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ON INFINITE SUMS OF CLOSED IDEALS IN F-LATTICES

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To the memory of Susan – Susanne Dierolf (1942–2009), an unforgettable true friend, a remarkable personality, a human in all respects, who loved all God's creatures (especially cats and rabbits) and, in all that, was such an excellent mathematician.

Abstract: The main result of the paper is that if (I_n) is a sequence of closed ideals in an F-lattice E, then also $\sum_{n=1}^{\infty} I_n$, the set of all elements $x \in E$ of the form $x = \sum_n x_n$, where $x_n \in I_n$ for every n, is a closed ideal in E.

Keywords: F-latices, closed ideals, infinite sums of ideals.

1. Introduction

Let E be a vector lattice [4] (or, in the terminology of [1], a Riesz space). We recall that an *ideal* in E is a vector subspace I of E such that if $y \in E$ and $|y| \leqslant |x|$ for some $x \in I$, then $y \in I$. (Obviously, I is then also a vector sublattice of E.) If $I_1, \dots, I_n$ are ideals in E, then it follows easily from the Riesz Decomposition Property (RDP) that also their sum $I = I_1 + \cdots + I_n$ is an ideal in E. For the reader's convenience, and to make a comparison with its infinite version (see Theorem 3.1 below) more easy, we quote this property as it is stated in [1, Th. 1.10].

Theorem (RDP). *Let $x_1, \dots, x_n \in E$ $(n \in \mathbb{N})$ and let $y \in E$ be such that $|y| \leqslant |x_1 + \cdots + x_n|$. Then there exist $y_1, \dots, y_n \in E$ with $y = y_1 + \cdots + y_n$ and $|y_i| \leqslant |x_i|$ for each i. In addition, if $y \geqslant 0$, then all the y_i's can be chosen to be $\geqslant 0$ as well.*

Now, suppose that E is a (Hausdorff) topological vector lattice, TVL for short (or a Hausdorff locally solid Riesz space) and that all the ideals I_j are closed in E. Then it is natural to ask if also the ideal I has to be closed in E. In general, it is not necessarily so (cf. [4, Ch. III, Exerc. 1(c)]; for more examples, see [6]), and to assure a positive answer one has usually to impose some completeness-type conditions involving E or the I_j's. We briefly discuss the known relevant results,

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along with their slight refinements, in Section 2. In particular, if E is an F-lattice (that is, a complete metrizable TVL), and the I_j's are as above, then the ideal I is closed in E (see Corollary 2.3).

Now, consider an *infinite* sequence $(I_n)_{n\in\mathbb{N}}$ of closed ideals in an F-lattice E. Then, by what was said just above, each of the sums $J_n = I_1 + \cdots + I_n$ is a closed ideal in E. Moreover, the union I of all the J_n's is obviously the smallest ideal in E that contains all the ideals I_n. Consequently, its closure $\overline{I}$ is the smallest closed ideal in E that contains all the I_n's. At first sight, it might appear unlikely to say anything more specific about the form of the elements of $\overline{I}$. What we prove in Theorem 3.2 is, therefore, somewhat surprising: $\overline{I}$ is precisely the set $\sum_{n=1}^{\infty} I_n$ of all elements $z \in E$ such that $z = \sum_{n=1}^{\infty} x_n$ for some $x_n \in I_n$ $(n \in \mathbb{N})$.

Our terminology and notation concerning TVL's is, in general, that of [1] and [4]. Throughout, the terms 'closed' and 'complete' are used to mean 'topologically closed' and 'topologically complete'.

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2. Finite sums of ideals

Let us first observe the following.

