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A REMARK ON THE CONDITIONAL ESTIMATE FOR THE SUM OF A PRIME AND A SQUARE

Yuta Suzuki

Abstract: Hardy and Littlewood conjectured that every sufficiently large integer is either a square or the sum of a prime and a square. Let $E(X)$ be the number of positive integers up to $X \geqslant 4$ for which this property does not hold. We prove

$$E(X) \ll X^{1/2}(\log X)^A(\log\log X)^4$$

with $A = 3/2$ under the Generalized Riemann Hypothesis. This is a small improvement on the previous remarks of Mikawa (1993) and Perelli-Zaccagnini (1995) which claim $A = 4, 3$ respectively.

Keywords: Hardy-Littlewood conjecture, circle method.

1. Introduction

In 1923, Hardy and Littlewood [2, Conjecture H] conjectured that every sufficiently large integer is either a square or the sum of a prime and a square. For a given real positive number $X \geqslant 4$, let $E(X)$ be the number of positive integers up to X which are neither a square nor the sum of a prime and a square. In the present note, we consider the conditional estimate of $E(X)$ under the Generalized Riemann Hypothesis (GRH). We always assume GRH below.

In 1985, Vinogradov [11] remarked that his method can be used to prove

$$E(X) \ll X^{2/3+\varepsilon}$$

under GRH where $\varepsilon > 0$ is an arbitrary positive constant and the implicit constant depends only on ε. However he did not publish the details for this result, and from today's point of view, his assertion is rather weak for the conditional estimates assuming GRH. The first detailed proof was published by Mikawa[6, Proposition]

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in 1993, and he obtained[1]

$$E(X) \ll X^{1/2}(\log X)^4(\log\log X)^4 \tag{1.1}$$

by the circle method. If we assume GRH, then the factor $X^{1/2}$ of this estimate looks like the best-possible one which can be attained by our current technology. However, as for the factor $(\log X)^4$, we can hope some improvements beyond Mikawa's result. Actually, in 1995, Perelli and Zaccagnini [8, p.191] asserted that one can obtain

$$E(X) \ll X^{1/2}(\log X)^{3+\varepsilon} \tag{1.2}$$

by refining Mikawa's calculations. However Perelli and Zaccagnini did not give the detailed calculations for this assertion.

In this note, we improve these conditional estimates to the following:

Theorem 1. *Assume GRH. Then we have*

$$E(X) \ll X^{1/2}(\log X)^{3/2}(\log\log X)^4. \tag{1.3}$$

For a positive integer n, we let

$$R(n) = \sum_{k+m^2=n} \Lambda(k),$$

where $\Lambda(k)$ is the von Mangoldt function. We also define the singular series

$$\mathfrak{S}(n) = \begin{cases} \displaystyle\prod_{p>2}\left(1 - \frac{(n/p)}{p-1}\right) & \text{(when n is not a square)}, \\ 0 & \text{(when n is a square)}, \end{cases}$$

where (n/p) is the Legendre symbol. Then the main part of this note is dedicated to the proof of the following estimate.

Theorem 2. *Assume GRH. For $X \geqslant 4$, we have*

$$\sum_{n \leqslant X} \left| R(n) - \mathfrak{S}(n)\sqrt{n}(1 + O(n^{-\eta}))\right|^2 \ll (X \log X)^{3/2}, \tag{1.4}$$

where η is an absolute positive constant.

[1] In his paper, Mikawa only claimed that

$$E(X) \ll X^{1/2}(\log X)^5,$$

but what he essentially proved is (1.1).

We prove this mean square estimate from Section 4 to Section 8, and deduce Theorem 1 from Theorem 2 in Section 9.

We generally follow Mikawa's argument. However there are mainly three points to refine his argument.

The first point is to use the technique of Languasco and Perelli [3]. According to them, we shall use power series as the generating functions rather than trigonometric polynomials which Mikawa used. It enables us to use the explicit formula directly, and to reduce the errors arising from the approximation of the generating function of prime numbers. By this method, we can obtain the result (1.2) which Perelli and Zaccagnini asserted. So it seems that they obtained (1.2) in this way.

The second point is to use the classical transformation formula of Jacobi. Once we decide to follow the technique of Languasco and Perelli, it is natural to use the power series

$$W(\alpha) = \sum_{n=1}^{\infty} e^{-n^2/X} e(n^2\alpha)$$

as the generating function of squares, where X is a positive real number, α is a real number, and $e(\alpha) := \exp(2\pi i\alpha)$. However this series is just a simple variant of Jacobi's ϑ-function, so we can use Jacobi's formula instead of the Weyl estimate or the truncated Jacobi formula [10, Theorem 4.1]. This enables us to save some more log powers.

