Functiones et Approximatio
60.1 (2019), 7–20
doi: 10.7169/facm/1610

ON VAN DER CORPUT'S METHOD FOR EXPONENTIAL SUMS

Hong-Quan Liu

Abstract: We give the best known error term of the B-process of van der Corput's method and we extend the class of exponential sums that can be estimated by using exponent pairs coming from the iteration of the A and B processes.

Keywords: analytic number theory, exponential sums, exponent pairs.

1. Introduction

Based on ideas of Weyl and others, van der Corput's method for estimating exponential sums was one of the distinguished advances in the theory of numbers that appeared in the 20th century. There are two themes in this paper. The first is to sharpen the error term of the B-process of the van der Corput method. We have

Theorem 1. *Let $f(x)$ be a real function such that $f^{(4)}(x)$ is a continuous function for $x \in [a,b]$ and let C_k $(1 \leqslant k \leqslant 4)$ be certain positive constants,*

$$C_1R^{-1} \leqslant |f''(x)| \leqslant C_2R^{-1}, \qquad |\beta_k(x)| \leqslant C_kU^{2-k} \quad \text{for } 3 \leqslant k \leqslant 4,$$

where $R > 0, U \geqslant 1$, and $\beta_k(x) = f^{(k)}(x)/f''(x)$. Then (here $e(\xi) = \exp(2\pi i\xi)$)

$$\sum_{a\leqslant m\leqslant b} e(f(m)) = \lambda \sum_{\alpha<v<\beta} |f''(x_v)|^{-1/2} e(f(x_v) - \nu x_v + 1/8) + E,$$

$$\begin{aligned} E = E_1 + E_2 &+ O(\log(2 + (b-a)R^{-1})) + O((b-a+R)U^{-1}) \\ &+ O\left(\min\left(R^{1/2}, \max\left(\frac{1}{\langle\alpha\rangle}, \frac{1}{\langle\beta\rangle}\right)\right)\right), \end{aligned}$$

where x_ν is defined by $f'(x_v) = v$, $\lambda = 1$ or $-i$ according as $f'' > 0$ or $f'' < 0$,

$$\begin{aligned} E_1 &= \lambda b_\alpha |f''(x_\alpha)|^{-1/2} e(f(x_\alpha) - \alpha x_\alpha + 1/8), \\ E_2 &= \lambda b_\beta |f''(x_\beta)|^{-1/2} e(f(x_\beta) - \beta x_\beta + 1/8), \end{aligned}$$

2010 Mathematics Subject Classification: primary: 11L07

$b_\alpha = 1/2$ *if* α *is an integer, otherwise* $b_\alpha = 0$, b_β *is defined similarly, and*

$$\langle \xi \rangle = \begin{cases} \beta - \alpha, & \text{if } \xi \text{ is an integer },\\ \|\xi\| = \min_{n\in Z} |n - \xi|, & \text{otherwise.} \end{cases}$$

In [6], Theorem 1, this result was derived having the additional condition $|3\beta_4(x) - 5\beta_3^2(x)| \gg U^{-2}$ (requiring also the continuous of the fifth-order derivative). By a comparison, Lemma 6 of [3], Lemma 3.6 of [2], and Theorem 10 of Chapter 3 of [7] all have the larger error term $O(\sqrt{R})$. Theorem 1.2 of [10] may be similar to our Theorem 1. However, its proof may be lengthy. Theorem 1 of [5] not only requires the additional conditions $1 \ll R \ll U$ but also its proof has gaps(otherwise his result is better than ours, for in case $f'(a)$ or $f'(b)$ is an integer, his error term does not contain $1/|\beta - \alpha|$); in fact on p.178 for the case $\|f'(a)\| = 0$ or $\|f'(b)\| = 0$, Karatsuba did not write down the details for treating the contribution of the right side of (9), which, taking into account the last two formulas of p.174 in showing Lemma 2, should give O-terms such as

$$O\left(\min\left(\sqrt{R}, \frac{1}{|f'(b) - f'(a)|}\right)\right).$$

Moreover when $\|f'(a)\| \neq 0$ and $\|f'(b)\| \neq 0$ for the contribution of $f'(a) - 1/2 \leqslant n \leqslant [f'(a)]$ to the right side of (9) of [5], obviously the integration by parts step was not correct, for we do not know how to estimate the expression (see p.177 of [5])

$$\int_a^b \frac{f''(x)}{(f'(x) - n)^2} e(f(x) - nx)dx,$$

as those O-terms given by the right side of our Theorem 1. Instead we should use Lemma 4.3 of [9] (which is obtained by using the second mean value theorem, but not simply by using the partial summation) to estimate

$$\int_a^b e(f(x) - nx)dx,$$

which then yields the error term

$$O\left(\min\left(\sqrt{R}, \frac{1}{\|f'(a)\|}\right)\right).$$

Thus, the final error term would involve

$$O\left(\min\left(R^{1/2}, \max\left(\frac{1}{\langle\alpha\rangle}, \frac{1}{\langle\beta\rangle}\right)\right)\right),$$

which is the same as ours.