Proposition 2.1. *Let $I_1, \ldots, I_n$ $(n \in \mathbb{N})$ be ideals in a* TVL *E such that $I_j \cap I_k = \{0\}$ (or, equivalently, $I_j \perp I_k$) whenever $j \neq k$. Then the topology of the ideal $I = I_1 + \cdots + I_n$ is the direct sum topology of its summands. In consequence, if all the ideals I_j are complete, so is I.*

Proof. Simply note that if $x = x_1 + \cdots + x_n$ with $x_j \in I_j$ for each j, then the summands are pairwise disjoint so that $|x| = |x_1| + \cdots + |x_n|$. Consequently, all the projections $x \to x_j : I \to I_j$ are continuous. ∎

In the proposition below, assertion (a) (except for the 'moreover' part) has been stated as Exerc. 1 in Ch. III of [4] and, for the case of Banach lattices, with full (but quite technical) proofs in [4, Ch. III, Th. 1.2] and [3, Prop. 1.2.2]. We note that those proofs (different from those given below) work also for F-lattices, and have some points in common with our proof of Theorem 3.2.

Proposition 2.2. *Let I_1 and I_2 be closed ideals in a* TVL *E, and denote $I_0 = I_1 \cap I_2$. Consider the ideal $I = I_1 + I_2$ in E.*

(a) *If the quotient* TVL *E/I_0 is complete, then the ideal I is closed in E. If, moreover, also the ideal I_0 is complete, so is I.*
(b) *If the ideal I_0 and both the quotients I_1/I_0 and I_2/I_0 are complete, then also the ideal I is complete (and hence closed) in E.*

Proof. Let $Q : E \to E/I_0$ be the quotient homomorphism. Then $J_1 = Q(I_1)$ and $J_2 = Q(I_2)$ are closed ideals in E/I_0 which can be identified with the quotients I_1/I_0 and I_2/I_0. By the assumptions of (a) or (b), these two ideals in E/I_0 are complete. Since they intersect only at 0, we conclude from the preceding proposition that $J_1 + J_2$ is complete in E/I_0. It follows that $I_1 + I_2 = Q^{-1}(J_1 + J_2)$ is closed in E.

To finish note that $(I_1 + I_2)/I_0 = J_1 + J_2$ so that, by the well know fact that topological completeness is a three-space property (cf. [2, Ch. III.3, Exerc. 9]), if I_0 is assumed to be complete, also $I_1 + I_2$ has to be complete. ∎

Corollary 2.3. *If $I_1, \dots, I_n$ are closed ideals in an F-lattice E, then also their sum $I_1 + \cdots + I_n$ is a closed ideal in E.*

It is not immediately clear how to extend Proposition 2.2 (b) to the case of more than two ideals. The extension contained in our next result, though somewhat formal, seems to be best possible.

Proposition 2.4. *Let $I_1, \dots, I_n$ be ideals in a* TVL *E. Consider the product* TVL *$F = I_1 \times \cdots \times I_n$ and the ideal $I = I_1 + \cdots + I_n$ in E with the induced topology. Then the continuous positive linear operator $A : F \to I$ defined by $A(x_1, \dots, x_n) = x_1 + \cdots + x_n$ is an open map. In other words, the associated operator $\hat{A} : F/N \to I$, where $N = \ker A$, is a topological isomorphism onto.*

Consequently, the ideal I is complete iff the quotient TVS *F/N is complete.*

Proof. Given any solid neighborhood U of zero in I, denote $U_j = U \cap I_j$ for $j = 1, \dots, n$ and $V = U_1 \times \cdots \times U_n$. Obviously, the sets V thus obtained form a base of solid neighborhoods of zero in F. Moreover, if $0 \leqslant x \in U$ then, by (RDP), $x = x_1 + \cdots + x_n$ for some $0 \leqslant x_j \in I_j$ and, clearly, $x_j \in U_j$ $(j = 1, \dots, n)$. It follows that $(x_1, \dots, x_n) \in V$ and $A(x_1, \dots, x_n) = x$. Thus $U \cap I_+ \subset A(V)$ and, consequently, $U \subset A(V + V)$. This proves that the map A is open. The other assertions are now obvious. ∎

