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SOME L^p INEQUALITIES FOR POLYNOMIALS

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Abstract: In this paper we establish some L^p inequalities for polynomials having no zeros in $|z| < k$, where $k \geqslant 1$. Our results not only generalizes some known polynomial inequalities, but also a variety of interesting results can be deduced from these by a fairly uniform procedure.

Keywords: Polynomials, Zygmund inequality, L^p inequalities, zeros.

1. Introduction and statement of results

Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ be a polynomial of degree at most n and $p'(z)$ its derivative, then

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)| \tag{1}$$

and for every $r \geqslant 1$,

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leqslant n \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \tag{2}$$

Inequality (1) is a classical result of Bernstein [13] (see also [16]), whereas inequality (2) is due to Zygmund [17] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $p(e^{i\theta})$. Arestov [1] proved that (2) remains true for $0 < r < 1$ as well. If we let $r \to \infty$ in inequality (2), we get (1).

If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then both the inequalities (1) and (2) can be sharpened. In fact, if $p(z) \neq 0$ in $|z| < 1$, then (1) and (2) can be respectively replaced by

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)| \tag{3}$$

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and

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leqslant nC_r\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{4}$$

where

$$C_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\alpha}|^r d\alpha\right\}^{-\frac{1}{r}}.$$

Inequality (3) was conjectured by Erdös and later verified by Lax [11], whereas inequality (4) was found out by De-Bruijn [6] for $r \geqslant 1$. Rahman and Schmeisser [15] have shown that (4) holds for $0 < r < 1$ also. If we let $r \to \infty$ in (4), we get (3).

As a generalization of (3) Malik [12] proved that if $p(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|, \tag{5}$$

whereas under the same hypothesis, Govil and Rahman [10] extended inequality (4) by showing that

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leqslant nE_r\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{6}$$

where

$$E_r = \left\{\frac{1}{2\pi}\int_0^{2\pi} |k+e^{i\alpha}|^r d\alpha\right\}^{-\frac{1}{r}}, \qquad r \geqslant 1.$$

It was shown by Gardner and Weems [9] that inequality (6) also holds for $0 < r < 1$.

Chan and Malik [5] generalized (5) in a different direction and proved that, if $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n which does not vanish in $|z| < k$, where $k \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k^t} \max_{|z|=1} |p(z)|. \tag{7}$$

Inequality (7) was independently proved by Qazi [14, Lemma 1] who also under the same hypothesis proved that

$$\max_{|z|=1} |p'(z)| \leqslant \left(\frac{n}{1+S_1}\right) \max_{|z|=1} |p(z)|, \tag{8}$$

where

$$S_1 = k^{t+1}\left(\frac{\left(\dfrac{t}{n}\right)\left|\dfrac{a_t}{a_0}\right|k^{t-1}+1}{\left(\dfrac{t}{n}\right)\left|\dfrac{a_t}{a_0}\right|k^{t+1}+1}\right). \tag{9}$$

If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v \neq 0$ in $|z| < k$, $k \geqslant 1$, then $\frac{t}{n}\left|\frac{a_t}{a_0}\right| k^t \leqslant 1$, which can also be taken as equivalent to $S_1 \geqslant k^t$. Hence inequality (8) is an improvement of inequality (7).

Recently, Aziz and Shah [4] investigated the dependence $\max_{|z|=1} |p(Rz) - p(z)|$ on $\max_{|z|=1} |p(z)|$ and proved that if $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n, $p(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for every $R > 1$ and $|z| = 1$,

$$|p(Rz) - p(z)| \leqslant \left\{ \frac{R^n - 1}{1 + \psi_1(R)} \right\} \max_{|z|=1} |p(z)|, \tag{10}$$

where

$$\psi_1(R) = k^{t+1} \left(\frac{\left(\frac{R^t - 1}{R^n - 1}\right)\left|\frac{a_t}{a_0}\right| k^{t-1} + 1}{\left(\frac{R^t - 1}{R^n - 1}\right)\left|\frac{a_t}{a_0}\right| k^{t+1} + 1} \right). \tag{11}$$

If we divide the two sides of (10) by $R - 1$, make $R \to 1$ and noting that $\psi_1(R) \to S_1$ as $R \to 1$, we get (8).