Our second theme is, roughly speaking, to show that for any given exponent pair (p, q) that comes from a suitable iteration of the A and B processes, apart from several special values of $\alpha > 1$, one can also use the exponent pair to estimate exponential sums of the form

$$S_f(N, M) = \sum_{N \leqslant n \leqslant M} e(f(n)), \qquad e(\xi) = \exp(2\pi i \xi),$$

where $1 \leqslant N < M \leqslant 2N$, $f(x)$ has continuous derivatives up to order J, and

$$f^{(j)}(x) = \lambda(\alpha)_j x^{\alpha - j}(1 + c_j(x)), \qquad \lambda \neq 0,\ \alpha > 1, \tag{1}$$

where $1 \leqslant j \leqslant J, (\alpha)_j = \alpha(\alpha-1)\cdots(\alpha-j+1), |c_j(x)| \leqslant C_j < 1/2$, and J and C_j depend on the exponent pair (p, q). Here we note that, by the definition using (p, q), one can only estimate $S_f(N, M)$ for which (1) holds with any $(0 \neq)\alpha < 1$. However, there are counterexamples such that (p, q) cannot be used to estimate $S_f(N, M)$ in the case that $\alpha > 1$. A counterexample is $f(x) = Cx^{3/2},\ C = 2(27)^{-1/2}$, and $M = 2N$. By the B-process we have

$$|S_f(N, M)| = \left| C_1 \sum_{u \leqslant r \leqslant v} r^{1/2} e(-r^3) + O(N^{1/4}) \right| \gg N^{3/4},$$

where $u \approx N^{1/2}$, $v \approx N^{1/2}$. If the exponent pair

$$(2/7, 4/7) = BA^2B(0, 1)$$

can be used, then

$$|S_f(N, 2N)| < (N^{1/2})^{2/7} N^{4/7} = N^{5/7} = o(N^{3/4}),$$

which is a contradiction (thus the theory of §2.3 of Ivic[4] has gaps). This reveals that in general we cannot use an exponent pair to estimate an exponential sum having as its exponent the function of (1) for an arbitrary $\alpha > 1$. Here we note that originally the accurate theory of exponent pairs can only be founded for the situation $(0 \neq)\alpha < 1$ (see [1, 8]). Can we find the condition of α such that a given exponent pair can be used? We shall resolve this problem in our Theorem 2. But the statement of Theorem 2 is lengthy. Thus, it will be given in §3.

To emphasize the importance of our results, we note that they are not merely of the theoretical significance. For example, the version of Theorem 1 has been used several times in our own works on estimating exponential sums of three or four variables.

2. Proof of Theorem 1

Assuming first that $f''(x) < 0$ for $x \in [a, b]$, then $\lambda = -i$. We use notations from the proof of Theorem 1 of [6]. It suffices to show that, for each $v \in [\alpha, \beta]$.

$$\int_0^d e(u) F(u) du = O(RU^{-1}). \tag{2}$$

Similarly to the formulas between (6) and (7) of [6], for any

$$t \in (0, c], \qquad c = \min(b - n, \varepsilon U),$$

by making the Taylor expansions we have ($n = n_v$ is defined by $f'(n_v) = v$)

$$\begin{aligned} -u = A(t) &= \frac{1}{2} f''(n) t^2 (1 + \tfrac{1}{2}\beta_3 t + O(t^2 U^{-2})), \\ A'(t) &= f'(n+t) - f'(n) = f''(n) t (1 + \tfrac{1}{2}\beta_3 t + O(t^2 U^{-2})), \\ A''(t) &= f''(n+t) = f''(n)(1 + \beta_3 t + O(t^2 U^{-2})). \end{aligned}$$

Thus, for $0 < u = -A(t) \leqslant d = -A(c)$ we get with $Y = (-\frac{f''(n)}{2})^{-1/2}$

$$\begin{aligned} F(u) = -\frac{1}{A'(t)} - \frac{1}{2} Y u^{-1/2} &= -\frac{1}{f''(n)t}(1 + O(tU^{-1})) \\ -\frac{1}{2}\left(-\frac{1}{2} f''(n)\right)^{-1/2} &\left(-\frac{1}{2} f''(n) t^2\right)^{-1/2} (1 + O(tU^{-1})) = O(RU^{-1}). \end{aligned} \tag{3}$$

From

$$\begin{aligned} -A^*(t)(A'(t))^{-3} &= -(f''(n))^{-2} t^{-3} (1 + \beta_3 t + O(t^2 U^{-2})) \\ &\quad \times \left(1 + \frac{1}{2}\beta_3 t + O(t^2 U^{-2})\right)^{-3} \\ &= -(f''(n))^{-2} t^{-3} \left(1 - \frac{1}{2}\beta_3 t + O(t^2 U^{-2})\right), \end{aligned}$$

we have

$$F'(u) = -\frac{A''(t)}{(A'(t))^3} + \frac{1}{4} Y u^{-3/2} = O(R^2 t^{-1} U^{-2}) = O(R^{3/2} u^{-1/2} U^{-2}). \tag{4}$$

Here we note that ε (depending on the constants C_k) is sufficiently small, so that the above arguments are valid(such as the expansion for $(A'(t))^{-3}$). Now let $\delta = \min(1, d)$. From (3) we get

$$\int_0^\delta e(u) F(u) du = O\left(\int_0^\delta |F(u)| du\right) = O(RU^{-1}). \tag{5}$$

Integrating by parts we also get from (3) and (4) that

$$\begin{aligned} \int_\delta^d e(u) F(u) du &= \frac{1}{2\pi i}(e(d)F(d) - e(\delta)F(\delta) - \int_\delta^d F'(u) e(u) du) \\ &= O(RU^{-1}) + O\left(\int_\delta^d R^{3/2} u^{-1/2} U^{-2} du\right) \\ &= O(RU^{-1}) + O(R^{3/2} d^{1/2} U^{-2}) = O(RU^{-1}). \end{aligned} \tag{6}$$

Thus (2) follows from (5) and (6). The case for $f''(x) > 0$ can be treated similarly. The proof is finished.

