ON THE MEAN SQUARE OF SHORT EXPONENTIAL SUMS RELATED TO CUSP FORMS

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Abstract: The purpose of the article is to estimate the mean square of a squareroot length exponential sum of Fourier coefficients of a holomorphic cusp form.

Keywords: exponential sums, Fourier coefficients, mean square.

1. Introduction

Let $f(z) = \sum_{n \geq 1} a(n) n^{(\kappa-1)/2} e(nz)$ be a holomorphic cusp form of weight κ with respect to the full modular group. Long exponential sums

$$\sum_{1 \leq n \leq M} a(n) e(n\alpha),$$

where α is a real number, have been widely studied. See e.g. Wilton [11] and Jutila [9]. Short sums

$$\sum_{M \leq n \leq M+\Delta} a(n) e(n\alpha),$$

where $\Delta \ll M^{3/4}$ have been studied for instance in [3] and [4]. However, it seems that very short sums, in particular, sums with $\Delta \asymp M^{1/2}$ seem to be extremely difficult to treat, even though this is an important special case. According to the results in [1] and the computer data in [2], it is plausible to believe the correct upper bound to be

$$\sum_{M \leq n \leq M+\sqrt{M}} a(n) e(n\alpha) \ll M^{1/4+\varepsilon}.$$

2010 Mathematics Subject Classification: primary: 11L07; secondary: 11F11, 11F30

However, anything like this seems to be hopeless to get by at the moment, and therefore, in the current paper, the aim is to consider the mean square of the sum in rational points. The mean square is a common way to consider sums that seem difficult to come by. See e.g. Jutila [7] or Ivić [6]. We prove the following theorem which shows this conjecture to be true on the average:

Theorem 1.1. *Let h and k, $0 \leqslant h < k \ll M^{1/4}$, be integers. Let $w(x)$ denote a smooth weight function that is supported on the interval $[M, M + \Delta]$ where $k^2 M^{1/2+\delta} \ll \Delta \ll M$ with δ an arbitrarily small fixed positive real number. Further assume that $w(M) = w(M + \Delta) = 0$, $0 \leqslant w(x) \leqslant 1$, and $w^{(n)}(x) \ll \Delta^{-n}$ for $1 \leqslant n \leqslant J$ for a sufficiently large J depending on δ. Then*

$$\int_M^{M+\Delta} \left| \sum_{x \leqslant n \leqslant x+\sqrt{x}} a(n) e\left(\frac{hn}{k}\right) \right|^2 w(x)\mathrm{d}x \ll \Delta M^{1/2+\varepsilon},$$

where the constant implied by the $\ll$ symbol depends only on ε.

On the other hand, the Omega results in [1] and [5] show that

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n) = \Omega(\sqrt{\Delta}),$$

where $f = \Omega(g)$ is to be understood to mean that $f = o(g)$ does not hold.

In Theorem 1.1 the weight function is necessary for the proofs of the lemmas. However, it is unclear whether one might be able to remove the weight function by using some clever trick. Anyway, as the square is positive, one may expect that the weight function does not effect the result too much.

Throughout the paper, ε denotes a real number which can be chosen to be arbitrarily small, however, ε is not necessarily the same at every incidence. The constants implied by $\ll$ depend only on ε. Also, let $w(x)$ denote a smooth weight function that is defined as in Theorem 1.1.

2. Lemmas

The following slightly modified version of Jutila and Motohashi's Lemma 6 in [10] is extremely useful while estimating oscillating integrals. The proof is similar to the proof of the original lemma.

Lemma 2.1. *Let A be a $P \geqslant 0$ times differentiable function which is compactly supported in a finite interval $[a, b]$. Assume also that there exist two quantities A_0 and A_1 such that for any non-negative integer $\nu \leqslant P$ and for any $x \in [a, b]$,*

$$A^{(\nu)}(x) \ll A_0 A_1^{-\nu}.$$

Moreover, let B be a function which is real-valued on $[a, b]$, and regular throughout the complex domain composed of all points within the distance ϱ from the interval; and assume that there exists a quantity B_1 such that

$$0 < B_1 \ll |B'(x)|$$

for any point x in the domain. Then we have

$$\int_a^b A(x) e\left(B(x)\right) \mathrm{d}x \ll A_0 \left(A_1 B_1\right)^{-P} \left(1 + \frac{A_1}{\varrho}\right)^P (b - a).$$