The following result which is due to Gardner, Govil and Weems [8] is of independent interest, because it provides generalizations and refinements of inequalities (3), (5), (7) and (8).

Theorem A. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$ where $k \geqslant 1$, then*

$$\max_{|z|=1} |p'(z)| \leqslant \left(\frac{n}{1 + S_0} \right) \left\{ \max_{|z|=1} |p(z)| - m \right\} \tag{12}$$

where $m = \min_{|z|=k} |p(z)|$ and

$$S_0 = k^{t+1} \left(\frac{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} k^{t-1} + 1}{\left(\frac{t}{n}\right) \frac{|a_t|}{|a_0| - m} k^{t+1} + 1} \right) \tag{13}$$

In this paper, we shall generalize inequalities (10) and (12) to the L^r norm of $p(z)$ for every $r > 0$. We first prove the following interesting generalization of (12).

Theorem 1. *Let $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geqslant 1$, be a polynomial of degree n which does not vanish in $|z| < k$, $k \geqslant 1$. Then for every complex number β with $|\beta| \leqslant 1$ and for each $r > 0$,*

$$\left\{ \int_0^{2\pi} \left| p'(e^{i\theta}) + \frac{mn\beta}{1 + S_0} \right|^r d\theta \right\}^{\frac{1}{r}} \leqslant nC_r \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{14}$$

where

$$m = \min_{|z|=k} |p(z)|, \qquad C_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S_0 + e^{i\alpha}|^r d\alpha \right\}^{-\frac{1}{r}}$$

and S_0 is as defined in Theorem A.

If we let $r \to \infty$ in (14), noting that $C_r \to \dfrac{1}{1+S_0}$ and choose argument of β with $|\beta| = 1$ suitably, we get (12). For $k = 1 = t$ and $\beta = 0$, Theorem 1 reduces to De-Bruijn's Theorem.

If we do not have the knowledge of $\min\limits_{|z|=k} |p(z)|$, we obtain the following result which is a special case of Theorem 1.

Corollary 1. *If $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for each $r > 0$,*

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leqslant n D_r \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{15}$$

where

$$D_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |S_1 + e^{i\alpha}|^r d\alpha \right\}^{-\frac{1}{r}} \tag{16}$$

and S_1 is defined by formula (9).

If we let $r \to \infty$ in (15), we get (8). Several other interesting results easily follow from Corollary 1. Here, we mention a few of these. Since it is well known that $S_1 \geqslant k^t$. Using this fact in inequality (15), we immediately get the following corollary.

Corollary 2. *If $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for each $r > 0$,*

$$\left\{ \int_0^{2\pi} |p'(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leqslant \frac{n}{\left\{ \frac{1}{2\pi} \int_0^{2\pi} |k^t + e^{i\alpha}|^r d\alpha \right\}^{\frac{1}{r}}} \times \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}. \tag{17}$$

For $t = 1$, inequality (17) reduces to inequality (6) for $r > 0$.

Instead of proving Theorem 1, we prove the following more general result which includes not only Theorem 1 and inequality (10) as special cases, but also leads to a standard development of interesting generalizations of some well known results.

Theorem 2. *If* $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geqslant 1$, *is a polynomial of degree* n *which does not vanish in* $|z| < k$, $k \geqslant 1$, *and* $m = \min_{|z|=k} |p(z)|$, *then for every complex number* β *with* $|\beta| \leqslant 1$, $r > 0$, $R > 1$ *and* α *real,*

$$\left\{ \int_0^{2\pi} \left| p(Re^{i\theta}) - p(e^{i\theta}) + \left(\frac{R^n - 1}{1 + \psi_0(R)} \right) m\beta \right|^r d\theta \right\}^{\frac{1}{r}} \leqslant (R^n - 1) B_r \left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}}, \tag{18}$$

where

$$B_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\psi_0(R) + e^{i\alpha}|^r d\alpha \right\}^{-\frac{1}{r}}$$

and

$$\psi_0(R) = k^{t+1} \left\{ \frac{\left(\frac{R^t - 1}{R^n - 1} \right) \frac{|a_t|}{|a_0| - m} k^{t-1} + 1}{\left(\frac{R^t - 1}{R^n - 1} \right) \frac{|a_t|}{|a_0| - m} k^{t+1} + 1} \right\}. \tag{19}$$