3. The statement of Theorem 2 and the resolution of the counterexample

Theorem 2. *Let* $(n(0),\ldots,n(k))$ *be a set of non-negative integers satisfying* $n(0)\geqslant 0, n(k)\geqslant 0$ *and* $n(i)\geqslant 1$ *if* $0\leqslant i\leqslant k-1$ *and* $k\geqslant 1$. *Let* $f(x)$ *be a real function defined on* $[N,M]$ *whose derivative functions of orders* 1 *to* $2+n(0)+\ldots+n(k)$ *are continuous on* $[N,M]$, *and satisfy*

$$f^{(j)}(x)=\lambda(\alpha)_i x^{\alpha-j}(1+O(\Delta)),\qquad 1\leqslant j\leqslant 2+n(0)+\ldots+n(k), \tag{7}$$

where $1\leqslant N\leqslant M\leqslant 2N$, $\lambda\neq 0,\alpha>1,(\alpha)_j=\alpha(\alpha-1)\cdots(\alpha-j+1)$ *for* $j\geqslant 1$, *and for* $\Delta\to 0$ *for* $N\to\infty$. *Let the exponent pair* (p,q) *be defined by*

$$(p,q)=A^{n(k)}B\cdots A^{n(1)}BA^{n(0)}B(0,1), \tag{8}$$

Suppose that

$$\begin{aligned}
\alpha(k)&=\alpha\neq 1,2,\cdots,n(k)+1,\\
\alpha(k-1)&=\frac{\alpha(k)-n(k)}{\alpha(k)-n(k)-1}\neq 2,\cdots,n(k-1)+1,\\
&\ \vdots\\
\alpha(j)&=\frac{\alpha(j+1)-n(j+1)}{\alpha(j+1)-n(j+1)-1}\neq 2,\cdots,n(j)+1,\\
&\ \vdots\\
\alpha(0)&=\frac{\alpha(1)-n(1)}{\alpha(1)-n(1)-1}\neq 2,\cdots,n(0)+1.
\end{aligned} \tag{9}$$

Then, for $F=|\lambda|N^{\alpha}$ *we have that*

$$S_f(N,M)=\sum_{N\leqslant n\leqslant M}e(f(n))\ll (FN^{-1})^pN^q+NF^{-1}. \tag{10}$$

In particular, for $k=0$ *the conditions (7) and (9) for which the estimate (10) holds can be relaxed to*

$$|f^{(j)}(x)|\approx FN^{-j},\qquad 1\leqslant j\leqslant n(0)+2,\ x\in[N,M]. \tag{11}$$

Taking more care, the term $O(\Delta)$ *in the condition (7) can be replaced by any number* c_j *that satisfies* $|c_j|<1/2$ *where* $|c_j|$ *is sufficiently small such that the following proof steps of Theorem 1 can be carried out.*

Using Theorem 2, we find immediately that we can use the exponent pair

$$(2/7,4/7)=BA^2B(0,1)$$

to estimate the exponential sum of (10) with $f(x)$ satisfying (7) for all $\alpha>1,\alpha\neq 2,3/2$, which explains the counterexample of Ivic's book given in our §1.

4. Lemmas

We need several lemmas.

Lemma 1. *Let $f(x)$ be a real function on $[a, b]$, $1 \leqslant a < b \leqslant 2a$, such that $f''(x)$ is continuous on $[a, b]$ and*

$$0 < \lambda \leqslant |f'(x)| < \lambda_1, \qquad |f''(x)| \approx \lambda N^{-1},$$

where $\lambda_1 = O(\lambda), a \approx N$. Then

(i) *For $\lambda_1 < 1$ we have*

$$S_f(a, b) = \sum_{a \leqslant n \leqslant b} e(f(n)) = O(\lambda^{-1}).$$

(ii) *There always holds*

$$S_f(a, b) = O(\lambda^{1/2} N^{1/2}) + O(\lambda^{-1}).$$

Proof. It is easy to observe that $f''(x)$ keeps a constant sign on $[a, b]$. We can assume that $f''(x) < 0$ on $[a, b]$ (otherwise we consider $-f''(x)$). Thus $f'(x)$ is strictly decreasing on $[a, b]$ and we find that (i) follows from Lemmas 4.2 and 4.8 of [9]. By Theorem 5.9 of [9] we get

$$S_f(a, b) = O(\lambda^{1/2} N^{1/2}) + O(N^{1/2} \lambda^{-1/2}). \tag{12}$$

If $\lambda_1 < 1$ then (ii) follows from (i), and if $\lambda_1 > 1$, then (ii) follows from (12), for we now have $N^{1/2} \lambda^{-1/2} < \lambda^{1/2} N^{1/2}$. ■

