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FACTORIZATION IN THE EXTENDED SELBERG CLASS

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Abstract: We prove that every function in the extended Selberg class $\mathcal{S}^\sharp$ can be factored into primitive functions. The proof is definitely more involved than in the case of the Selberg class $\mathcal{S}$.
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1. Introduction

We denote by $\mathcal{S}$ the Selberg class of Dirichlet series with functional equation and Euler product. It is well known that $\mathcal{S}$ contains several classical L-functions, and it is expected that $\mathcal{S}$ essentially coincides with the class of automorphic L-functions. We refer to the survey paper [5] for definitions, notation and basic properties of $\mathcal{S}$ and related classes of Dirichlet series, such as the extended Selberg class $\mathcal{S}^\sharp$ of Dirichlet series with functional equation, but not necessarily with Euler product. We recall that a function $F(s)$ in $\mathcal{S}$ is primitive if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}$ implies $F_1 = 1$ or $F_2 = 1$. It is well known that every function in $\mathcal{S}$ can be factored into primitive functions; see Conrey-Ghosh [2]. The proof is an immediate consequence, by a simple induction on the degree, of the following three facts:

i) the degree is additive, *i.e.*, $d_{FG} = d_F + d_G$ for $F, G \in \mathcal{S}$;
ii) there are no functions $F \in \mathcal{S}$ with degree $0 < d_F < 1$;
iii) the only function of degree 0 in $\mathcal{S}$ is the constant 1.

The notion of primitive function is defined in the extended Selberg class $\mathcal{S}^\sharp$ as well, and hence the problem of the factorization into primitive functions can also be raised in the framework of $\mathcal{S}^\sharp$. In view of Lemma 1 below, in this case we consider only factorizations up to constants, since the non-zero constants are invertible in $\mathcal{S}^\sharp$. Note that the first two of the above facts still hold in $\mathcal{S}^\sharp$, see [4], but $\mathcal{S}_0^\sharp$ is not any more reduced to the single function $F(s) = 1$ identically. We refer to Theorem 1 of [4] for the characterization of functions in $\mathcal{S}_0^\sharp$. As a consequence, the above simple induction on the degree is not enough to show

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the existence of the factorization in $\mathcal{S}^\sharp$. However, the argument can be suitably modified to prove the following

Theorem 1. *Every function in the extended Selberg class $\mathcal{S}^\sharp$ can be factored into primitive functions.*

The basic tools in the proof of Theorem 1 are the notion of conductor of $F \in \mathcal{S}^\sharp$ and the characterization of the functions of degree 0 in $\mathcal{S}^\sharp$, see [4]. Indeed, we recall that the conductor q_F of $F \in \mathcal{S}^\sharp$ is defined as

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{k} \lambda_j^{2\lambda_j},$$

see [6]. Note that q_F is multiplicative, *i.e.*, $q_{FG} = q_F q_G$ if $F, G \in \mathcal{S}^\sharp$, and $q_F = Q^2$ if $F \in \mathcal{S}_0^\sharp$. Moreover, if $F \in \mathcal{S}_0^\sharp$ then q_F is a positive integer and $F(s)$ is a Dirichlet polynomial of the form

$$F(s) = \sum_{n \mid q_F} a(n) n^{-s}. \tag{1.1}$$

Further, $\mathcal{S}_d^\sharp = \emptyset$ for $0 < d < 1$, and $F(s)$ is constant if and only if $d_F = 0$ and $q_F = 1$; see Theorem 1 of [4] for the above results.

We call *almost-primitive* a function $F \in \mathcal{S}^\sharp$ such that if $F(s) = F_1(s)F_2(s)$ with $F_1, F_2 \in \mathcal{S}^\sharp$, then $d_{F_1} = 0$ or $d_{F_2} = 0$. We have

Theorem 2. *If $F \in \mathcal{S}^\sharp$ is almost-primitive, then $F(s) = G(s)P(s)$ with $G, P \in \mathcal{S}^\sharp$, $d_G = 0$ and $P(s)$ primitive.*

We remark that Theorem 1 is a simple consequence of Theorem 2 and of the above recalled results. In fact, an induction on the degree shows that every $F \in \mathcal{S}^\sharp$ can be written as

$$F(s) = F_1(s) \cdots F_k(s),$$

where each $F_j(s)$ is almost-primitive. Therefore, by Theorem 2 we have

$$F(s) = G(s)P_1(s) \cdots P_k(s)$$

with primitive $P_j(s)$ and $d_G = 0$. Since the functions in $\mathcal{S}_0^\sharp$ have integer conductor and those with conductor equal to 1 are constant, an induction on the conductor shows that $G(s)$ is a product of primitive functions, and Theorem 1 follows.