The third point is a careful treatment on the extension of major arcs. Mikawa estimated the errors arising from this extension by using the large sieve. However in our case, where we are asking about log powers, his estimate is insufficient. So we shall devide these extended arcs into two parts, and we use the Bessel inequality besides the large sieve.

We also consider carefully the decay of the generating function of squares on the major arcs.

2. Notation

Throughout the letters α, η denote real numbers, X, P, Q denote large positive real numbers, m, n, k denote integers, p denotes a prime number, and z, w denote complex numbers. For any real number α, $e(\alpha) = e^{2\pi i\alpha}$. The arithmetic function $\varphi(n)$ denotes the Euler totient function, $\Lambda(n)$ denotes the von Mangoldt function, $\mu(n)$ denotes the Möbius function, $\tau(n)$ denotes the number of divisors of n, and (n/p) denotes the Legendre symbol. The letters a, q denote positive integers satisfying $(a, q) = 1$ and the expressions

$$\sideset{}{^*}\sum_{a \pmod q}, \qquad \sideset{}{^*}\coprod_{a \pmod q}$$

denote a sum and a disjoint sum over all reduced residues $a \pmod q$ respectively.

3. Preliminary lemmas

Following Languasco and Perelli [3], we use the following power series for the generating functions of prime numbers and squares:

$$S(\alpha)=\sum_{n=1}^{\infty}\Lambda(n)e^{-n/X}e(n\alpha),\qquad W(\alpha)=\sum_{n=1}^{\infty}e^{-n^2/X}e(n^2\alpha).$$

In this section, we summarize some lemmas on these generating functions.

For $S(\alpha)$, we use the following variant of the estimate of Languasco and Perelli. For the proof, see [3, 4]. Let $z=1/X-2\pi i\alpha$.

Lemma 1. *Assume GRH. For $1\leqslant q\leqslant X$ and $0\leqslant\xi\leqslant 1/2$, we have*

$$\sideset{}{^*}\sum_{a \pmod q}\int_{-\xi}^{\xi}\left|S\left(\frac{a}{q}+\alpha\right)-\frac{\mu(q)}{\varphi(q)}\frac{1}{z}\right|^2 d\alpha \ll q\xi X(\log X)^2.$$

For $W(\alpha)$, we use Jacobi's transformation formula in the following form:

Lemma 2. *For any real number α and any complex number z with $\Re z>0$, we have*

$$\sum_{n=-\infty}^{\infty}\exp\left(-\pi(n+\alpha)^2 z\right)=\frac{1}{\sqrt{z}}\sum_{n=-\infty}^{\infty}\exp\left(-\pi n^2/z-2\pi i n\alpha\right),$$

where the branch of $\sqrt{z}$ is chosen as its value at 1 equals 1.

This lemma is classical. For the proof, see [7, Theorem 10.1]. Before applying Jacobi's formula to $W(\alpha)$, we need to estimate the generalized Gaussian sum

$$G(a,n;q)=\sum_{k=1}^{q}e\left(\frac{ak^2+nk}{q}\right),\qquad G(a,q)=G(a,0;q).$$

The following estimate can be deduced immediately by Weyl differencing.

Lemma 3. *If $(a,q)=1$, then we have $G(a,n;q)\ll q^{1/2}$.*

We can now prove our approximation of $W(\alpha)$. The following estimate is also classical. For example, see [9, Lecture 33]. However, we include its proof for completeness.

Lemma 4. *If $(a,q)=1$ and $|\alpha|\leqslant 1/2$, then we have*

$$W\left(\frac{a}{q}+\alpha\right)=\frac{\sqrt{\pi}}{2}\frac{G(a,q)}{q}\frac{1}{\sqrt{z}}+O\left(q^{1/2}+q^{1/2}X^{1/2}|\alpha|^{1/2}\right),$$

where the branch of $\sqrt{z}$ is chosen as its value at 1 equals 1.