Lemma 2. *Let $f(x)$ be a real function such that $f^{(4)}(x)$ is continuous on the interval $[a, b]$ and*

$$|f''(x)| \approx R^{-1}, \qquad \beta_k(x) = f^{(k)}(x)/f''(x) = O(U^{2-k}),$$

where $R > 0, U \geqslant 1, 3 \leqslant k \leqslant 4, 1 \leqslant a < b \leqslant 2a$. Then

$$\sum_{a \leqslant n \leqslant b} ef(n) = \lambda \sum_{\alpha < v < \beta} |f''(x_v)|^{-1/2} e(f(x_v) - v x_v + 1/8)$$
$$+ O(R^{1/2}) + O(\log(2 + aR^{-1}) + (a + R)U^{-1}).$$

Proof. It follows from Theorem 1. ■

The next lemma prepares the inductive steps.

Lemma 3. *Let (p, σ) be a pair of numbers with $0 < p \leqslant 1/2 \leqslant \sigma < 1$ such that if a real function $f(x)$ has continuous derivatives up to order K on an interval $[a, b] \subseteq [N, 2N]$, and satisfies*

$$|f^{(r)}(x)| \approx \xi N^{1-r}, \qquad 1 \leqslant r \leqslant K,\ \xi > 0,$$

then

$$S_f(a,b) = O(\xi^p N^\sigma) + O(\xi^{-1}).$$

Then $(k,\lambda) = A(p,\sigma) = (\frac{p}{2(p+1)}, \frac{1}{2} + \frac{\sigma}{2(1+p)})$ *is a pair of numbers such that if* $g(x)$ *is a real function defined on an interval* $[c,d] \subseteq [M,2M]$ *having continuous derivatives up to order* $K+1$*, and satisfying*

$$|g^{(r)}(x)| \approx \eta M^{1-r}, \qquad 1 \leqslant r \leqslant K+1,\ \eta > 0,$$

then we have the estimate

$$S_g(c,d) = O(\eta^k M^\lambda) + O(\eta^{-1}). \tag{13}$$

Here $t \approx T$ *means that* $C_1 \leqslant t/T \leqslant C_2$ *for two suitable absolute constants* $C_1 > 0$, $C_2 > 0$ *(*M *and* N *may not be those stated in Theorem 2).*

Proof. If $M < 100$ then (13) is obvious. Let $M \geqslant 100$ and assume that $g(x)$ fulfill the asserted properties. By Weyl's inequality(see for instance Lemma 1 of [3]), for any $Q \in [10, M/5]$ we have

$$|S_g(c,d)|^2 = O\left(M^2Q^{-1} + MQ^{-1} \sum_{1\leqslant|w|\leqslant Q} \left|\sum_{n\in I} e(G(n,w))\right|\right),$$

where $I = [c,d]\bigcap[c-w,d-w], G(n,w) = g(n+w) - g(n)$. Suppose that $|I| > 10$. Here $|I|$ denotes the length of I. Then obviously the function $G(x,w)(x \in I)$ has continuous derivative function of order K. Additionally for $1 \leqslant r \leqslant K$,

$$G^{(r)}(x,w) = g^{(r)}(x+w) - g^{(r)}(x) = wg^{(r+1)}(x+\theta w), \qquad 0 < \theta < 1.$$

Consequently, by the hypothesis about $g(x)$ we have

$$|G^{(r)}(x,w)| \approx \eta|w|M^{-r}, \qquad 1 \leqslant r \leqslant K.$$

Thus, by the hypothesis for the pair of numbers (p,σ) we have

$$\left|\sum_{n\in I} e(G(n,w))\right| \ll (|w|\eta M^{-1})^p M^\sigma + |w|^{-1}\eta^{-1}M.$$

For $|I| \leqslant 10$, this estimate is trivial. Consequently

$$|S_g(c,d)|^2 \ll M^2Q^{-1} + M^{1-p+\sigma}\eta^p + M^2\eta^{-1}Q^{-1}\log(2M). \tag{14}$$

If $\eta \leqslant \log(2M)$ then from Lemma 1(ii) we get the estimate (13), for we have $\lambda > 1/2$. If $\eta > \log(2M)$, then the third term on the right of (14) can be neglected, for it is smaller than M^2Q^{-1}. Let

$$Q = \min\left(M/5, (M^{1+p-\sigma}\eta^{-p})^{1/(1+p)}\right). \tag{15}$$

It is easy to observe that (14) holds for all $Q \in [0, M/5]$. Thus we can put the choice (15) in (14), arriving at the estimate

$$|S_g(c,d)|^2 \ll \eta^{2k} M^{2\lambda} + M, \tag{16}$$

where

$$(k,\lambda) = A(p,\sigma) = \left(\frac{p}{2(p+1)}, \frac{1}{2} + \frac{\sigma}{2(p+1)}\right).$$