A well known conjecture states that $\mathcal{S}$ has unique factorization into primitive functions. Moreover, it is well known that the Selberg orthonormality conjecture implies such a conjecture; see section 4 of [5]. Note that the analog of the Selberg orthonormality conjecture does not hold in $\mathcal{S}^\sharp$. Indeed, let $\chi_1(n)$ and $\chi_2(n)$ be two primitive Dirichlet characters with the same modulus and parity, and consider

$F(s) = L(s,\chi_1) + L(s,\chi_2)$ and $G(s) = L(s,\chi_1)$. Then $F(s)$ and $G(s)$ belong to $S_1^\sharp$ and are primitive (since the functions of S are linearly independent over the p-finite Dirichlet series, see [3]), but it is easily checked that the Selberg orthonormality conjecture does not hold for $F(s)$ and $G(s)$. It remains open the problem of determining if the unique factorization holds in $S^\sharp$. We conclude with another interesting problem related with the factorization in $S^\sharp$: is it true that a primitive function in S is primitive in $S^\sharp$ as well ?

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2. Proof of Theorem 2

We first characterize the invertible elements of $S^\sharp$.

Lemma 1. *The invertible functions in* $S^\sharp$ *are the non-zero constants.*

Proof. Clearly, the non-zero constants are invertible in $S^\sharp$. Let now $F \in S^\sharp$ be invertible, and let $G(s) = F(s)^{-1}$. Then $d_F + d_G = 0$, and hence both $F(s)$ and $G(s)$ are Dirichlet polynomials (see the Introduction). Denoting by n_0 and m_0 the largest indexes of non-zero coefficients of $F(s)$ and $G(s)$, respectively, we have that the coefficient of index $n_0 m_0$ of $F(s)G(s)$ is non-zero. Therefore $n_0 m_0 = 1$, and Lemma 1 follows. ■

It is well known that every $F \in S^\sharp$ has a zero-free half-plane, say $\sigma > \sigma_F$. By the functional equation, $F(s)$ has no zeros for $\sigma < -\sigma_F$, apart from the trivial zeros coming from the poles of the Γ-factors. We denote by $\rho = \beta + i\gamma$ the generic zero of $F(s)$, and write

$$N_F(T) = \#\{\rho : F(\rho) = 0,\ |\beta| \leqslant \sigma_F,\ |\gamma| < T\}.$$

The classical proof of the Riemann-von Mangoldt formula can be adapted to show that

$$N_F(T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T) \tag{2.1}$$

with a certain constant c_F, for $T \geqslant 2$ and any fixed $F \in S^\sharp$ with $d_F > 0$; see section 2 of [5]. The proof of Theorem 2 is based on the following uniform estimate for the number of zeros of functions in $S_0^\sharp$, *i.e.*, for Dirichlet polynomials of type (1.1).

Proposition 1. *We have*

$$N_F(T) = \frac{T}{\pi} \log q_F + O_{\sigma_F}(\log^6 q_F)$$

uniformly for $T \geqslant 2$ *and* $F \in S_0^\sharp$ *with* $a(1) = 1$ *and* $q_F \geqslant 2$.

A similar result already appears as Proposition 1 of Bombieri-Friedlander [1]. However, Proposition 1 of [1] deals with more general Dirichlet polynomials but gives only an upper bound for $N_F(T)$, while we need a lower bound. We first show how Theorem 2 follows from our Proposition 1, and in the next section we prove Proposition 1.

Assume that $F \in \mathcal{S}^\sharp$ is almost-primitive. If $F(s)$ is not primitive, it can be written as

$$F(s) = L_1(s)F_1(s)$$

with $d_{L_1} = 0$, $q_{L_1} \geqslant 2$ and $F_1(s)$ almost-primitive. If $F_1(s)$ is not primitive we apply inductively the same reasoning, and hence arguing by contradiction we may assume that for every $n \in \mathbb{N}$

$$F(s) = L_1(s) \cdots L_n(s)F_n(s) \tag{2.2}$$

with $d_{L_j} = 0$, $q_{L_j} \geqslant 2$ and $F_n(s)$ almost-primitive, $j = 1, ..., n$. Moreover, looking at the Dirichlet series of both sides of (2.2), we have that only a finite number of $L_j(s)$ can have first coefficient $a_{L_j}(1) = 0$. Therefore, by a normalization, for n sufficiently large we can rewrite (2.2) as

$$F(s) = H(s)H_1(s) \cdots H_n(s)F_n(s)$$

with $d_H = 0$, $d_{H_j} = 0$, $q_{H_j} \geqslant 2$, $a_{H_j}(1) = 1$ and $F_n(s)$ almost-primitive, $j = 1, ..., n$. Writing $G_n(s) = H_1(s) \cdots H_n(s)$, for large n we finally obtain

$$F(s) = H(s)G_n(s)F_n(s) \tag{2.3}$$

with $d_H = 0$, $d_{G_n} = 0$, $q_{G_n} \to \infty$ as $n \to \infty$, $a_{G_n}(1) = 1$ and $F_n(s)$ almost-primitive.