Proof. We first consider a kind of Jacobi's theta function

$$\Theta(\alpha)=\sum_{n=-\infty}^{\infty}e^{-n^2/X}e(n^2\alpha)$$

instead of $W(\alpha)$. Then the original $W(\alpha)$ can be written as

$$W(\alpha)=\frac{1}{2}\Theta(\alpha)+O(1).$$

Let $w=z/\pi$. Then we have

$$\Theta\left(\frac{a}{q}+\alpha\right)=\sum_{n=-\infty}^{\infty}e^{-\pi n^2 w}e\left(\frac{an^2}{q}\right).$$

We now divide the above series according to the residue $n \pmod{q}$. Then

$$\Theta\left(\frac{a}{q}+\alpha\right)=\sum_{k=1}^{q}e\left(\frac{ak^2}{q}\right)\sum_{m=-\infty}^{\infty}\exp\left(-\pi(m+k/q)^2q^2w\right).$$

Applying Lemma 2 to the inner sum, we find

$$\begin{aligned}\Theta\left(\frac{a}{q}+\alpha\right)&=\sum_{k=1}^{q}e\left(\frac{ak^2}{q}\right)\frac{1}{q\sqrt{w}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi n^2}{q^2w}-2\pi i\frac{nk}{q}\right)\\&=\frac{1}{q\sqrt{w}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi n^2}{q^2w}\right)\sum_{k=1}^{q}e\left(\frac{ak^2+nk}{q}\right).\end{aligned}$$

We now pick up the term with $n=0$ as the main term. Then

$$\Theta\left(\frac{a}{q}+\alpha\right)=\frac{1}{q\sqrt{w}}\left\{G(a,q)+R(a,q;\alpha)\right\}, \tag{3.1}$$

where

$$R(a,q;\alpha)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}G(a,n;q)\exp\left(-\frac{\pi n^2}{q^2w}\right).$$

By Lemma 3, we can estimate the error term $R(a,q;\alpha)$ as

$$R(a,q;\alpha)\ll\sqrt{q}\sum_{n=1}^{\infty}\exp\left(-\frac{\pi n^2\delta}{q^2}\right), \tag{3.2}$$

where δ is given by

$$\delta=\Re\left(\frac{1}{w}\right)\asymp\frac{1}{X|w|^2}.$$

Then

$$R(a,q;\alpha) \ll \sqrt{q}\int_0^\infty \exp\left(-\frac{\pi t^2\delta}{q^2}\right) dt \ll q^{3/2}X^{1/2}|w|.$$

Substituting this estimate into (3.1), we get

$$\Theta\left(\frac{a}{q}+\alpha\right) = \frac{G(a,q)}{q\sqrt{w}} + O\left(q^{1/2}X^{1/2}|w|^{1/2}\right).$$

Returning to the original notation $W(\alpha)$ and $z=\pi w$, we have

$$W\left(\frac{a}{q}+\alpha\right) = \frac{\sqrt{\pi}}{2}\frac{G(a,q)}{q}\frac{1}{\sqrt{z}} + O\left(1+q^{1/2}X^{1/2}|z|^{1/2}\right).$$

Since $|z|^{1/2} \ll X^{-1/2}+|\alpha|^{1/2}$, we obtain the lemma. ■

4. The Farey dissection

We take $Q=(X\log X)^{1/2}$ as the order of the dissection, and let I be the unit interval $I=[1/Q,1+1/Q]$. For a pair of positive integers a,q such that $a \leqslant q$ and $(a,q)=1$, we shall denote by $\mathfrak{M}_{a,q}$ the Farey arc around a/q which is defined by

$$\mathfrak{M}_{a,q} = \left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right].$$

Let $P=X/Q$. Then the Farey arcs $\mathfrak{M}_{a,q}$ with $q \leqslant P$ are pairwise disjoint for sufficiently large X. So let us define the major arcs:

$$\mathfrak{M} = \coprod_{q\leqslant P}\ \sideset{}{^*}\coprod_{a \pmod q} \mathfrak{M}_{a,q}$$

and define the minor arcs: $\mathfrak{m} = I \setminus \mathfrak{M}$.

We introduce the following approximants for our generating functions

$$T(\alpha) := \begin{cases} \dfrac{\mu(q)}{\varphi(q)}\dfrac{1}{(z+2\pi i a/q)} & (\text{when } \alpha \in \mathfrak{M}_{a,q},\ q \leqslant P), \\ 0 & (\text{when } \alpha \in \mathfrak{m}), \end{cases} \tag{4.1}$$

$$U(\alpha) := \begin{cases} \dfrac{\sqrt{\pi}}{2}\dfrac{G(a,q)}{q}\dfrac{1}{(z+2\pi i a/q)^{1/2}} & (\text{when } \alpha \in \mathfrak{M}_{a,q},\ q \leqslant P), \\ 0 & (\text{when } \alpha \in \mathfrak{m}). \end{cases} \tag{4.2}$$