As $\lambda > 1/2$ we now have $\eta > \log(2M)$. It follows from (16) that

$$|S_g(c,d)|^2 \ll \eta^{2k} M^{2\lambda},$$

which show that the estimate (13) holds also in case $\eta > \log(2M)$. ■

5. Proof of Theorem 2

We use mathematical induction on k. When $k = 0$, if $n(0) = 0$ then (10) follows from Lemma 1(ii), for we have $B(0,1) = (1/2, 1/2)$. If $n(0) \geqslant 1$ by the condition (11), we can repeatedly apply Lemma 3, starting from Lemma 1(ii), and we know that the estimate (10) holds for

$$(p,q) = A^{n(0)} B(0,1) = A^{n(0)}(1/2, 1/2).$$

Assume that Theorem 2 is true for $k-1, k \geqslant 1$. Then we need to show that Theorem 2 holds also for k. Let

$$\begin{aligned} A^{n(k-1)} B \cdots B A^{n(0)} B(0,1) &= (\omega, \tau), \\ B(\omega,\tau) = (\tau - 1/2, \omega + 1/2) &= (a,b), n(k) = n. \end{aligned} \tag{17}$$

Obviously $(\omega,\tau) \neq (0,1)$, Thus $(a,b) \neq (1/2,1/2), 0 < a < 1/2 < b < 1$. Let

$$(p,q) = (p_k, q_k) = A^{n(k)}(a,b).$$

Our purpose is to deduce the estimate (10) from (7), (9) and the inductive hypothesis for $k-1$. If $N \leqslant F^p$, then we trivially get

$$S_f(N,M) = O(N) = O(F^p)$$

and (10) follows. Assume that $N > F^p$. It suffices for us to consider the difficult case with $n = n(k) \geqslant 1$. We can suppose that $N(\log N)^{-1} \geqslant 100$. Let Q_i $(1 \leqslant i \leqslant n)$ be parameters satisfying

$$10 \leqslant Q_i \leqslant N \max(\Delta, (\log N)^{-1}). \tag{18}$$

Utilizing repeatedly Weyl's inequality for $1 \leqslant r \leqslant n$ we obtain

$$|S_{r-1}|^2 \ll N^2 Q_r^{-1} + N Q_r^{-1} \sum_{1 \leqslant |q_r| \leqslant Q_r} |S_r|, \tag{19}$$

where

$$S_0 = S_f(N, M), S_r = \sum_{x \in I_r} e(f_r(x, q_1, \cdots, q_r)),$$

the functions $f_r (0 \leqslant r \leqslant n)$ are defined by $f_0(x) = f(x)$ and

$$f_r(x, q_1, \cdots, q_r) = f_{r-1}(x + q_r, q_1, \cdots, q_{r-1}) - f_{r-1}(x, q_1, \cdots, q_{r-1}),$$

and the intervals $I_r (1 \leqslant r \leqslant n)$ are defined by

$$I_1 = \{x | N \leqslant x, x + q_1 \leqslant M\}, I_{r+1} = \{x | x \in I_r, (x + q_{r+1}) \in I_r\}.$$

For a set of fixed integers $q_1, \cdots, q_n$, assume that $|I_r| > 10$. Let

$$f_r(x, q_1, \cdots, q_r) = f_r(x), \qquad g(x) = f_n(x).$$

Suppose that

$$1 \leqslant j \leqslant n(0) + \cdots + n(k-1) + 2. \tag{20}$$

For all real numbers $x_r \in I_r (1 \leqslant r \leqslant n)$, by the definition and the mean-value theorem of calculus we have

$$\begin{aligned} f_r^{(j)}(x_r) &= f_{r-1}^{(j)}(x_r + q_r) - f_{r-1}^{(j)}(x_r) = q_r f_{r-1}^{(j+1)}(y_r), \\ y_r x_r^{-1} &= 1 + O(Q_r N^{-1}), \qquad y_r \in I_{r-1} \end{aligned} \tag{21}$$

where $I_0 = [N, M]$. Thus, from (7), (18) and (21) we get

$$g^{(j)}(x) = f_n^{(j)}(x) = \lambda(\alpha)_{n+j} q_1 \cdots q_n x^{\alpha - n - j}(1 + O(\tilde{\Delta})), \tag{22}$$

where $\tilde{\Delta} = \Delta + (\log N)^{-1}$. If $n(0) = 1, k = 1$, then

$$A^{n(1)} BAB(0,1) = A^{n(1)+1} B(0,1).$$

Thus, (10) follows from the proof for the case $k = 0$ given at the beginning of our proof. Let $n(0) \geqslant 2$ or $n(0) = 1$ and $k \geqslant 2$. Then

$$n(0) + n(1) + \cdots = n(k-1) + 2 \geqslant 4. \tag{23}$$

From (9), we get $(\alpha)_{n+2} \neq 0$. Let N be large enough such that (21) and (22) yield

$$1/2 < g^{(j)}(x) \left(\lambda(\alpha)_{n+j} q_1 \cdots q_n x^{\alpha - n - j}\right)^{-1} < 3/2 \tag{24}$$

for $1 \leqslant j \leqslant 2$ and let $x \in I_n = [N_1, N_2]$. From (24) we know that $g'(x)$ keeps its sign on I_n. Suppose $g'(x) > 0$ on I_n (otherwise we consider $-g(x)$). By (20), (22)–(24), and Lemma 2, we get (here $g'(x_v) = v$)