Since the conductor of the functions in $\mathcal{S}_0^\sharp$ is integer and $\mathcal{S}_d^\sharp = \emptyset$ for $0 < d < 1$, from (2.2) we immediately have that $d_F \geqslant 1$. Hence we may use (2.1) and Proposition 1 to show that (2.3) is impossible. Indeed, for n sufficiently large we have

$$N_F(T) \geqslant N_{G_n}(T),$$

and $G_n(s) \neq 0$ for $\sigma > \sigma_F$. Therefore, from (2.1) and Proposition 1 we have

$$\frac{d_F}{\pi} T \log T \geqslant \frac{1}{2\pi} T \log q_{G_n} + O(\log^6 q_{G_n})$$

for sufficiently large T, and hence we get a contradiction as $n \to \infty$ by choosing $T = T_n = q_{G_n}^\delta$ with a small $\delta > 0$.

3. Proof of Proposition 1

Since $a(1) = 1$, we can find a sufficiently large $\sigma_0 > \sigma_F$ such that

$$|F(s) - 1| \leqslant \frac{1}{4} \qquad \text{for } \sigma \geqslant \sigma_0; \tag{3.1}$$

we will choose σ_0 later on. Moreover, we may assume that $\pm T$ is not the ordinate of a zero of $F(s)$ and that $q_F \geqslant 2$. Recalling that $q_F = Q^2$ for $F \in \mathcal{S}_0^\sharp$, by a standard technique based on the argument principle, the functional equation and (3.1) we have

$$\begin{aligned} N_F(T) &= \frac{1}{2\pi}\Delta_{\partial R} \arg\big(Q^s F(s)\big) = \frac{1}{\pi}\Delta_{L_1\cup L_2\cup L_3} \arg\big(Q^s F(s)\big) \\ &= \frac{T}{\pi}\log q_F + O(1) + O\big(|\Delta_{L_1\cup L_3} \arg\big(Q^s F(s)\big)|\big), \end{aligned} \tag{3.2}$$

where R is the rectangle of vertices $\sigma_0 \pm iT$, $1 - \sigma_0 \pm iT$ and $L_1 \cup L_2 \cup L_3$ is the right half of its perimeter, L_2 being the vertical side.

The second error term in (3.2) does not exceed π times the number of zeros of

$$\tfrac{1}{2}\big(F(s \pm iT) + \overline{F}(s \pm iT)\big)$$

in the circle with center σ_0 and radius $\sigma_0 - \frac{1}{2}$. Therefore, by Jensen's inequality such an error term is

$$\ll \sigma_0 \log\big(\max_{|s-\sigma_0|\leqslant\sigma_0} |F(s \pm iT)|\big),$$

and hence from (3.2) we have

$$N_F(T) = \frac{T}{\pi}\log q_F + O\big(\sigma_0 \log\big(\max_{\sigma\geqslant 0}|F(s)|\big)\big). \tag{3.3}$$

Writing

$$M = \max_{n|q_F} |a(n)| \tag{3.4}$$

(and assuming that $M \geqslant 2$) we have

$$\max_{\sigma\geqslant 0} |F(s)| \ll q_F^\varepsilon M,$$

and hence (3.3) becomes

$$N_F(T) = \frac{T}{\pi}\log q_F + O\big(\sigma_0 \log(q_F^\varepsilon M)\big). \tag{3.5}$$

Suppose now that

$$M \ll_{\sigma_F} e^{10\log^3 q_F}. \tag{3.6}$$

Then (3.1) holds with the choice

$$\sigma_0 = c\log^3 q_F \tag{3.7}$$

for a suitable constant $c > 0$, and hence Proposition 1 follows immediately from (3.5)–(3.7). Therefore, in order to conclude the proof of Proposition 1 we need the following