Consider the Fourier coefficients $\widehat{TU}(n)$ of $T(\alpha)U(\alpha)$ which is defined by

$$e^{-n/X}\widehat{TU}(n) = \int_0^1 T(\alpha)U(\alpha)e(-n\alpha)\,d\alpha. \tag{4.3}$$

Then we get by the Parseval identity that

$$\sum_{n=1}^{\infty} e^{-2n/X} \left| R(n) - \widehat{TU}(n) \right|^2 = \int_0^1 |S(\alpha)W(\alpha) - T(\alpha)U(\alpha)|^2 \, d\alpha. \tag{4.4}$$

We devide this last integral as

$$\int_0^1 |S(\alpha)W(\alpha) - T(\alpha)U(\alpha)|^2 \, d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}}, \tag{4.5}$$

and then we shall estimate these integrals separately.

5. The minor arcs

On the minor arcs, the variable α can be approximated as

$$\left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{qQ}$$

by some Farey fraction a/q with $P < q \leqslant Q$. Since

$$|z| \gg \max\left(X^{-1}, |\alpha| \right), \tag{5.1}$$

Lemmas 3 and 4 give the estimate

$$W(\alpha)^2 \ll XP^{-1} + Q + XQ^{-1} \ll Q.$$

Hence by the Parseval identity, we find that

$$\begin{aligned} \int_{\mathfrak{m}} |S(\alpha)W(\alpha)|^2 \, d\alpha &\ll \sup_{\alpha \in \mathfrak{m}} |W(\alpha)|^2 \int_0^1 |S(\alpha)|^2 \, d\alpha \\ &\ll Q \sum_{n=1}^{\infty} \Lambda(n)^2 e^{-2n/X}. \end{aligned}$$

Using the prime number theorem, we have

$$\int_{\mathfrak{m}} \ll QX \log X = (X \log X)^{3/2}. \tag{5.2}$$

6. The major arcs

We divide the integrand as

$$\begin{aligned} S(\alpha)W(\alpha) - T(\alpha)U(\alpha) &= (S(\alpha)W(\alpha) - T(\alpha)W(\alpha)) \\ &\quad + (T(\alpha)W(\alpha) - T(\alpha)U(\alpha)) \\ &= E_1 + E_2, \text{ say.} \end{aligned}$$

And we separate the integral over the major arcs as

$$\int_{\mathfrak{M}} \ll \int_{\mathfrak{M}} |E_1|^2 \, d\alpha + \int_{\mathfrak{M}} |E_2|^2 \, d\alpha = I_1 + I_2.$$

We first dissect these integrals into the integrals over small arcs:

$$\int_{\mathfrak{M}} = \sum_{q \leqslant P} \sideset{}{^*}\sum_{a \pmod{q}} \int_{\mathfrak{M}_{a,q}}.$$

We first treat I_1. By Lemmas 3 and 4, we have

$$W\left(\frac{a}{q} + \alpha\right)^2 \ll \frac{1}{q|z|} + q + qX|\alpha|.$$

By the estimate (5.1), we obtain the following estimates:

$$W\left(\frac{a}{q} + \alpha\right)^2 \ll \frac{X}{q}, \tag{6.1}$$

$$W\left(\frac{a}{q} + \alpha\right)^2 \ll \frac{1}{q|\alpha|}, \tag{6.2}$$

on each small arc $\mathfrak{M}_{a,q}$ with $q \leqslant P$. We have to estimate

$$I_1 = \sum_{q \leqslant P} \sideset{}{^*}\sum_{a \pmod{q}} \int_{-1/qQ}^{1/qQ} \left| S\left(\frac{a}{q} + \alpha\right) - \frac{\mu(q)}{\varphi(q)} \frac{1}{z} \right|^2 \left| W\left(\frac{a}{q} + \alpha\right) \right|^2 d\alpha.$$

We divide these integrals into two parts as

$$\int_{-1/qQ}^{1/qQ} = \int_{-1/X}^{1/X} + \int_{1/X < |\alpha| \leqslant 1/qQ} = J_1(a,q) + J_2(a,q).$$

For $J_1(a,q)$, we use the estimate (6.1) and Lemma 1. Then we have

$$\begin{aligned} \sum_{q \leqslant P} \sideset{}{^*}\sum_{a \pmod{q}} J_1(a,q) &\ll \sum_{q \leqslant P} \frac{X}{q} \sideset{}{^*}\sum_{a \pmod{q}} \int_{-1/X}^{1/X} \left| S\left(\frac{a}{q} + \alpha\right) - \frac{\mu(q)}{\varphi(q)} \frac{1}{z} \right|^2 d\alpha \\ &\ll PX(\log X)^2. \end{aligned} \tag{6.3}$$