$$\begin{aligned} S_n = \sum_{N_1 \leqslant x \leqslant N_2} e(g(x)) &= \lambda \sum_{V_1 \leqslant v \leqslant V_2} |g''(x_v)|^{-1/2} e(g(x_v) - v x_v + 1/8) \\ &\quad + O(R^{1/2}) + O(\log(2 + NR^{-1})) + O(RN^{-1}), \end{aligned} \tag{25}$$

where for $V_1 \leqslant v \leqslant V_2, x_v$ is determined by $g'(x_v) = v, R = N^2 F_n^{-1}, F_n = FN^{-n}|q_1 \cdots q_n|$,

$$V_1 = \min(g'(N_1), g'(N_2)), \qquad V_2 = \max(g'(N_1), g'(N_2)),$$

and $\lambda = 1$ or $-i$ according as $g''(x) > 0$ or $g''(x) < 0$ on I_n. Since $g''(x) > 0$ on I_n, from (24) we find that

$$\lambda(\alpha)_{n+1} q_1 \cdots q_n > 0,$$

and $g''(x) > 0$ or $g''(x) < 0$ on I_n follow from $\alpha > n+1$ or $\alpha < n+1$ respectively. We suppose without loss of generality that $\alpha > n+1$. Then $g''(x) > 0$ holds on I_n, and consequently $\lambda = 1, V_1 = g'(N_1), V_2 = g'(N_2)$. Let

$$K(v) = (g''(x_v))^{-1/2}, \qquad v_1 \leqslant v \leqslant v_2,$$

where v is a real variable. From $g'(x_v) = v$ we get

$$(x_v)' = (g''(x_v))^{-1}. \tag{26}$$

From (22) and (26) we get

$$K_v = O(R^{1/2}), \qquad K'(v) = O((g''(x_v))^{-5/2}|g^{(3)}(x_v)|) = O(R^{1/2}V^{-1}), \tag{27}$$

where $V = F_n N^{-1}$. We have $V_i \asymp V, i = 1, 2$. Thus, if $V \leqslant 100$, from $K(V) = O(R^{1/2})$ we get

$$\sum_{V_1 \leqslant v \leqslant V_2} |g''(x_v)|^{-1/2} e(g(x_v) - vx_v + 1/8) = O(R^{1/2}). \tag{28}$$

Assuming that $V > 100$. From (27) and a partial summation we obtain

$$\sum_{v_1 \leqslant v \leqslant v_2} |g''(x_v)|^{-1/2} e(g(x_v) - vx_v + 1/8) = O(R^{1/2}|S_G(u_1, u_2)|) + O(R^{1/2}), \tag{29}$$

where $G(v) = g(x_v) - vx_v$, $[u_1, u_2]$ is a suitable closed interval contained in $[V_1, V_2]$, and

$$S_G(u_1, u_2) = \sum_{u_1 \leqslant v \leqslant u_2} e(G(v)).$$

Let $F(x)$ be a real function defined on the interval $I_n = [N_1, N_2]$, which has continuous derivatives from orders 1 to $2 + n(0) + n(1) + \cdots + n(k-1)$, and $F''(x) > 0$. Thus $F'(x)$ is strictly increasing. For each real variable v with

$$F'(N_1) = V_1 \leqslant v \leqslant V_2 = F'(N_2),$$

let $x_v \in I_n$ be defined by

$$F'(x_v) = v \tag{30}$$

and let $W(v) = F(x_v) - vx_v$. From (30) it is easy to verify that

$$W'(v) = -x_v, \qquad W''(v) = -(x_v)' = -(F''(x_v))^{-1}. \tag{31}$$

Thus, by taking derivatives consecutively, we obtain

$$W^{(r+2)}(v) = (\Sigma_1 p_r(c_1, \cdots, c_r) F^{(c_1)}(x_v) \cdots F^{(c_r)}(x_v))(F''(x_v))^{-1-2r}, \tag{32}$$

where Σ_1 means a summation for lattice points $(c_1, \cdots, c_r)$ satisfying

$$r + 2 \geqslant c_1 \geqslant c_2 \geqslant \cdots \geqslant c_r \geqslant 2, c_1 + \cdots + c_r = 3r,$$

and $1 \leqslant r \leqslant n(0) + \cdots + n(k-1)$, $p_r(c_1, \cdots, c_r)$ is an integer. To explain (32) more carefully, we should use mathematical induction for r and both (30) and (31). Choosing in (32) respectively

$$F(x) = g(x), \qquad W(v) = G(v) = g(x_v) - vx_v$$

and

$$F(x) = \tilde{g}(x) = \Phi x^{\alpha - n}, \qquad \Phi = \lambda(\alpha)_n q_1 \cdots q_n, \qquad W(v) = \tilde{G}(v) = \tilde{g}(x_v) - vy_v,$$

for any real variable $v, v_1 \leqslant v \leqslant v_2$, we get respectively

$$G^{(r+2)}(v) = \left(\Sigma_1 p_r(c_1, \cdots, c_r) g^{(c_1)}(x_v) \cdots g^{(c_r)}(x_v)\right)(g''(x_v))^{-1-2r} \tag{33}$$