Proposition 2. *Let $F \in S_0^\sharp$ have $a(1) = 1$ and $q_F \geqslant 2$. Then, with the notation in (3.4), we have*

$$M = O_{\sigma_F}\left(e^{10\log^3 q_F}\right).$$

We first prove a lemma. Let $\Omega(n)$ denote the total number of prime factors of n and, given $\delta_0 \geqslant 1$, define the sequence $a(n,\delta_0)$ by induction as $a(1,\delta_0) = \delta_0$ and

$$a(n,\delta_0) = \delta_0 + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1\cdots n_l = n}} a(n_1,\delta_0)\cdots a(n_l,\delta_0)$$

for $n \geqslant 2$, an empty sum being equal to 0. We have

Lemma 2. *For $n \geqslant 1$*

$$\delta_0 \leqslant a(n,\delta_0) \leqslant \delta_0^{\Omega(n)} 2^{\Omega(n)^3}.$$

Proof. We first note that for $l \geqslant 2$ and $a_1,\dots,a_l \geqslant 1$ we have

$$a_1^3 + \cdots + a_l^3 \leqslant (a_1 + \cdots + a_l)^3 - (a_1 + \cdots + a_l)^2. \tag{3.8}$$

Indeed, (3.8) holds for $l = 2$ since $3a_1a_2^2 + 3a_1^2a_2 \geqslant a_1^2 + a_2^2 + 2a_1a_2 = (a_1 + a_2)^2$. Moreover, by induction we have

$$\begin{aligned}(a_1 + \cdots + a_l + a_{l+1})^3 &\geqslant (a_1 + \cdots + a_l)^3 + a_{l+1}^3 + (a_1 + \cdots + a_l + a_{l+1})^2 \\ &\geqslant a_1^3 + \cdots + a_l^3 + a_{l+1}^3 + (a_1 + \cdots + a_l + a_{l+1})^2.\end{aligned}$$

Note that the lemma is trivial when n is a prime number. We prove the lemma by induction, and we may assume that $n \geqslant 4$ and $\Omega(n) \geqslant 2$. Assuming that the lemma holds for $m \leqslant n-1$ and using (3.8) we have

$$\begin{aligned}\delta_0 \leqslant a(n,\delta_0) &= \delta_0 + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1\cdots n_l = n}} a(n_1,\delta_0)\cdots a(n_l,\delta_0) \\ &\leqslant \delta_0 + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1\cdots n_l = n}} \delta_0^{\Omega(n)} 2^{\Omega(n_1)^3 + \cdots + \Omega(n_l)^3} \\ &\leqslant \delta_0 + \delta_0^{\Omega(n)} 2^{\Omega(n)^3 - \Omega(n)^2} \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1\cdots n_l = n}} 1.\end{aligned}$$

Note that we have at most $2^{\Omega(n)}$ possible choices for each n_j in the last sum, and hence

$$\begin{aligned}\sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1\cdots n_l = n}} 1 \leqslant \sum_{l=2}^{\Omega(n)} \frac{1}{l} 2^{\Omega(n)l} &\leqslant \sum_{l=2}^{\Omega(n)-1} 2^{\Omega(n)l} + \frac{2^{\Omega(n)^2}}{\Omega(n)} \\ &\leqslant \frac{2^{\Omega(n)^2}}{2^{\Omega(n)} - 1} + \frac{2^{\Omega(n)^2}}{\Omega(n)} - 1 \leqslant 2^{\Omega(n)^2} - 1,\end{aligned}$$

and the lemma follows. ∎

Proof of Proposition 2. For σ sufficiently large we can write

$$\log F(s) = \sum_{n=2}^{\infty} b(n) n^{-s}, \tag{3.9}$$

the series being absolutely convergent. We may assume that $\sigma_F > 1$, and we first bound $\log F(s)$ for $\sigma \geqslant \sigma_F + \delta$, δ being a small positive constant. For $\sigma \geqslant \sigma_F + \frac{\delta}{2}$ we have

$$F(s) \ll_\delta M,$$

and hence

$$\Re \log F(s) = \log |F(s)| \leqslant c_1(\delta) \log M$$

with some $c_1(\delta) > 0$. Moreover, for every $\varepsilon > 0$ there exists $c_2(\varepsilon) > 0$ such that

$$F(s) = 1 + O(\varepsilon)$$

for $\sigma > c_2(\varepsilon) \log M$, and hence

$$\log F(s) = O(1).$$

Therefore, by the Borel-Carathéodory theorem we have

$$\log F(s) = O_\delta(\log^2 M) \tag{3.10}$$

for $\sigma \geqslant \sigma_F + \delta$.

