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COMPARING $L(s,\chi)$ WITH ITS TRUNCATED EULER PRODUCT AND GENERALIZATION

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Abstract: We show that any L-function attached to a non-exceptionnal Hecke Grossencharakter Ξ may be approximated by a truncated Euler product when s lies near the line $\Re s = 1$. This leads to some refined bounds on $L(s,\Xi)$.

Keywords: Hecke Grossencharakter, Dirichlet L-functions.

1. Introduction and results

We first need to fix some terminology. We select a number field $\mathbb{K}/\mathbb{Q}$ be a number field of degree d and discriminant Δ. We denote its norm by N, as a shortcut to $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}$. We shall consider Hecke Grossencharakters Ξ to (finite) ideal $\mathfrak{f}$, of norm q, and associated with some finite set of infinite places. The conductor $\mathfrak{f}$ being fixed, the main Theorem of [5] tells us there exists an absolute constant $C > 0$ such that no L-function $L(s,\Xi)$ has a zero ρ in the region

$$\Re\rho \geqslant 1 - \frac{C}{\operatorname{Log}\max(q\Delta, q\Delta|\Im s|)} \tag{1}$$

except at most one such L-function; this potential exception is associated to a real valued character and may have at most one real zero β in this region. We refer to this hypothetical character as the exceptional character and term the remaining ones as being non-exceptional. See also [11]. In the case of Dirichlet characters, i.e. $\mathbb{K} = \mathbb{Q}$, we know from [13] that we may take $C = 1/6.3958$.

Theorem 1. *Let Ξ be a non-exceptional Hecke Grossencharacter with (finite) conductor $\mathfrak{f}$ of norm $q > 1$. We have*

$$L(s,\Xi) \asymp \prod_{\mathrm{N}\,\mathfrak{p} \leqslant q\Delta|s|} \big(1 - \Xi(\mathfrak{p})/\mathrm{N}\,\mathfrak{p}^s\big)^{-1}$$

when $1 \geqslant (\Re s - 1)\operatorname{Log}(q\Delta(2+|s|)) \geqslant -C/2$, *the constant* C *being the one from* (1).

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The restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \geqslant 1/\operatorname{Log}(q\Delta)$. Under the Riemann hypothesis for the implied L-function, we can restrict the above product to $p \leqslant \operatorname{Log}\operatorname{Log}(q\Delta|s|)$. As trivial consequences, we find via (a generalization of) Mertens theorems (see (9) below) that, under these conditions

$$\frac{q/\phi(q)}{\operatorname{Log}(q\Delta|s|)} \ll |L(s,\Xi)| \ll \frac{\phi(q)}{q}\operatorname{Log}(q\Delta|s|). \tag{2}$$

The upper bound is classical in the case of Dirichlet characters but improves considerably in the general case on the one given in Theorem 5 of [17], albeit being less explicit. The factor $q/\phi(q)$ in the lower one appears to be novel, even in the case of Dirichlet characters. For instance, it supersedes the one of Corollary 2 of [11] by the factor $q/\phi(q)$ and by the fact that it is valid for any non-exceptional character. From a historical viewpoint, [14] shows that $|L(1,\chi)| \gg 1/\operatorname{Log}^5 q$ for non-real characters, and improves this in $|L(1,\chi)| \gg 1/\operatorname{Log} q$ in [15]. The proof is somewhat more delicate than expected.

Note also (by again invoking Mertens' theorems) that we can restrict the product to $p \leqslant (q|s|)^a$ for any positive a.

Granville & Soundararajan investigated in [6] (see also [7]) the distribution of values of $L(1,\chi)$ (χ being a Dirichlet character) via an approximation by an Euler product and in particular,they show in their Proposition 1 that the Euler product may be truncated to $p \leqslant \operatorname{Log} q$ for all but $\mathcal{O}(q^{1-2/\operatorname{Log}\operatorname{Log} q})$ characters. Note however that they aim at an exact approximation of $L(1,\Xi)$ while we only seek to recover its order of magnitude. For $L(1,\chi)$, see also [8], [16] and [1].