Lemma 2.2. *Let $0 \leqslant h < k \leqslant M^{1/4}$. Now*

$$\frac{k}{2\pi^2} \sum_{n \leqslant M} \frac{|a(n)|^2}{n^{3/2}} \int_M^{M+\Delta} w(x) x^{1/2}$$

$$\times \left(\cos\left(\frac{4\pi\sqrt{n(x + \sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(4\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right)\right)^2 \mathrm{d}x \ll k^\varepsilon \Delta M^{1/2}.$$

Proof. Notice first that

$$\left(\cos\left(\frac{4\pi\sqrt{n(x + \sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(4\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right)\right)^2$$

$$= 4\sin^2\left(2\pi\frac{\sqrt{n(x + \sqrt{x})}}{k} + 2\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right) \sin^2\left(2\pi\frac{\sqrt{n(x + \sqrt{x})}}{k} - 2\pi\frac{\sqrt{nx}}{k}\right).$$

Since $\frac{\sqrt{n(x+\sqrt{x})}}{k} - \frac{\sqrt{nx}}{k} \ll \frac{\sqrt{n}}{k}$, we have

$$\sin^2\left(2\pi\frac{\sqrt{n(x + \sqrt{x})}}{k} + 2\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right) \sin^2\left(2\pi\frac{\sqrt{n(x + \sqrt{x})}}{k} - 2\pi\frac{\sqrt{nx}}{k}\right)$$

$$\ll \sin^2\left(2\pi\frac{\sqrt{n(x + \sqrt{x})}}{k} + 2\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right) \cdot \frac{n}{k^2},$$

when $n \leqslant k^2$. For $n > k^2$, estimate the sine-part of the integral to be $\leqslant 1$. We

obtain

$$\frac{k}{2\pi^2} \sum_{n \leqslant M} |a(n)|^2 \, n^{-3/2} \int_M^{M+\Delta} w(x) x^{1/2}$$

$$\times \left(\cos \left(\frac{4\pi \sqrt{n(x + \sqrt{x})}}{k} - \frac{\pi}{4} \right) - \cos \left(4\pi \frac{\sqrt{nx}}{k} - \frac{\pi}{4} \right) \right)^2 \mathrm{d}x$$

$$\ll k \sum_{n \leqslant k^2} n^{\varepsilon - 3/2 + 1} k^{-2} \int_M^{M+\Delta} w(x) x^{1/2} \mathrm{d}x$$

$$+ k \sum_{k^2 < n \leqslant M} n^{\varepsilon - 3/2} \int_M^{M+\Delta} w(x) x^{1/2} \mathrm{d}x \ll k^\varepsilon \Delta M^{1/2}. \qquad \blacksquare$$

Using Lemma 2.1, we get the following estimates

Lemma 2.3. *Let* $1 \leqslant m, n \leqslant M$. *Then*

$$\int_M^{M+\Delta} w(x) x^{1/2} e \left(\pm \left(2\frac{\sqrt{n T_1(x)}}{k} + 2\frac{\sqrt{m T_2(x)}}{k} \right) \right) \mathrm{d}x$$

$$\ll \left(\sqrt{n} + \sqrt{m} \right)^{-P} \Delta^{1-P} k^P M^{P/2+1/2},$$

where $T_1(x)$ *and* $T_2(x)$ *are* x *or* $x + \sqrt{x}$ *(not necessarily but possibly the same).*

Lemma 2.4. *Let* $1 \leqslant m < n \leqslant M$. *Then*

$$\int_M^{M+\Delta} w(x) x^{1/2} e \left(\pm 2\frac{\sqrt{n T(x)}}{k} \mp 2\frac{\sqrt{m T(x)}}{k} \right) \mathrm{d}x$$

$$\ll \left(\sqrt{n} - \sqrt{m} \right)^{-P} \Delta^{1-P} k^P M^{P/2+1/2},$$

where $T(x) = x$ *or* $T(x) = x + \sqrt{x}$.