For $J_2(a,q)$, we use (6.2) and Lemma 1. Then integration by parts gives

$$\begin{aligned} \sideset{}{^*}\sum_{a \pmod{q}} J_2(a,q) &\ll \sideset{}{^*}\sum_{a \pmod{q}} \int_{1/X < |\alpha| \leqslant 1/qQ} \left| S\left(\frac{a}{q} + \alpha\right) - \frac{\mu(q)}{\varphi(q)} \frac{1}{z} \right|^2 \frac{d\alpha}{q|\alpha|} \\ &\ll X(\log X)^2 \left(\log \frac{2X}{qQ}\right). \end{aligned}$$

Therefore

$$\sum_{q\leqslant P}\ \sideset{}{^*}\sum_{a \pmod q} J_2(a,q) \ll X(\log X)^2 \sum_{q\leqslant P} \log\frac{2X}{qQ} \ll PX(\log X)^2. \tag{6.4}$$

Here we used the following estimate:

$$\sum_{q\leqslant P} \log\frac{2X}{qQ} = \sum_{q\leqslant P} \log\frac{2P}{q} = \sum_{q\leqslant P} \int_q^{2P} \frac{du}{u} \ll P.$$

By (6.3) and (6.4), we have

$$I_1 \ll PX(\log X)^2 = (X\log X)^{3/2}. \tag{6.5}$$

We now estimate I_2. On each small arc $\mathfrak{M}_{a,q}$, Lemma 4 gives that

$$|W(\alpha) - U(\alpha)|^2 \ll P.$$

And since

$$\begin{aligned}
\int_{\mathfrak{M}_{a,q}} |T(\alpha)|^2\, d\alpha &\ll \frac{1}{\varphi(q)^2} \int_{-1/qQ}^{1/qQ} \frac{d\alpha}{|z|^2} \\
&\ll \frac{1}{\varphi(q)^2} \left(\int_{|\alpha|\leqslant 1/X} \frac{d\alpha}{|z|^2} + \int_{1/X<|\alpha|\leqslant 1/qQ} \frac{d\alpha}{|z|^2} \right) \\
&\ll \frac{1}{\varphi(q)^2} \left(X + \int_{1/X}^{1/qQ} \frac{d\alpha}{\alpha^2} \right) \ll \frac{X}{\varphi(q)^2},
\end{aligned}$$

we can estimate I_2 as

$$\begin{aligned}
I_2 &\ll P \sum_{q\leqslant P}\ \sideset{}{^*}\sum_{a \pmod q} \int_{\mathfrak{M}_{a,q}} |T(\alpha)|^2\, d\alpha \\
&\ll PX \sum_{q\leqslant P} \frac{1}{\varphi(q)} \ll PX\log X = X^{3/2}(\log X)^{1/2}.
\end{aligned} \tag{6.6}$$

The above estimates (6.5), (6.6) for I_1, I_2 give

$$\int_{\mathfrak{M}} \ll (X\log X)^{3/2}$$

for the integral over the major arcs. Now, we combine (5.2) with this inequality and recall (4.4) and (4.5) to get

$$\sum_{n=1}^{\infty} e^{-2n/X} \left| R(n) - \widehat{TU}(n) \right|^2 \ll (X\log X)^{3/2}.$$

In particular, we have

$$\sum_{n\leqslant X} \left| R(n) - \widehat{TU}(n) \right|^2 \ll (X\log X)^{3/2} \tag{6.7}$$

since $e^{-2n/X} \gg 1$ for $n \leqslant X$.

7. Extension of the major arcs

The next task is to extend each of the small arcs in (4.3) to the whole arc $[-1/2, 1/2]$. Since $T(\alpha)$ and $U(\alpha)$ are zero on the minor arcs, we can divide (4.3) into the integrals over the major arcs:

$$
\begin{aligned}
&= \frac{\sqrt{\pi}}{2} \sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) \int_{\mathfrak{M}_{a,q}} \frac{e(-n\alpha)}{(z+2\pi i a/q)^{3/2}}\, d\alpha \\
&= \frac{\sqrt{\pi}}{2} \sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \int_{-1/qQ}^{1/qQ} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha.
\end{aligned}
$$