Our main ingredient is the following Lemma of independent interest.

Lemma 1. *Under the conditions above,* $|L'/L(s,\Xi)| \ll \operatorname{Log}(q\Delta(2+|s|))$.

In this Lemma also, the restriction to non-exceptional characters can be dispensed with if we assume $|\Im s| \geqslant 1/\operatorname{Log}(q\Delta)$. The inequality $-\Re L'/L(s,\Xi) \leqslant c\operatorname{Log}(q\Delta(2+|s|))$ when $\Re s > 1$ is a classical element of the proof of the zero-free region for $L(\cdot,\Xi)$ (see [4, chapter 14] for instance); by using his local method, Landau shows in [15, page 30] that $\Re L'/L(s,\Xi) \leqslant c\operatorname{Log}(q\Delta(2+|s|))$. The above Lemma shows that much more is true and that only invoking a one-sided bound for the real part does not lead to any improvement.

Under the Riemann hypothesis for $L(\cdot,\Xi)$, the upper bound becomes $\operatorname{Log}\operatorname{Log}(q\Delta(2+|s|))$.

Generalization

Like many properties of Dirichlet L-functions, this one generalizes to a wide class of L-functions. We shall not describe such a general context but refer the reader to chapter 5 of [12]. We work under the conditions of their Theorem 5.10: $L(f,s)$ is an L-function fo degree d such that the Rankin-Selberg convolutions $L(f\otimes f,s)$ and $L(f\otimes\overline{f},s)$ exist, the latter having a simple pole at $s=1$ while the former is entire when $f \neq \overline{f}$. We further suppose that $|\alpha_j(p)|^2 \leqslant p/2$ at the ramified primes.

The notion of exceptional character is more complicated to define in a general context, since it requires a way of defining families of L-functions. Assuming that our candidate has no real zero in the classical zero-free region, we find that

$$L(f,s) \asymp \prod_{p\leqslant \mathfrak{q}(f,s)} (1-\alpha_1(p)p^{-s})^{-1}\cdots(1-\alpha_d(p)p^{-s})^{-1} \tag{3}$$

where the analytical conductor is defined there in equation (5.7).

Notations

We need some names for our variables, and the easiest path is to keep a fixed point $s_0=\sigma_0+it_0$, which will be s in the Theorem, and a running $s=\sigma+it$. We define

$$\mathcal{L} = \operatorname{Log}(q\Delta(|s_0|+2)). \tag{4}$$

The point $s_1=\sigma_1+it_0$ with $\sigma_1 = 1+1/\mathcal{L}$ will be of special interest.

2. Some material on primes in number fields

We can use the prime number Theorem for $\mathbb{K}/\mathbb{Q}$, but we prefer to sketch an elementary approach to the classical results we need. Such material is also contained in [18]. Assume we have an asymptotic estimate:

$$\sum_{\mathrm{N}\,\mathfrak{a}\leqslant X} 1 = c_0X + \mathcal{O}(X/\operatorname{Log}(2X)) \tag{5}$$

where $\mathfrak{a}$ ranges the integral ideals of $\mathbb{K}$. Such an estimate is linked with the fact that the Dedekind zeta function $\zeta_{\mathbb{K}}$ of $\mathbb{K}$ has a simple pole at $s=1$. In particular c_0 is the residue of this function at $s=1$. The results we seek also hold with the error term being simply $o(X)$, but our proof would require a modification. From this we deduce that

$$\sum_{\mathrm{N}\,\mathfrak{a}\leqslant X} \operatorname{Log}\mathrm{N}\,\mathfrak{a} = c_0X\operatorname{Log}X + \mathcal{O}(X). \tag{6}$$