If we let $r \to \infty$ in (18) and choose argument of β with $|\beta| = 1$ suitably, we get

$$\max_{|z|=1} |p(Rz) - p(z)| \leqslant \left(\frac{R^n - 1}{1 + \psi_0(R)} \right) \left\{ \max_{|z|=1} |p(z)| - m \right\}. \tag{20}$$

Dividing the two sides of (20) by $R-1$, letting $R \to 1$ and noting that $\psi_0(R) \to S_0$ as $R \to 1$, we get Theorem A.

From inequality (20), it follows that

$$\max_{|z|=R} |p(z)| \leqslant \left(\frac{R^n + \psi_0(R)}{1 + \psi_0(R)} \right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{1 + \psi_0(R)} \right) m. \tag{21}$$

It can be easily verified that for every n and $R \geqslant 1$, the function $\left(\frac{R^n + x}{1 + x} \right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{1 + x} \right) m$, is a non-increasing function of x. If we combine this fact with Lemma 6 (stated in Section 2), according to which $\psi_0(R) \geqslant k^t$ for $t \geqslant 1$, we get

$$\max_{|z|=R} |p(z)| \leqslant \left(\frac{R^n + k^t}{1 + k^t} \right) \max_{|z|=1} |p(z)| - \left(\frac{R^n - 1}{1 + k^t} \right) m, \tag{22}$$

which is a generalization of a result due to Aziz [2, Theorem 4].

If we divide the two sides of (18) by $R - 1$, make $R \to 1$ and note that $\psi_0(R) \to S_0$ as $R \to 1$, we get inequality (14) of Theorem 1.

2. Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 1. *If $p(z) = a_0 + \sum_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then for $|z| = 1$ and $R > 1$,*

$$|q(Rz) - q(z)| \geqslant k^{t+1} \left\{ \frac{\left(\frac{R^t - 1}{R^n - 1}\right) \left|\frac{a_t}{a_0}\right| k^{t-1} + 1}{\left(\frac{R^t - 1}{R^n - 1}\right) \left|\frac{a_t}{a_0}\right| k^{t+1} + 1} \right\} |p(Rz) - p(z)| \tag{23}$$

where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

The above lemma is due to Aziz and Shah [4].
The following lemma is due to Aziz and Rather [3].

Lemma 2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant t$, where $t \leqslant 1$, then*

$$|p(Rz) - p(z)| \geqslant \left(\frac{R^n - 1}{t^n}\right) \min_{|z|=t} |p(z)|, \qquad \text{for} \quad |z| = 1 \quad \text{and} \quad R \geqslant 1\,.$$

Lemma 3. *The function*

$$S(x) = k^{t+1} \left\{ \frac{\left(\frac{R^t - 1}{R^n - 1}\right) \left(\frac{|a_t|}{x}\right) k^{t-1} + 1}{\left(\frac{R^t - 1}{R^n - 1}\right) \left(\frac{|a_t|}{x}\right) k^{t+1} + 1} \right\}, \qquad R > 1,$$

is a non-decreasing function of x.

Proof of Lemma 3. The proof follows by considering the first derivative test for $S(x)$. ■

Lemma 4. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < k$ then $|p(z)| > m$ for $|z| < k$, and in particular*

$$|a_0| > m\,,$$

where $m = \min_{|z|=k} |p(z)|$.