Here we have to extend the range of the integral

$$
\int_{-1/qQ}^{1/qQ} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha
$$

to the whole arc $[-1/2, 1/2]$. The error arising from this extension is

$$
r_n = \frac{\sqrt{\pi}}{2} \sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \int_{1/qQ<|\alpha|\leqslant 1/2} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha,
$$

and we shall estimate its squared mean value

$$
\sum_{n \leqslant X} |r_n|^2.
$$

We divide each of the above integrals into two parts as

$$
\int_{1/qQ<|\alpha|\leqslant 1/2} = \int_{1/qQ<|\alpha|\leqslant 1/4qP} + \int_{1/4qP<|\alpha|\leqslant 1/2}.
$$

Then the former short extended arcs

$$
\mathfrak{M}^o_{a,q} := \left[\frac{a}{q} - \frac{1}{4qP}, \frac{a}{q} - \frac{1}{qQ}\right] \sqcup \left[\frac{a}{q} + \frac{1}{qQ}, \frac{a}{q} + \frac{1}{4qP}\right]
$$

for $q \leqslant P$ are pairwise disjoint since for two distinct Farey fraction $a/q, a'/q'$ with $q, q' \leqslant P$, we have

$$
\left|\frac{a}{q} - \frac{a'}{q'}\right| \geqslant \frac{1}{qq'} \geqslant \frac{1}{2qP} + \frac{1}{2q'P}.
$$

Thus we introduce

$$
\mathfrak{n} = \coprod_{q \leqslant P} \sideset{}{^*}\coprod_{a \pmod q} \mathfrak{M}^o_{a,q}.
$$

Then we find that

$$
r_n = r_n^{(1)} + r_n^{(2)},
$$

where

$$r_n^{(1)} = \int_{\mathfrak{n}} T_1(\alpha)U_1(\alpha)e(-n\alpha)d\alpha,$$

$$T_1(\alpha) := \frac{\mu(q)}{\varphi(q)} \frac{1}{(z+2\pi ia/q)} \qquad \left(\text{when } \alpha \in \mathfrak{M}^o_{a,q},\ q \leqslant P\right),$$

$$U_1(\alpha) := \frac{\sqrt{\pi}}{2} \frac{G(a,q)}{q} \frac{1}{(z+2\pi ia/q)^{1/2}} \qquad \left(\text{when } \alpha \in \mathfrak{M}^o_{a,q},\ q \leqslant P\right),$$

and

$$r_n^{(2)} = \frac{\sqrt{\pi}}{2} \sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \int_{1/4qP<|\alpha|\leqslant 1/2} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha.$$

We first treat $r_n^{(1)}$. For this integral, we use the Bessel inequality. Then

$$\sum_{n \leqslant X} \left| r_n^{(1)} \right|^2 \ll \int_{\mathfrak{n}} |T_1(\alpha)U_1(\alpha)|^2\, d\alpha.$$

Dissecting into small arcs, we have

$$\ll \sum_{q \leqslant P} \frac{\mu^2(q)}{q^2\varphi(q)^2} \sideset{}{^*}\sum_{a \pmod q} |G(a,q)|^2 \int_{1/qQ<|\alpha|\leqslant 1/4qP} \frac{1}{|z|^3} d\alpha$$

$$= \sum_{q \leqslant P} \frac{\mu^2(q)}{q\varphi(q)} \int_{1/qQ<|\alpha|\leqslant 1/4qP} \frac{1}{|\alpha|^3} d\alpha \ll Q^2 \sum_{q \leqslant P} \frac{q}{\varphi(q)} \ll Q^2 P = QX.$$

Thus we have

$$\sum_{n \leqslant X} \left| r_n^{(1)} \right|^2 \ll X^{3/2} (\log X)^{1/2}. \tag{7.1}$$

We now deal with $r_n^{(2)}$. Following Mikawa [6], we use the large sieve for these integrals. Our mean square error

$$\sum_{n \leqslant X} \left| r_n^{(2)} \right|^2$$

is a constant multiple of

$$\sum_{n \leqslant X} \left| \sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \int_{1/4qP<|\alpha|\leqslant 1/2} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha \right|^2.$$

In the absolute sign of each above term, we first change the order of summation and integration:

$$\sum_{q \leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \int_{1/4qP<|\alpha|\leqslant 1/2} \frac{e(-n\alpha)}{z^{3/2}}\, d\alpha$$

$$= \int_{1/X<|\alpha|\leqslant 1/2} \frac{e(-n\alpha)}{z^{3/2}} \sum_{1/4|\alpha|P<q\leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) d\alpha.$$