Writing $\zeta_{\mathbb{K}}'/\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{a})/\mathrm{N}\,\mathfrak{a}^s$ we find that $\sum_{\mathfrak{b}|\mathfrak{a}} \Lambda_{\mathbb{K}}(\mathfrak{b}) = \operatorname{Log}\mathrm{N}\,\mathfrak{a}$ and plugging this into (6), we get

$$c_0X\operatorname{Log}X + \mathcal{O}(X) = \sum_{\mathrm{N}\,\mathfrak{b}\leqslant X} \Lambda_{\mathbb{K}}(\mathfrak{b}) \sum_{\mathrm{N}\,\mathfrak{c}\leqslant X/\mathrm{N}\,\mathfrak{b}} 1 = X\sum_{\mathrm{N}\,\mathfrak{b}\leqslant X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{\mathrm{N}\,\mathfrak{b}}$$

by appealing to (5), from which we infer

$$\sum_{\mathrm{N}\,\mathfrak{b}\leqslant X} \frac{\Lambda_{\mathbb{K}}(\mathfrak{b})}{\mathrm{N}\,\mathfrak{b}} = \operatorname{Log}X + \mathcal{O}(1). \tag{7}$$

Using the expression of $\zeta_{\mathbb{K}}$ as an Euler product, we find that $\Lambda_{\mathbb{K}}(\mathfrak{b})$ is zero except when $\mathfrak{b}$ is a power of a prime $\mathfrak{p}$, at which point it takes the value $\operatorname{Log} \mathrm{N}\,\mathfrak{p}$. This finally leads us to the estimate

$$\sum_{\mathrm{N}\,\mathfrak{p}\leqslant X} \frac{\operatorname{Log} \mathrm{N}\,\mathfrak{p}}{\mathrm{N}\,\mathfrak{p}} = \operatorname{Log} X + \mathcal{O}(1). \tag{8}$$

We infer $\sum_{\mathrm{N}\,\mathfrak{p}\leqslant X} 1/\mathrm{N}\,\mathfrak{p} = \operatorname{Log}\operatorname{Log} X + \mathcal{O}(1)$ and thus

$$\prod_{\mathrm{N}\,\mathfrak{p}\leqslant X} (1 - 1/\mathrm{N}\,\mathfrak{p}) \asymp 1/\operatorname{Log} X \tag{9}$$

which is enough for our purpose. This is not what is referred to as Mertens' Theorem, since we do not have a proper asymptotic, but these estimates are enough for our purpose. We refer the reader to [3] for related material on explicit Mertens' Theorem in abelian number fields.

3. Proof of Lemma 1

We start from Linnik's density lemma which the reader may find in [5, Lemma 7] or in [2, chapter 6] in case of Dirichlet characters. We define $n(1+it, r)$ to be the number of zeros ρ of $L(s,\chi)$ in the disc $|\rho - i - it| \leqslant r$. We have

$$\frac{L'}{L}(s,\Xi) = \frac{-\delta_\Xi}{s-1} + \sum_{|\rho-1-it_0|\leqslant 1/3} \frac{1}{s-\rho} + \mathcal{O}(\mathcal{L}) \qquad \big(|s-1-it_0| \leqslant 1/4\big), \tag{10}$$

where δ_Ξ is 1 if Ξ is principal, and 0 otherwise. This is for instance Lemma 6 of [5]; In case of Dirichlet characters, this is (4) of chapter 16 of [4], and in a general context (5.28) of [12]. These two last proofs relie on a global representation of L'/L, while Fogel's one follows the local method of Landau. The latest refinements of this method may be found in [10] and [9].