Lemma 2.5. *Let* $1 \leqslant m \neq n \leqslant M$. *Then*

$$\int_M^{M+\Delta} x^{1/2} w(x) e \left(\pm 2\frac{\sqrt{m(x + \sqrt{x})}}{k} \mp 2\frac{\sqrt{nx}}{k} \right) \mathrm{d}x$$

$$\ll \Delta^{1-P} \left| \sqrt{m} - \sqrt{n} \right|^{-P} k^P M^{P/2+1/2}.$$

Proof. When $m > n$, the proof is similar to Lemma 2.4. Therefore, it is sufficient to concentrate on the case $n > m$. We may also assume the first sign to be plus, and the second one to be minus, as the other case can be treated similarly. Write

$$B(x) = 2\frac{\sqrt{m(x + \sqrt{x})}}{k} - 2\frac{\sqrt{nx}}{k}.$$

Now

$$B'(x) = \frac{\sqrt{m}}{k\sqrt{(x+\sqrt{x})}}\left(1 + \frac{1}{2}x^{-1/2}\right) - \frac{\sqrt{n}}{k\sqrt{x}}$$

$$= \frac{\sqrt{m}}{k}\left(\frac{1 + x^{-1/2} + \frac{1}{4}x^{-1}}{x + \sqrt{x}}\right)^{1/2} - \frac{\sqrt{n}}{k\sqrt{x}}$$

$$= \frac{\sqrt{m}}{k\sqrt{x}}\left(1 + \frac{1}{4(x+\sqrt{x})}\right)^{1/2} - \frac{\sqrt{n}}{k\sqrt{x}}$$

$$= \frac{\sqrt{m}}{k\sqrt{x}}\left(1 + \frac{1}{8(x+\sqrt{x})} + O\left(\frac{1}{x^2}\right)\right) - \frac{\sqrt{n}}{k\sqrt{x}}$$

$$= \frac{\sqrt{m} - \sqrt{n}}{k\sqrt{x}} + \frac{\sqrt{m}}{8k\sqrt{x}(x+\sqrt{x})} + O\left(\frac{\sqrt{m}}{kx^{5/2}}\right).$$

Let us now estimate the second term and the error term. When x is sufficiently large, we have

$$\frac{\sqrt{m}}{8k\sqrt{x}(x+\sqrt{x})} + O\left(\frac{\sqrt{m}}{kx^{5/2}}\right) \leqslant \frac{1}{4kx} \leqslant \left|\frac{\sqrt{m} - \sqrt{n}}{4k\sqrt{x}}\right|.$$

Therefore,

$$|B'(x)| = \left|\frac{\sqrt{m}}{k\sqrt{(x+\sqrt{x})}}\left(1 + \frac{1}{2}x^{-1/2}\right) - \frac{\sqrt{n}}{k\sqrt{x}}\right| \geqslant \left|3\frac{(\sqrt{m} - \sqrt{n})}{4k\sqrt{x}}\right|.$$

Using Lemma 2.1 we obtain the estimate

$$\int_M^{M+\Delta} x^{1/2} w(x) e\left(\pm\left(2\frac{\sqrt{m(x+\sqrt{x})}}{k} - 2\frac{\sqrt{nx}}{k}\right)\right) dx$$

$$\ll \Delta^{1-P}\left|\sqrt{m} - \sqrt{n}\right|^{-P} k^P M^{P/2+1/2},$$

as desired.   ∎

The estimates above are very useful if $|\sqrt{m} - \sqrt{n}|$ is large enough. When it is not large enough, we have to use absolute values to estimate the integral.

3. Proof of the main theorem

Let us first use a modification of Theorem 1.1 [8] (proof is similar than that of the original theorem, just the Fourier coefficients have been normalized):

$$\sum_{1\leqslant n\leqslant x} a(n)e\left(\frac{hn}{k}\right) = \left(\pi\sqrt{2}\right)^{-1}k^{1/2}x^{1/4}\sum_{n\leqslant N} a(n)e_k(-n\bar{h})n^{-3/4}\cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right)$$

$$+ O\left(kx^{1/2+\varepsilon}N^{-1/2}\right).$$

Choose $N \asymp x \asymp M$. Now

$$\sum_{x \leqslant n \leqslant x + \sqrt{x}} a(n) e\left(\frac{hn}{k}\right)$$

$$= \left(\pi\sqrt{2}\right)^{-1} k^{1/2} \sum_{n \leqslant M} a(n) e_k(-n\bar{h}) n^{-3/4}$$

$$\times \left(\cos\left(\frac{4\pi\sqrt{n(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) (x+\sqrt{x})^{1/4} \right.$$

$$\left. - \cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right) x^{1/4} \right) + O\left(kx^{\varepsilon}\right)$$

$$= \frac{k^{1/2}}{\pi\sqrt{2}} \sum_{n \leqslant M} a(n) e_k(-n\bar{h}) n^{-3/4} x^{1/4}$$

$$\times \left(\cos\left(\frac{4\pi\sqrt{n(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right) \right) + O\left(kM^{\varepsilon}\right).$$