Remark 2.5. Proposition 2.2 (b) can be viewed as a corollary to the result above. To see this first note that, in the setting of that proposition, the product $(I_1/I_0) \times (I_2/I_0)$, which can be identified with the quotient $(I_1 \times I_2)/(I_0 \times I_0)$, is complete. Second, note that the latter can be identified with the quotient $\big((I_1 \times I_2)/N\big)/\big((I_0 \times I_0)/N\big)$, where $N = \ker A = \{(z, -z) : z \in I_0\}$. Third, note that the the map $z \to (z, 0) + N$ is an isomorphism from I_0 onto $(I_0 \times I_0)/N$ so that also the latter is complete. Therefore, as completeness is a three space property, $(I_1 \times I_2)/N$ is complete so that the preceding proposition can be applied.

3. Infinite sums of ideals

We start this section by proving an infinite version of the Riesz Decomposition Property for topological vector lattices.

Theorem 3.1. *Let E be a* TVL, *and suppose an element $x \in E$ is the sum of a convergent series $\sum_{n=1}^{\infty} x_n$ in E. If $y \in E$ and $|y| \leqslant |x|$, then there is a sequence (y_n) in E such that $y = \sum_{n=1}^{\infty} y_n$ and, for each n, $|y_n| \leqslant |x_n|$ and $|s_n| \leqslant |r_n|$, where $r_n = \sum_{k=n}^{\infty} x_k$ and $s_n = \sum_{k=n}^{\infty} y_k$. Moreover, if y is positive, then one can choose all the y_n's to be positive as well.*

Proof. We simply apply inductively the usual (RDP). Since $|y| \leqslant |x_1 + r_2|$, one can write y as $y = y_1 + s_2$, for some $y_1, s_2 \in E$ with $|y_1| \leqslant |x_1|$ and $|s_2| \leqslant |r_2|$. Then $|y - y_1| = |s_2| \leqslant |x_2 + r_3|$ so that $y - y_1 = y_2 + s_3$ for some $y_2, s_3 \in E$ with $|y_2| \leqslant |x_2|$ and $|s_3| \leqslant |r_3|$.

Proceeding by an obvious induction, we find sequences (y_n) and (s_n) in E such that, for each n, $|y_n| \leqslant |x_n|$, $y = y_1 + \cdots + y_n + s_{n+1}$, and $|s_{n+1}| \leqslant |r_{n+1}|$. Then, clearly, $s_{n+1} \to 0$ as $n \to \infty$ so that $y = \sum_{n=1}^{\infty} y_n$, and all the y_n's and s_n's are as required.

If $y \geqslant 0$, then at each step above one can choose $y_n \geqslant 0$ and $s_{n+1} \geqslant 0$. ■

Our main result seems to be the following.

Theorem 3.2. *Let E be an F-lattice, and let (I_n) be a sequence of closed ideals in E. Define $I = \sum_{n=1}^{\infty} I_n$ to be the set of elements $z \in E$ that are of the form*

$$z = \sum_{n=1}^{\infty} x_n, \qquad \text{where } x_n \in I_n \text{ for every } n. \tag{$*$}$$

Then I is the smallest closed ideal in E that contains all the ideals I_n. Moreover,

(a) *every positive element $z \in I$ can be represented in the above form with all $x_n \geqslant 0$;*
(b) *every element $z \in E$ can be represented in the form $(*)$ so that also the series $\sum_{n=1}^{\infty} |x_n|$ converges in E.*

Proof. From the previous theorem it follows immediately that I is an ideal in E and that the assertion (a) holds. To prove (b), apply (a) to both x^+ and x^-. Thus it remains to be shown that I is closed or, equivalently, complete. Let $\|\cdot\|$ be a monotone F-norm defining the topology of E.

Let $(z_k)_{k \in \mathbb{N}}$ be a sequence in I such that $\sum_k \|z_k\| < \infty$. We need to show that the series $\sum_k z_k$ converges in E and that its sum, z, is in I. Obviously, we may assume that $z_k \geqslant 0$ for each k. Then, by the final assertion of the preceding theorem, $z_k = \sum_{n=1}^{\infty} x_{k,n}$ for some $0 \leqslant x_{k,n} \in I_n$ $(k, n \in \mathbb{N})$.