and

$$\tilde{G}^{(r+2)}(v) = \left(\Sigma_1 p_r(c_1, \cdots, c_r) \tilde{g}^{(c_1)}(x_v) \cdots \tilde{g}^{(c_r)}(x_v)\right)(\tilde{g}''(x_v))^{-1-2r}, \tag{34}$$

where $x_v \in [N_1, N_2]$ and $y_v \approx N$ are determined by $g'(x_v) = v$ and $\tilde{g}'(y_v) = v$, respectively (note that $g'(x_v) > 0$ implies that $(\alpha - n)\Phi > 0$). Thus,

$$y_v = (v(\Phi(\alpha - n))^{-1})^{1/(\alpha - n - 1)} \tag{35}$$

and from (22) we get

$$x_v = (v(\Phi(\alpha - n))^{-1})^{1/(\alpha - n - 1)}(1 + O(\tilde{\Delta})) = y_v(1 + O(\tilde{\Delta})). \tag{36}$$

Let $1 \leqslant c \leqslant 2 + n(0) + \cdots + n(k-1)$. From (20), (22), (35) and (36) we get

$$g^{(c)}(x_v) = \tilde{g}^{(c)}(y_v)(1 + O(\tilde{\Delta})). \tag{37}$$

From (9) and (35) we obtain

$$\tilde{G}(v) = \Phi y_v^{\alpha - n} - vy_v = \sigma v^{\alpha(k-1)}, \qquad \sigma = -(\alpha(k-1))^{-1}(\Phi(\alpha - n))^{1 - \alpha(k-1)}.$$

Thus for $1 \leqslant r \leqslant n(0) + \cdots + n(k-1)$ we get

$$\tilde{G}^{(r+2)}(v) = \sigma(\alpha(k-1))_{r+2} v^{\alpha(k-1) - r - 2}. \tag{38}$$

From (34) and (38) we get

$$\sigma(\alpha(k-1))_{r+2}v^{\alpha(k-1)-r-2} = (\Sigma_1 p_r(c_1,\cdots,c_r)\tilde{g}^{(c_1)}(y_v)\cdots\tilde{g}^{(c_r)}(y_v))(\tilde{g}''(y_v))^{-1-2r}. \tag{39}$$

For an integer $c, 2 \leqslant c \leqslant r+2$, we have

$$g^{(c)}(x_v) = O(F_nN^{-c}), \qquad \tilde{g}^{(c)}(y_v) = O(F_nN^{-c}), \qquad \tilde{g}''(y_v) \gg F_nN^{-2}.$$

Thus, for a lattice point $(c_1,\cdots,c_r)$ counted in Σ_1, from (37) we get

$$g^{(c_1)}(x_v)\cdots g^{(c_r)}(x_v) = \tilde{g}^{(c_1)}(x_v)\cdots\tilde{g}^{(c_r)}(x_v) + O(\tilde{\Delta}F_n^rN^{-3r}).$$

Consequently, from (33) and (39) we get

$$\begin{aligned} G^{(r+2)}(v) &= \frac{\left(\Sigma_1 p_r(c_1,\cdots,c_r)\tilde{g}^{(c_1)}(x_v)\cdots\tilde{g}^{(c_r)}(x_v) + O(\tilde{\Delta}F_n^rN^{-3r})\right)}{(\tilde{g}''(y_v))^{1+2r}}(1+O(\tilde{\Delta})) \\ &= \sigma(\alpha(k-1))_{r+2}v^{\alpha(k-1)-r-2}\left(1+O(\tilde{\Delta})\right). \end{aligned}$$

Thus we have

$$G^{(j)}(v) = \sigma(\alpha(k-1))_j v^{\alpha(k-1)-j}(1+O(\tilde{\Delta})) \tag{40}$$

for $3 \leqslant j \leqslant 2+n(0)+n(1)+\cdots+n(k-1)$. It is easy to verify (40) for $j=1,2$ in view of (22), (36), and

$$G'(v) = -x_v, \qquad G''(v) = -(g''(x_v))^{-1}.$$

From $N > F^p$ (cf. the arguments between (17) and (18)), we have

$$N^{p^{-1}-1} > FN^{-1} > F_nN^{-1} = V.$$

Thus, $\tilde{\Delta} \to 0$ as $V \to \infty$. Evidently, we can assume that $1 \leqslant u_1 < u_2 \leqslant 2u_1$. From

$$|\sigma v^{\alpha(k-1)}| \approx F_n,$$

we get by (40), (9), and the inductive hypothesis for $k-1$ the estimate

$$|S_G(u_1,u_2)| \ll (F_nV^{-1})^{\omega}V^{\tau} + VF_n^{-1}, \tag{41}$$

where (ω,τ) is given by (17). From (25), (28), (29), and (41), we get

$$S_n = O\left(N^b(F_nN^{-1})^a + NF_n^{-1/2} + L + NF_n^{-1}\right), \tag{42}$$

where $L = \log N$ and $(a,b) = (\tau-1/2, \omega+1/2)$. If $F_n \leqslant \delta N$, then δ is a sufficiently small positive constant such that