Then by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\left|\sum_{q\leqslant P}\frac{\mu(q)}{q\varphi(q)}\sideset{}{^*}\sum_{a \pmod q} G(a,q)e\left(-\frac{an}{q}\right)\int_{1/4qP<|\alpha|\leqslant 1/2}\frac{e(-n\alpha)}{z^{3/2}}\,d\alpha\right|^2\\ &\ll\left(\int_{1/X<|\alpha|\leqslant 1/2}\frac{1}{|z|}d\alpha\right)\\ &\quad\times\left(\int_{1/X<|\alpha|\leqslant 1/2}\frac{1}{|z|^2}\left|\sum_{1/4|\alpha|P<q\leqslant P}\frac{\mu(q)}{q\varphi(q)}\sideset{}{^*}\sum_{a \pmod q} G(a,q)e\left(-\frac{an}{q}\right)\right|^2 d\alpha\right).\end{aligned}$$

The first integral can be estimated as

$$\int_{1/X<|\alpha|\leqslant 1/2}\frac{1}{|z|}d\alpha\ll\int_{1/X<|\alpha|\leqslant 1/2}\frac{1}{|\alpha|}d\alpha\ll\log X.$$

And for the second integral, after taking the summation over n, we use the following estimate obtained via the large sieve:

$$\begin{aligned}&\sum_{n\leqslant X}\left|\sum_{1/4|\alpha|P<q\leqslant P}\frac{\mu(q)}{q\varphi(q)}\sideset{}{^*}\sum_{a \pmod q} G(a,q)e\left(-\frac{an}{q}\right)\right|^2\\ &\qquad\ll\left(X+P^2\right)\sum_{1/4|\alpha|P<q\leqslant P}\ \sideset{}{^*}\sum_{a \pmod q}\frac{\mu^2(q)\,|G(a,q)|^2}{q^2\varphi(q)^2}\\ &\qquad\ll X\sum_{1/|\alpha|P<q\leqslant P}\frac{\mu^2(q)}{q\varphi(q)}.\end{aligned}$$

Combining the above estimates, we get the estimate

$$\begin{aligned}&\sum_{n\leqslant X}\left|\sum_{q\leqslant P}\frac{\mu(q)}{q\varphi(q)}\sideset{}{^*}\sum_{a \pmod q} G(a,q)e\left(-\frac{an}{q}\right)\int_{1/4qP<|\alpha|\leqslant 1/2}\frac{e(-n\alpha)}{z^{3/2}}\,d\alpha\right|^2\\ &\qquad\ll X\log X\int_{1/X<|\alpha|\leqslant 1/2}\frac{1}{|\alpha|^2}\sum_{1/4|\alpha|P<q\leqslant P}\frac{\mu^2(q)}{q\varphi(q)}d\alpha\\ &\qquad\ll X\log X\sum_{q\leqslant P}\frac{\mu^2(q)}{q\varphi(q)}\int_{1/4qP<|\alpha|\leqslant 1/2}\frac{d\alpha}{|\alpha|^2}\\ &\qquad\ll PX\log X\sum_{q\leqslant P}\frac{\mu^2(q)}{\varphi(q)}\ll PX(\log X)^2,\end{aligned}$$

i.e.

$$\sum_{n\leqslant X}\left|r_n^{(2)}\right|^2\ll(X\log X)^{3/2}. \tag{7.2}$$

Combining (7.1) and (7.2), we have

$$\sum_{n\leqslant X} |r_n|^2 \ll \sum_{n\leqslant X} \left|r_n^{(1)}\right|^2 + \sum_{n\leqslant X} \left|r_n^{(2)}\right|^2 \ll (X\log X)^{3/2}. \tag{7.3}$$

8. Completion of the proof of Theorem 2

We next calculate explicitly the extended integral

$$V(n,P) = \left\{ \frac{\sqrt{\pi}}{2} \sum_{q\leqslant P} \frac{\mu(q)}{q\varphi(q)} \sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \right\} \int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^{3/2}} d\alpha.$$

We use the following integral formula

$$\int_{-1/2}^{1/2} \frac{e(-n\alpha)}{z^{3/2}} d\alpha = e^{-n/N} \frac{2\sqrt{n}}{\Gamma(1/2)} + O\left(\frac{1}{n}\right). \tag{8.1}$$

This is a simple corollary of the Hankel integral formula. For the proof, see [5, Lemma 4]. Since it holds that [1, Lemma 3, 4]

$$\sideset{}{^*}\sum_{a \pmod q} G(a,q) e\left(-\frac{an}{q}\right) \ll q$$

for any square-free number q, the contribution of the error term of (8.1) is

$$\ll \frac{1}{n} \sum_{q\leqslant P} \frac{1}{\varphi(q)} \ll \frac{\log X}{n}.$$