The above lemma is due to Gardner, Govil and Musukula [7, Lemma 2.6], however for the sake of completeness we present the brief outline of the proof. For this, we can assume without loss of generality that $p(z)$ has no zeros on $|z| = k$, for otherwise the result holds trivially. Since $p(z)$, being a polynomial, is analytic in $|z| \leqslant k$ and has no zeros in $|z| < k$, by the minimum modulus principle,

$$|p(z)| \geqslant m \qquad \text{for } |z| \leqslant k,$$

which in particular implies $|a_0| = |p(0)| > m$.

Lemma 5. *If $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$ and $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$ and $R > 1$*

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz)-p(z)| \leqslant |q(Rz)-q(z)| - (R^n-1)m\,, \tag{24}$$

where $m = \min\limits_{|z|=k} |p(z)|$.

Proof of Lemma 5. Since $m \leqslant |p(z)|$ for $|z| = k$.

Hence, it follows by Rouche's Theorem that for $m > 0$ and for every complex number α with $|\alpha| \leqslant 1$, the polynomial $h(z) = p(z) - \alpha m$ does not vanish in $|z| < k$.

Applying Lemma 1 to the polynomial $h(z) = p(z) - \alpha m$, we get for every complex number α with $|\alpha| \leqslant 1$,

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0-\alpha m|}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0-\alpha m|}k^{t+1}+1}\right\}|p(Rz)-p(z)| \leqslant |q(Rz)-q(z)-m\bar{\alpha}(R^n-1)z^n| \tag{25}$$

for $|z| = 1$ and $R > 1$. Since for every α, $|\alpha| \leqslant 1$ we have

$$|a_0 - \alpha m| \geqslant |a_0| - |\alpha| m \geqslant |a_0| - m \tag{26}$$

and $|a_0| > m$ by Lemma 4, we get on combining (25), (26) and Lemma 3 that for every α where $|\alpha| \leqslant 1$,

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz)-p(z)| \leqslant |q(Rz)-q(z)-m\bar{\alpha}(R^n-1)z^n| \tag{27}$$

for $|z| = 1$ and $R > 1$.

Also all the zeros of $q(z)$ lie in $|z| \leqslant \dfrac{1}{k} \leqslant 1$, it follows by Lemma 2 (with $p(z)$ replaced by $q(z)$ and t by $1/k$) that

$$|q(Rz) - q(z)| \geqslant (R^n - 1)k^n \min_{|z|=\frac{1}{k}} |q(z)|.$$

But

$$\min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)|,$$

therefore, we have

$$|q(Rz) - q(z)| \geqslant (R^n - 1)m \qquad \text{for } \ |z| = 1 \ \text{ and } \ R > 1\,. \tag{28}$$

Now choosing argument of α with $|\alpha| = 1$ on the right hand side of (27) such that for $|z| = 1$ and $R > 1$,

$$|q(Rz) - q(z) - m\bar{\alpha}(R^n - 1)z^n| = |q(Rz) - q(z)| - (R^n - 1)m$$

which is possible by (28), we conclude that

$$k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p(Rz) - p(z)| \leqslant |q(Rz) - q(z)| - (R^n - 1)m$$

for $|z| = 1$ and $R > 1$, which is inequality (24) and that proves Lemma 5 completely. ■

Lemma 6. *If $p(z) = a_0 + \sum\limits_{v=t}^{n} a_v z^v$, $t \geqslant 1$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$ and $m = \min\limits_{|z|=k} |p(z)|$, then*

$$\psi_0(R) = k^{t+1}\left\{\frac{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\dfrac{R^t-1}{R^n-1}\right)\dfrac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\} \geqslant k^t\,, \qquad R > 1\,.$$

Proof of Lemma 6. Since, we have

$$\frac{R^t - 1}{R^n - 1} \leqslant \frac{t}{n} \tag{29}$$

holds for all $R > 1$ and $1 \leqslant t \leqslant n$ by considering the first derivative test for the function $\varphi(R) = nR^t - tR^n$.