$$F_nN^{-1} \ll g'(x) \leqslant 1/2$$

for $x \in I_n = [N_1, N_2]$. Then by Lemma 1(i), we get

$$S_n = O(NF_n^{-1}).$$

If $F_n > \delta N$, then from (42) we obtain that

$$S_n = O(N^{b-a}F_n^a),$$

since $b > 1/2$. Thus, we always have

$$S_n = O(N^{b-a}F_n^a + NF_n^{-1}). \tag{43}$$

From (19) and (43) we have

$$|S_{n-1}|^2 \ll N^2Q_n^{-1} + N^{b-2a+1}F_{n-1}^aQ_n^a + N^3F_{n-1}^{-1}Q_n^{-1}L, \tag{44}$$

where $F_{n-1} = F_nN|q_n|^{-1}$. For $1 \leqslant r \leqslant n$, let

$$(a(r), b(r)) = A^r(a, b).$$

By (7), (18) and (21) we have

$$f_{n-1}^{(j)}(x) = \lambda(\alpha)_{n-1+j}q_1 \cdots q_{n-1}x^{\alpha-n+1-j}(1+O(\tilde{\Delta})), \quad x \in I_{n-1}, 1 \leqslant j \leqslant 2. \tag{45}$$

If $F_{n-1} \leqslant 1$ or $1 < F_{n-1} \leqslant NL$ from $b(1) - a(1) > 1/2$, by Lemma 1(ii) and (45) we get

$$\begin{aligned} |S_{n-1}| &\ll F_{n-1}^{1/2} + NF_{n-1}^{-1} \ll (NL)^{1/2} + NF_{n-1}^{-1} \\ &\ll (F_{n-1}N^{-1})^{a(1)}N^{b(1)} + NF_{n-1}^{-1}. \end{aligned} \tag{46}$$

If $F_{n-1} > NL$, by (44) we get

$$|S_{n-1}|^2 \ll N^2Q_n^{-1} + N^{b-2a+1}F_{n-1}^aQ_n^a. \tag{47}$$

Note that (47) holds obviously for $Q_n \in [0, 10]$. Thus, we can choose

$$Q_n = N \min\Big(\max\big(\Delta, (\log N)^{-1}\big), \big(N^{a-b}F_{n-1}^{-a}\big)^{1/(a+1)}\Big)$$

in (47). This value of Q_n satisfies (18), so we obtain from (47) that

$$|S_{n-1}|^2 \ll NL + (F_{n-1}^aN^{b+1})^{1/(a+1)} \ll (F_{n-1}N^{-1})^{2a(1)}N^{2b(1)}.$$

Thus (46) is always true. By repeating similar arguments based on (7), (19), (21), (46), and Lemma 1(ii), we can deduce for $r \geqslant 2$ that

$$|S_{n-r}| \ll (F_{n-r}N^{-1})^{a(r)}N^{b(r)} + NF_{n-r}^{-1}, \tag{48}$$

where $F_0 = F$ and for $n \geqslant r + 1$

$$F_{n-r} = F|q_1 \cdots q_{n-r}|N^{r-n}.$$

Especially, from $S_0 = S_f(N, M)$ and

$$(a(n), b(n)) = A^n(a, b) = A^{n(k)}BA^{n(k-1)}B \cdots A^{n(0)}B(0, 1),$$

the required estimate (10) follows from (48) by taking $r = n$. This completes the mathematical induction. The proof of Theorem 2 is finished.

Remark. Here we remark that the arguments leading to (40) are essentially the same as in the proof of Lemma 7 of [1], and we give a detailed proof of (40) here, for it represents a key step in van der Corput's theory of exponent pairs; Lemma 7 of [1] is just Lemma 4 of [8], which was not proved in [8], and such a result was also neglected in [4] which caused the erroneous theory of exponent pairs of §2.3 of [4] as that counterexample shows in the Introduction of our paper.

References

[1] J.G. van der Corput, *Verscharfung der Abschatzung beim Teilerproblem*, Math. Ann. **87** (1922), 39–65.

[2] S.W. Graham and G. Kolesnik, *Van der Corput's method of exponential sums*, London Math. Soc. Lecture Note Ser. **126**, Cambridge University Press, 1991.

[3] D.R. Heath-Brown, *The Pjateckii-Sapiro prime number theorem*, J. Number Theory **16** (1983), 242–266.

[4] A.Ivic, *The Riemann zeta-function*, New York, 1985 (see also its new version, Dover 2003).

[5] A.A. Karatsuba, *Approximation of exponential sums by shorter ones*, Proc. Indian Acad. Sci. (Math.Sci.) **97** (1987), 167–178.

[6] H.-Q. Liu, *On a fundamental result in van der Corput's method of estimating exponential sums*, Acta Arith. **90** (1999), 357–370.

[7] H.L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, Washington, 1994.

[8] E.Phillips, *The zeta-function of Riemann, further developments of van der Corput's method*, Quart. J. Math. **4** (1933), 209–225.

[9] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edition, revised by D. R. Heath-Brown, Oxford, 1986.

[10] J. Vandehey, *Error term improvements for van der Corput transforms*, Q. J. Math. **65**(4) (2014), 1461–1502.

Address: Hong-Quan Liu: Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China.

E-mail: teutop@163.com

Received: 29 May 2016; **revised:** 30 July 2017