One of the consequences of (10) is Linnik's density lemma:

$$n(1+it, r) \ll r\mathcal{L} + 1. \tag{11}$$

Apply (10) to $s = \sigma + it_0$ with $\sigma \geqslant 1 - C/2\mathcal{L}$ and to s_1 and substract. For any zero ρ in the summation above, we have $|s-\rho| \geqslant |1 + it_0 - \rho|/2$ and thus, with $r_k = 2^k/\mathcal{L}$

$$\begin{aligned}\left|\frac{L'}{L}(s,\Xi)-\frac{L'}{L}(s_1,\Xi)\right| &\leqslant \sum_{|\rho-1-it_0|\leqslant 1/3}\frac{4|\sigma-\sigma_1|}{|1+it_0-\rho|^2}+\mathcal{O}(\mathcal{L})\\ &\leqslant |\sigma-\sigma_1|\sum_{0\leqslant k\leqslant \operatorname{Log}\mathcal{L}}\ \sum_{2^k\leqslant|\rho-1-it_0|\mathcal{L}\leqslant 2^{k+1}}\frac{4}{r_k^2}+\mathcal{O}(\mathcal{L})\\ &\leqslant |\sigma-\sigma_1|\sum_{0\leqslant k\leqslant \operatorname{Log}\mathcal{L}}\frac{4n(1+it_0,r_k)}{r_k^2}+\mathcal{O}(\mathcal{L})\\ &\ll |\sigma-\sigma_1|\sum_{0\leqslant k\leqslant \operatorname{Log}\mathcal{L}}\left(\frac{\mathcal{L}}{r_k}+\frac{1}{r_k^2}\right)+\mathcal{O}(\mathcal{L})\ll|\sigma-\sigma_1|\mathcal{L}^2+\mathcal{L}.\end{aligned}$$

Notice furthermore that $|L'/L(s_1,\Xi)|\leqslant -\zeta'/\zeta(\sigma_1)\ll\mathcal{L}$, so that, when $\sigma\leqslant 1+\mathcal{L}$, the above inequality reduces to

$$\left|\frac{L'}{L}(s,\Xi)\right|\ll\mathcal{L}. \tag{12}$$

This ends the proof in case of non-exceptional characters. In case of an exceptional character, we simply consider separately in (10) its contribution, namely $1/(s-\beta)$ which is again $\mathcal{O}(\mathcal{L})$. Under the Riemann hypothesis, we simply invoke Theorem 5.17 of [12].

4. Proof of the Theorem

Define $R=q\Delta|s_0|$. We check that, on using (8),

$$\left|\frac{L'}{L}(s,\Xi)+\sum_{\mathrm{N}\,\mathfrak{p}\leqslant R}\frac{\Xi(\mathfrak{p})\operatorname{Log}\mathrm{N}\,\mathfrak{p}}{\mathrm{N}\,\mathfrak{p}^s-\Xi(\mathfrak{p})}\right|\ll\mathcal{L}+\operatorname{Log}R\ll\mathcal{L} \tag{13}$$

when $s=\sigma+it_0$ and $1\geqslant(\sigma-1)\mathcal{L}\geqslant -C/2$. We integrate (12) between s_1 and s_0 and find that

$$|\operatorname{Log}L_R(s_0,\Xi)-\operatorname{Log}L_R(s_1,\Xi)|\ll 1 \tag{14}$$

with $L_R(s,\Xi)=\prod_{\mathrm{N}\,\mathfrak{p}>R}(1-\Xi(\mathfrak{p})/\mathrm{N}\,\mathfrak{p}^s)^{-1}$. Next we note that

$$\begin{aligned}|L_R(s_1,\Xi)|&\leqslant\prod_{\mathrm{N}\,\mathfrak{p}>R}(1-\mathrm{N}\,\mathfrak{p}^{-\sigma_1})^{-1}\leqslant\exp\sum_{\mathrm{N}\,\mathfrak{p}>R}\mathrm{N}\,\mathfrak{p}^{-\sigma_1}\\ &\ll\exp\int_R^\infty\frac{dt}{t^{\sigma_1}\operatorname{Log}t}=\exp\int_{R^{\sigma_1-1}}^\infty\frac{dv}{v^2\operatorname{Log}v}\end{aligned}$$

by setting $v=t^{\sigma_1-1}$, and where we have again invoked (8). The last quantity is bounded since so is R^{σ_1-1}. Considering only real parts in (14), the Theorem readily follows.

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