We verify that the family $(x_{k,n} : k, n \in \mathbb{N})$ is unconditionally summable (or satisfies the corresponding Cauchy condition) in E. Fix any $\varepsilon > 0$. Next, choose any m such that $\sum_{k \in N_m} \|z_k\| < \varepsilon/2$, where $N_m = \{k \in \mathbb{N} : k > m\}$. Then $\|\sum_{(k,n) \in A} x_{k,n}\| < \varepsilon/2$ for every finite set $A \subset N_m \times \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} x_{k,n}$ are unconditionally convergent for $k = 1, \ldots, m$, there is p such that $\|\sum_{(k,n) \in B} x_{k,n}\| < \varepsilon/2$ whenever B is a finite set contained in $\{1, \ldots, p\} \times N_p$. It follows that $\|\sum_{(k,n) \in C} x_{k,n}\| < \varepsilon$ for every finite set C contained in $\mathbb{N} \times \mathbb{N} \setminus (\{1, \ldots, m\} \times \{1, \ldots, p\})$.

Finally, by the unrestricted associative law for summable families, denoting $x_n = \sum_k x_{k,n}$ (which is an element of I_n) for each n, we have $\sum_k z_k = \sum_{k,n} x_{k,n} = \sum_n x_n$, which concludes the proof. ∎

Corollary 3.3. *Let E be an F-lattice. If (u_n) is a sequence in E, then the elements of the form $z = \sum_{n=1}^{\infty} x_n$ where, for each n, $x_n \in E$ and $|x_n| \leqslant a_n|u_n|$ for some $a_n \in \mathbb{R}_+$, form the closed ideal in E generated by the sequence (u_n).*

A more general result can easily be deduced from the preceding theorem.

Corollary 3.4. *Let E be an F-lattice, and let $(I_\gamma)_{\gamma\in\Gamma}$ be a family of closed ideals in E. Define $I = \sum_{\gamma\in\Gamma} I_\gamma$ to be the set of all elements $z \in E$ that are of the form $z = \sum_{n=1}^{\infty} x_n$, where $x_n \in I_{\gamma_n}$ for every $n \in \mathbb{N}$, and (γ_n) is a sequence in Γ. Then I is the smallest closed ideal in E that contains all the ideals I_γ. Moreover, also the assertions* (a) *and* (b) *of the theorem, suitably modified, hold.*

We close this section with the following.

Theorem 3.5. *Let E be a complete* TVL, *and let (I_n) be a sequence of closed ideals in E such that $I_m \cap I_n = \{0\}$ (or, equivalently, $I_m \perp I_n$) whenever $m \neq n$. Then $I = \sum_{n=1}^{\infty} I_n$ is a closed ideal in E, each I_n is a projection band in I and, denoting by $P_n : I \to I_n$ the associated band projection, one has $z = \sum_{n=1}^{\infty} P_n z$ as a unique representation of z in the form $(*)$ for every $z \in I$.*

Proof. From the mutual orthogonality of the I_n's it follows easily that each $z \in I$ has a unique representation in the form $(*)$. This leads to the natural projections $P_n : I \to I_n$ and $Q_n = P_1 + \cdots + P_n : I \to J_n = I_1 + \cdots + I_n$, and since $|P_n z| \leqslant |z|$ and $|Q_n z| \leqslant |z|$ ($n \in \mathbb{N}$, $z \in I$), all these projections are equicontinuous. Moreover, their ranges are complete ideals in E (cf. Proposition 2.1). Therefore, they extend in a unique way to equicontinuous projections, still denoted by P_n and Q_n, to all of $\overline{I}$, and one still has $Q_n = P_1 + \cdots + P_n$ for each n. Since $Q_n z \to z$ for all $z \in I$, the same holds for all $z \in \overline{I}$. Hence, for each $z \in \overline{I}$ one has $z = \sum_{n=1}^{\infty} P_n z$ so that $z \in I$. This proves that I is closed. The other assertions of the theorem are fairly obvious. ∎

