r_2 = r$.

With regard to b_Γ, defined by (2), the factor corresponding to ℓ is 1 when $r_\ell = r$, and less than 1 when $r_\ell < r$. Hence

$$b_\Gamma \leqslant \prod_{i=1}^{r} \left(1 - \frac{\ell_i^{r - r_{\ell_i}} - 1}{\ell_i^r(\ell_i - 1) - 1}\right),$$

where the equality holds if and only if $r_\ell = r$ for every $\ell \notin \{2, \ell_1, \dots, \ell_r\}$.

Thus

$$b_\Gamma c_\Gamma \leqslant \prod_{i=1}^{r} \left(1 - \frac{\ell_i^{r - r_{\ell_i}} - 1}{\ell_i^r(\ell_i - 1) - 1}\right)\left(1 - \frac{1}{2^r} \prod_{i=1}^{r} \left(1 - \frac{1}{\ell_i^{r_{\ell_i}}(\ell_i - 1) - 1}\right)\right),$$

and the equality holds if and only if (9) holds and $r_\ell = r$ for every $\ell \notin \{\ell_1, \dots, \ell_r\}$. Putting

$$x_i = \ell^{r_{\ell_i}}(\ell_i - 1), \quad y_i = \ell_i^r(\ell_i - 1),$$

we have $x_i \leqslant y_i$, and the bound for $b_\Gamma c_\Gamma$ can be written as

$$\prod_{i=1}^{r} \frac{y_i}{y_i - 1} \prod_{i=1}^{r} \frac{x_i - 1}{x_i} \left(1 - \prod_{i=1}^{r} \frac{x_i - 2}{2(x_i - 1)}\right).$$

We let

$$\begin{aligned} g_r(x_1, \dots, x_r) &= \prod_{i=1}^{r} \frac{x_i - 1}{x_i} \left(1 - \prod_{i=1}^{r} \frac{x_i - 2}{2(x_i - 1)}\right) \\ &= \prod_{i=1}^{r} \left(1 - \frac{1}{x_i}\right) - \prod_{i=1}^{r} \left(\frac{1}{2} - \frac{1}{x_i}\right). \end{aligned}$$

For $r = 1$, $g_1(x_1)$ is constant, equal to $1/2$. For $r \geqslant 2$, we highlight the dependency on x_1 by noting that

$$\begin{aligned} g_r(x_1, \dots, x_r) = &\prod_{i=2}^{r} \left(1 - \frac{1}{x_i}\right) - \frac{1}{2} \prod_{i=2}^{r} \left(\frac{1}{2} - \frac{1}{x_i}\right) \\ &- \frac{1}{x_1}\left(\prod_{i=2}^{r} \left(1 - \frac{1}{x_i}\right) - \prod_{i=2}^{r} \left(\frac{1}{2} - \frac{1}{x_i}\right)\right). \end{aligned}$$

By symmetry in $x_1,\dots,x_r$, we see that $g_r(x_1,\dots,x_r)\leqslant g_r(y_1,\dots,y_r)$ when $r\geqslant 2$, and the equality holds if and only if $x_i=y_i$ for $i=1,\dots,r$, that is $r_{\ell_i}=r$ for $i=1,\dots,r$. In conclusion

$$
\begin{aligned}
b_\Gamma c_\Gamma \leqslant g_r(y_1,\dots,y_r)\prod_{i=1}^r \frac{y_i}{y_i-1} &= 1-\prod_{i=1}^r \frac{y_i-2}{2(y_i-1)}\\
&= 1-\frac{1}{2^r}\prod_{i=1}^r\Big(1-\frac{1}{\ell_i^r(\ell_i-1)-1}\Big).
\end{aligned}
$$

Moreover, the equality holds if and only if (9) holds, and $r_\ell=1$ for every $\ell\neq\ell_1$ when $r=1$, whereas $r_\ell=r$ for every ℓ when $r\geqslant 2$.

We remind that $(p_i)_{i\geqslant 1}$ is the sequence of all the odd primes. Then

$$
1-\frac{1}{2^r}\prod_{i=1}^r\Big(1-\frac{1}{\ell_i^r(\ell_i-1)-1}\Big)\leqslant 1-\frac{1}{2^r}\prod_{i=1}^r\Big(1-\frac{1}{p_i^r(p_i-1)-1}\Big),
$$

where the equality holds if and only if $\ell_i=p_i$, for $i=1,\dots,r$. This completes the proof of the characterization of optimal free groups in Theorem 1. It is plain that $\big\langle(-1/p_i)p_i : i=1,\dots,r\big\rangle$ is the simplest optimal free group.

5. Optimal torsion groups

We repeat the same arguments as in the case of free groups, except that now we have $r_2\leqslant r+1$. Therefore there exist r primes $\ell_1,\dots,\ell_r$ such that

$$
c_\Gamma\leqslant 1-\frac{1}{2^{r_2}}\prod_{i=1}^r\big(1-f(\ell_i)\big)\leqslant 1-\frac{1}{2}\prod_{i=1}^r\frac{x_i-2}{2(x_i-1)}
$$

and

$$
b_\Gamma c_\Gamma\leqslant \prod_{i=1}^r\frac{y_i}{y_i-1}\prod_{i=1}^r\frac{x_i-1}{x_i}\Big(1-\frac{1}{2}\prod_{i=1}^r\frac{x_i-2}{2(x_i-1)}\Big).
$$