Now

$$\int_M^{M+\Delta} \left| \sum_{x \leqslant n \leqslant x + \sqrt{x}} a(n) e\left(\frac{hn}{k}\right) \right|^2 w(x) \mathrm{d}x$$

$$= O\left(\frac{k}{2\pi^2} \int_M^{M+\Delta} \left| \sum_{n \leqslant M} a(n) e_k(-n\bar{h}) n^{-3/4} x^{1/4} \right. \right.$$

$$\left. \left. \times \left(\cos\left(\frac{4\pi\sqrt{n(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right) \right) \right|^2 w(x) \mathrm{d}x \right)$$

$$+ O\left(k^2 \Delta M^{\varepsilon}\right)$$

$$= O\left(\frac{k}{2\pi^2} \sum_{m \neq n} \frac{a(n)a(m)}{(nm)^{3/4}} e_k\left(-n\bar{h} + m\bar{h}\right) \int_M^{M+\Delta} x^{1/2} w(x) \right.$$

$$\times \left(\cos\left(\frac{4\pi\sqrt{n(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(\frac{4\pi\sqrt{nx}}{k} - \frac{\pi}{4}\right) \right)$$

$$\times \left(\cos\left(\frac{4\pi\sqrt{m(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(\frac{4\pi\sqrt{mx}}{k} - \frac{\pi}{4}\right) \right) \mathrm{d}x$$

$$+ \frac{k}{2\pi^2} \sum_{n \leqslant M} \frac{|a(n)|^2}{n^{3/2}} \int_M^{M+\Delta} w(x) x^{1/2}$$

$$\left. \times \left(\cos\left(\frac{4\pi\sqrt{n(x+\sqrt{x})}}{k} - \frac{\pi}{4}\right) - \cos\left(4\pi\frac{\sqrt{nx}}{k} - \frac{\pi}{4}\right) \right)^2 \mathrm{d}x \right) + O\left(k^2 \Delta M^{\varepsilon}\right)$$

The second sum (containing the diagonal terms) has been treated in Lemma 2.2. The cosines in the integral in the first sum can be written as exponential terms.

We have to separate the treatment of the remaining terms in two cases. The first case is the one where $|\sqrt{m} - \sqrt{n}|$ is large, ie.

$$|m - n| > \sqrt{m} \frac{kM^{1/2+\varepsilon}}{\Delta},$$

because now

$$\left|\sqrt{m} - \sqrt{n}\right| = \frac{|m - n|}{\sqrt{m} + \sqrt{n}} \gg \frac{kM^{1/2+\varepsilon}}{\Delta},$$

and therefore, Lemmas 2.3, 2.4 and 2.5 give a non-trivial result. It is sufficient to estimate the sum over the terms estimated in Lemma 2.5 as all the other sums go similarly.

$$\frac{k}{2\pi^2} \sum_{1 \leqslant m \neq n \leqslant M, |m-n| \gg \sqrt{m} \frac{kM^{1/2+\varepsilon}}{\Delta}} \frac{|a(n)a(m)|}{(nm)^{3/4}} \left|\sqrt{m} - \sqrt{n}\right|^{-P}$$
$$\times \Delta^{1-P} M^{1/2+P/2} k^{1+P} = O(1),$$

when P is large enough. Finally, we need to treat the terms with

$$|m - n| < \sqrt{m} \frac{kM^{1/2+\varepsilon}}{\Delta}.$$

Here we will benefit from the shortness of the interval on which the possible values of n have to lie. First of all, if $\sqrt{m} \frac{jkM^{1/2+\varepsilon}}{\Delta} < 1$, there are no $n \neq m$ on the interval. Therefore, we may limit ourselves on the case $m > \frac{\Delta^2}{k^2 M^{1+\varepsilon}}$. We use the trivial estimate for the integral: it is at most $\ll M^{1/2}\Delta$:

$$\frac{k}{2\pi^2} \sum_{\frac{\Delta^2}{k^2 M^{1+\varepsilon}} < m \leqslant M, |m-n| < \sqrt{m} \frac{kM^{1/2+\varepsilon}}{\Delta}} \left|\frac{a(n)a(m)}{(nm)^{3/4}}\right| M^{1/2}\Delta$$
$$\ll k \sum_{1 \leqslant m \leqslant M} |a(m)| m^{\varepsilon-1} k^2 M^{1+\varepsilon} \ll k^2 M^{1+\varepsilon} \ll M^{1/2+\varepsilon}\Delta$$

as desired. This proves the theorem.

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