4. The case of Banach lattices $C(S)$, S a compact space

We now proceed to the special case of closed ideals in the Banach lattice $C(S)$ of all continuous functions $f : S \to \mathbb{R}$ (with the supnorm $\|\cdot\|_\infty$), where S is a compact Hausdorff space. It is well known that any closed ideal I in $C(S)$ can uniquely be represented in the form

$$I = C_0(S\|K) := \{f \in C(S) : f = 0 \text{ on } K\},$$

where K is a compact subset of S, and conversely (see, e.g., [3, Prop. 2.1.9]). In this case one has an alternative description of the closed ideal generated by a given family of closed ideals (comp. Corollary 3.4).

Theorem 4.1. *Let $(I_\gamma)_{\gamma\in\Gamma}$ be a family of closed ideals in $C(S)$. For each $\gamma\in\Gamma$, let K_γ be the compact set in S such that $I_\gamma = C_0(S\|K_\gamma)$. Also, denote by K the intersection of the K_γ's, and by I the smallest closed ideal in $C(S)$ containing all the ideals I_γ ($\gamma\in\Gamma$). Then $I = C_0(S\|K)$.*

Proof. It is clear that I is the closure of the union of all the ideals $I_{\gamma_1}+\cdots+I_{\gamma_n}$, where $\{\gamma_1,\dots,\gamma_n\}$ is a finite subset of Γ. From this it follows immediately that $I\subset C_0(S\|K)$.

As for the converse inclusion, we first show that it holds when we deal with a finite family of closed ideals. In fact, in that case it suffices to show that if $I_j = C_0(S\|K_j)$ for $j=1,2$, then $C_0(S\|K)\subset I = I_1+I_2$, where $K = K_1\cap K_2$ (and having done that, proceed by induction). Now, take any $f\in C_0(S\|K)$ and any $\varepsilon>0$. Denote $U_1 = S\setminus K_1$, $U_2 = S\setminus K_2$, and $U_3 = \{s\in S : |f(s)|<\varepsilon\}$. Clearly, these sets form an open covering of S. Let $\varphi_1,\varphi_2,\varphi_3 : S\to[0,1]$ be a partition of unity that is subordinated to the covering U_1,U_2,U_3 of S. Thus each φ_k is continuous, $\operatorname{supp}\varphi_k\subset U_k$, and $\varphi_1+\varphi_2+\varphi_3 = 1$ on S. Denote $f_k = f\varphi_k$ for $k=1,2,3$. Then $f_1\in I_1$, $f_2\in I_2$, and $\|f-(f_1+f_2)\|_\infty = \|f_3\|_\infty<\varepsilon$. It follows that $f\in I$.

Now, let us treat the general case. By what was shown above, we have $I_{\gamma_1}+\cdots+I_{\gamma_n} = C_0(S\|K_{\gamma_1}\cap\cdots\cap K_{\gamma_n})$ for every finite set $\{\gamma_1,\dots,\gamma_n\}\subset\Gamma$. Therefore, we may assume that the family (I_γ) of ideals is directed upward, and the corresponding family (K_γ) of compact sets is directed downward. Again, take any $f\in C_0(S\|K)$ and any $\varepsilon>0$. Let $V = \{s\in S : |f(s)|<\varepsilon\}$. Since $K_\gamma\downarrow K$, there is $\gamma\in\Gamma$ such that $K_\gamma\subset V$. Denote $U = S\setminus K_\gamma$, and let φ,ψ be a partition of unity on S corresponding to the open covering U,V of S. Denote $g = f\varphi$ and $h = f\psi$. Then $g\in I_\gamma$ and $\|f-g\|_\infty = \|h\|_\infty<\varepsilon$. It follows that $f\in I$. ■

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