In the latter bound the equality holds if and only if (9) holds, $r_\ell=r$ for every $\ell\notin\{2,\ell_1,\dots,\ell_r\}$, and $r_2=r+1$. We recall that for torsion groups (9) implies $r_2=r+1$. We set

$$
\begin{aligned}
h_r(x_1,\dots,x_r) &= \prod_{i=1}^r\frac{x_i-1}{x_i}\Big(1-\frac{1}{2}\prod_{i=1}^r\frac{x_i-2}{2(x_i-1)}\Big)\\
&= \prod_{i=1}^r\Big(1-\frac{1}{x_i}\Big)-\frac{1}{2}\prod_{i=1}^r\Big(\frac{1}{2}-\frac{1}{x_i}\Big).
\end{aligned}
$$

We underline the dependency on x_1 by noting that

$$h_r(x_1,\dots,x_r)=\prod_{i=2}^{r}\Big(1-\frac{1}{x_i}\Big)-\frac{1}{4}\prod_{i=2}^{r}\Big(\frac{1}{2}-\frac{1}{x_i}\Big)$$
$$-\frac{1}{x_1}\Bigg(\prod_{i=2}^{r}\Big(1-\frac{1}{x_i}\Big)-\frac{1}{2}\prod_{i=2}^{r}\Big(\frac{1}{2}-\frac{1}{x_i}\Big)\Bigg).$$

We observe that $h_1(x_1)$ is not constant, being equal to $3/4-x_1/2$. By symmetry in $x_1,\dots,x_r$, we see that $h_r(x_1,\dots,x_r)\leqslant h_r(y_1,\dots,y_r)$, where the equality holds if and only if $x_i=y_i$ for $i=1,\dots,r$, or, equivalently, $r_{\ell_i}=r$ for $i=1,\dots,r$. In conclusion

$$b_\Gamma c_\Gamma\leqslant h_r(y_1,\dots,y_r)\prod_{i=1}^{r}\frac{y_i}{y_1-1}=1-\frac{1}{2}\prod_{i=1}^{r}\frac{y_i-2}{2(y_i-1)}$$
$$=1-\frac{1}{2^{r+1}}\prod_{i=1}^{r}\Big(1-\frac{1}{\ell_i^r(\ell_i-1)-1}\Big).$$

Moreover, the equality holds if and only if (9) holds, and $r_\ell=r$ for every $\ell>2$ (and $r_2=r+1$). Finally,

$$1-\frac{1}{2^{r+1}}\prod_{i=1}^{r}\Big(1-\frac{1}{\ell_i^r(\ell_i-1)-1}\Big)\leqslant 1-\frac{1}{2^{r+1}}\prod_{i=1}^{r}\Big(1-\frac{1}{p_i^r(p_i-1)-1}\Big),$$

where equality holds if and only $\ell_i=p_i$, for $i=1,\dots,r$. This concludes the proof of the characterization of optimal torsion groups in Theorem 2. Obviously, $\big\langle -1,p_i : i=1,\dots,r\big\rangle$ is the simplest optimal torsion group.

6. Optimal positive groups

We follow the same arguments as in the case of free groups. However, we now select only primes in $\mathrm{Supp}(\widetilde{\Gamma})$ which are congruent to 1 (mod 4). By Lemma 4, using induction, there exist u primes $\ell_1,\dots,\ell_u\in\mathrm{Supp}(\widetilde{\Gamma})$, for some $u\in\{0,\dots,t\}$, and a subgroup $\widetilde{\Gamma}_0$ of $\widetilde{\Gamma}$ with 2^{t-u} elements, such that: $\ell_i\equiv 1\pmod 4$, for $i=1,\dots,u$; every $\ell\in\mathrm{Supp}(\widetilde{\Gamma}_0)$ satisfies $\ell\equiv 3\pmod 4$; and

$$S(\widetilde{\Gamma},\mu_+,f)\geqslant\prod_{i=1}^{u}\big(1-f(\ell_i)\big)S(\widetilde{\Gamma}_0,\mu_+,f).$$

The equality holds if and only if $\ell_1\mathbb{Q}^{*2},\dots,\ell_u\mathbb{Q}^{*2}\in\widetilde{\Gamma}$. If $m\mathbb{Q}^{*2}\in\widetilde{\Gamma}_0$, then $m>0$, $m\equiv 1\pmod 4$, and $\ell\equiv 3\pmod 4$ for all ℓ dividing m. Therefore m is the product of an even number of primes, whence $\mu(m)=1$. It follows that

$$S(\widetilde{\Gamma}_0,\mu_+,f)\geqslant 1,$$

and the equality holds if and only if $\widetilde{\Gamma}_0 = \{\mathbb{Q}^{*2}\}$, or, equivalently, $u = t$. Therefore

$$S(\widetilde{\Gamma}, \mu_+, f) \geqslant \prod_{i=1}^{u} \big(1 - f(\ell_i)\big),$$

and the equality holds if and only if $u = t$ and $\widetilde{\Gamma} = \big\langle \ell_1\mathbb{Q}^{*2}, \dots, \ell_t\mathbb{Q}^{*2} \big\rangle$. Hence there always exist r (instead of u) primes $\ell_1, \dots, \ell_r$ such that $\ell_i \equiv 1 \pmod 4$ for $i = 1, \dots, r$, and

$$S(\widetilde{\Gamma}, \mu_+, f) \geqslant \prod_{i=1}^{r} \big(1 - f(\ell_i)\big),$$

where the equality holds if and only if $\widetilde{\Gamma} = \big\langle \ell_1\mathbb{Q}^{*2}, \dots, \ell_r\mathbb{Q}^{*2} \big\rangle$.

The proof continues exactly as in Section 4, the only difference being that in the last inequality we have to consider just the primes $q \equiv 1 \pmod 4$. We add that $\big\langle q_i : i = 1, \dots, r \big\rangle$ is the simplest optimal positive group.

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