Thus $V(n,P)$ is calculated explicitly as

$$V(n,P) = e^{-n/N} \mathfrak{S}(n,P)\sqrt{n} + O\left(\frac{\log X}{n}\right) \tag{8.2}$$

where

$$\mathfrak{S}(n,P) := \sum_{q\leqslant P} \sideset{}{^*}\sum_{a \pmod q} \frac{\mu(q)}{q\varphi(q)} G(a,q) e\left(-\frac{an}{q}\right).$$

By (6.7) and (7.3), we get

$$\sum_{n\leqslant X} \left|R(n) - \mathfrak{S}(n,P)\sqrt{n}\right|^2 \ll (X\log X)^{3/2}.$$

Assuming GRH, we can show [1, Lemma 21] that there exists a positive absolute constant η such that

$$\mathfrak{S}(n,P) = \mathfrak{S}(n)(1 + O(n^{-\eta}))$$

for $n \neq m^2$. Therefore Theorem 2 follows.

9. Proof of Theorem 1

First we shall modify Theorem 2 to

$$\sum_{n\leqslant X}\left|\sum_{p+m^2=n}(\log p)-\mathfrak{S}(n)\sqrt{n}(1+O(n^{-\eta}))\right|^2 \ll (X\log X)^{3/2}. \tag{9.1}$$

In order to justify this modification, we proceed as follows. We remove the restriction that p is a prime. Then we have to estimate:

$$\sum_{n\leqslant X}\Bigg(\sum_{\substack{p^k+m^2=n\\ k\geqslant 2}}(\log p)\Bigg)^2$$
$$\ll (\log X)^2\left(\sum_{n\leqslant X}\Bigg(\sum_{m_1^2+m_2^2=n}1\Bigg)^2+\sum_{n\leqslant X}\Bigg(\sum_{\substack{m_1^k+m_2^2=n\\ 3\leqslant k\ll \log X}}1\Bigg)^2\right).$$

Let us denote these sums by

$$=(\log X)^2\left(\sum\nolimits_1+\sum\nolimits_2\right).$$

For $\sum_1$, we use Jacobi's two-square theorem to get

$$\sum\nolimits_1 \ll \sum_{n\leqslant X}\tau(n)^2 \ll X(\log X)^3.$$

For $\sum_2$, we first notice that for $k\geqslant 3$ and $n\leqslant X$

$$\sum_{m_1^k+m_2^2=n}1 \ll \sum_{m^k\leqslant n}1 \ll \sum_{m^3\leqslant n}1 \ll X^{1/3}.$$

Hence we have

$$\sum\nolimits_2 \ll X^{1/3}\sum_{n\leqslant X}\sum_{\substack{m_1^k+m_2^2=n\\ 3\leqslant k\ll\log X}}1 \ll X^{1/3}\sum_{\substack{m^k\leqslant X\\ 3\leqslant k\ll\log X}}\sum_{m^2\leqslant X}1 \ll X^{7/6}(\log X).$$

These estimates gives

$$\sum_{n\leqslant X}\Bigg(\sum_{\substack{p^k+m^2=n\\ k\geqslant 2}}(\log p)\Bigg)^2 \ll X^{7/6}(\log X)^3 \ll (X\log X)^{3/2}.$$

This justifies (9.1).

Now assuming GRH, we know that

$$\mathfrak{S}(n) \gg (\log\log n)^{-2}$$

holds for $n \neq m^2$ and $n \geqslant 4$ [6, p. 304]. Hence (9.1) gives

$$X(\log\log X)^{-4} \sum_{\substack{X/2<n\leqslant X \\ n\neq m^2,\, p+m^2}} 1 \ll (X\log X)^{3/2}$$

or

$$E(X) - E(X/2) \ll X^{1/2}(\log X)^{3/2}(\log\log X)^4.$$

Therefore we finally obtain that

$$\begin{aligned} E(X) &\ll X^{1/2} + \sum_{k=1}^{O(\log X)} \big(E(X/2^{k-1}) - E(X/2^k)\big) \\ &\ll X^{1/2}(\log X)^{3/2}(\log\log X)^4. \end{aligned}$$

This completes the proof of Theorem 1.

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Address: Yuta Suzuki: Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan.

E-mail: m14021y@math.nagoya-u.ac.jp

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