From (3.10) we deduce that the Lindelöf μ-function of $\log F(s)$ satisfies $\mu(\sigma) = 0$ for $\sigma > \sigma_F$. Moreover, $\log F(s)$ is holomorphic for $\sigma > \sigma_F$. Therefore, by a general result in the theory of Dirichlet series, see chapter 9 of [7], we have that the Dirichlet series (3.9) converges for $\sigma > \sigma_F$, and hence it is absolutely convergent for $\sigma > \sigma_F + 1$. By the formula for the n-th coefficient of a Dirichlet series, see again chapter 9 of [7], for $\sigma > \sigma_F + 1$ we have

$$b(n) n^{-\sigma} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log F(\sigma + it) n^{it} dt \ll \log^2 M$$

in view of (3.10). Hence

$$|b(n)| \leqslant \delta_0 n^\sigma \log^2 M \tag{3.11}$$

for some $\delta_0 \geqslant 1$ and every $\sigma > \sigma_F + 1$.

Now we express the coefficients $b(n)$ in terms of the coefficients $a(n)$. For σ sufficiently large we have

$$F(s) = 1 + G(s) \qquad \text{with} \qquad |G(s)| \leqslant \tfrac{1}{2},$$

and hence

$$\log F(s) = \log\big(1+G(s)\big) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} G(s)^l$$
$$= \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} \sum_n n^{-s} \Big(\sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1 \cdots n_l = n}} a(n_1)\cdots a(n_l) \Big).$$

Therefore, comparing Dirichlet coefficients we obtain

$$b(n) = a(n) + \sum_{l=2}^{\Omega(n)} \frac{(-1)^{l+1}}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1 \cdots n_l = n}} a(n_1)\cdots a(n_l). \tag{3.12}$$

By induction, from (3.11) and (3.12) we obtain

$$|a(n)| \leqslant n^\sigma a(n,\delta_0) \log^{2\Omega(n)} M \tag{3.13}$$

for $\sigma > \sigma_F + 1$, where $a(n,\delta_0)$ is the sequence defined before Lemma 2, starting with the δ_0 in (3.11). Indeed, for $n=2$ we have

$$|a(2)| = |b(2)| \leqslant \delta_0 2^\sigma \log^2 M \leqslant 2^\sigma a(2,\delta_0) \log^{2\Omega(2)} M.$$

Moreover, assuming (3.13) for $2 \leqslant m \leqslant n-1$ we get

$$|a(n)| \leqslant |b(n)| + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1 \cdots n_l = n}} |a(n_1)\cdots a(n_l)|$$
$$\leqslant \delta_0 n^\sigma \log^2 M + \sum_{l=2}^{\Omega(n)} \frac{1}{l} \sum_{\substack{n_1,\dots,n_l \geqslant 2 \\ n_1 \cdots n_l = n}} a(n_1,\delta_0)\cdots a(n_l,\delta_0) n^\sigma \log^{2\Omega(n)} M$$
$$\leqslant n^\sigma a(n,\delta_0) \log^{2\Omega(n)} M$$

by the inductive definition of the sequence $a(n,\delta_0)$, and (3.13) follows. Note that (3.13) implies

$$M \leqslant q_F^\sigma \max_{n|q_F} \big(a(n,\delta_0) \log^{2\Omega(n)} M \big). \tag{3.14}$$

Now we are ready to conclude the proof of Proposition 2. If $M \leqslant \exp(\log^3 q_F)$ the result follows, and hence we may assume that $M > \exp(\log^3 q_F)$, *i.e.*,

$$\log M > \log^3 q_F. \tag{3.15}$$

Since $\Omega(n) \leqslant \frac{\log x}{\log 2}$ for $n \leqslant x$, from (3.14), (3.15) and Lemma 2 we have

$$\begin{aligned} M &\ll q_F^{\sigma}(\log M)^{2^{\frac{\log q_F}{\log 2}}} \delta_0^{\frac{\log q_F}{\log 2}} e^{4\log^3 q_F} \\ &\ll q_F^{\sigma} M^{\frac{2}{\log 2}\frac{\log q_F \log\log M}{\log M}} \delta_0^{\frac{\log q_F}{\log 2}} e^{4\log^3 q_F} \\ &\ll q_F^{\sigma} M^{\frac{2}{\log 2}\frac{1}{\log q_F}} \delta_0^{\frac{\log q_F}{\log 2}} e^{4\log^3 q_F} \\ &\ll q_F^{\sigma} M^{\frac{1}{2}} \delta_0^{\frac{\log q_F}{\log 2}} e^{4\log^3 q_F}. \end{aligned}$$

Therefore, choosing for example $\sigma = \sigma_F + 2$ we obtain

$$M \ll q_F^{2\sigma} \delta_0^{2^{\frac{\log q_F}{\log 2}}} e^{8\log^3 q_F} \ll_{\sigma_F} e^{10\log^3 q_F}$$

and the result follows. ■

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