Also, we have by an inequality (see [8, Proof of Lemma 3]),

$$\frac{|a_t|k^t}{|a_0|-m} \leqslant \frac{n}{t}\,, \qquad t \geqslant 1\,. \tag{30}$$

Considering (29) and (30), we get

$$\frac{|a_t|k^t}{|a_0|-m} \leqslant \frac{R^n-1}{R^t-1}\,.$$

The above inequality is clearly equivalent to

$$\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|k^t}{|a_0|-m}(k-1) \leqslant (k-1)\,,$$

which implies

$$\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|k^{t+1}}{|a_0|-m}+1 \leqslant \left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|k^t}{|a_0|-m}+k\,,$$

from which Lemma 6 follows. ■

Lemma 7. *If A, B and C are non-negative real numbers such that $B+C \leqslant A$, then for every real number α,*

$$|(A-C)e^{i\alpha}+(B+C)| \leqslant |Ae^{i\alpha}+B|\,.$$

Lemma 8. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then for each $r>0$, $R \geqslant 1$ and α real,*

$$\left\{\int_0^{2\pi} |(p(Re^{i\theta})-p(e^{i\theta}))+e^{i\alpha}\left(R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right)|^r d\theta\right\}^{\frac{1}{r}} \leqslant (R^n-1)\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}.$$

The result is best possible and equality holds for $p(z)=\lambda z^n+\mu$, $|\lambda|=|\mu|$.

The above two lemmas are due to Aziz and Rather [3].

3. Proofs of the Theorems

Proof of Theorem 2. Since $p(z) \neq 0$ in $|z|<k$, $k \geqslant 1$, therefore, by Lemma 5, for each θ, $0 \leqslant \theta < 2\pi$ and $R>1$, we have

$$\psi_0(R)|p(Re^{i\theta})-p(e^{i\theta})| \leqslant \left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right| - m(R^n-1),$$

where $\psi_0(R)$ is as defined in inequality (19).

This implies

$$\psi_0(R)\left\{|p(Re^{i\theta})-p(e^{i\theta})|+\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\right\} \leqslant \left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|-\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\,. \tag{31}$$

Taking $A=\left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|$, $B=|p(Re^{i\theta})-p(e^{i\theta})|$ and $C=\left(\frac{R^n-1}{1+\psi_0(R)}\right)m$ in Lemma 7 and noting by Lemma 6 that $\psi_0(R)\geqslant k^t\geqslant 1$,

$$B+C\leqslant \psi_0(R)(B+C)\leqslant A-C\leqslant A\,,$$

we get for every real α,

$$\left|\left\{\left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|-\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\right\}e^{i\alpha} + \left\{|p(Re^{i\theta})-p(e^{i\theta})|+\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\right\}\right| \leqslant \left|\left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|e^{i\alpha}+|p(Re^{i\theta})-p(e^{i\theta})|\right|.$$

This implies for each $r>0$,

$$\int_0^{2\pi}|F(\theta)+e^{i\alpha}G(\theta)|^r d\theta \leqslant \int_0^{2\pi}\left|\left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|e^{i\alpha}+|p(Re^{i\theta})-p(e^{i\theta})|\right|^r d\theta, \tag{32}$$

where

$$F(\theta)=|p(Re^{i\theta})-p(e^{i\theta})|+\left(\frac{R^n-1}{1+\psi_0(R)}\right)m$$

and

$$G(\theta)=\left|R^n p\left(\frac{e^{i\theta}}{R}\right)-p(e^{i\theta})\right|-\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\,.$$

Integrating both sides of (32) with respect to α from 0 to 2π, we get with the help of Lemma 8, for each $r > 0$, $R > 1$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r d\theta d\alpha \\
&\quad \leqslant \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|R^n p\left(\frac{e^{i\theta}}{R}\right) - p(e^{i\theta})\right|e^{i\alpha} + |p(Re^{i\theta}) - p(e^{i\theta})|\right|^r d\alpha\right\}d\theta \\
&\quad = \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(R^n p\left(\frac{e^{i\theta}}{R}\right) - p(e^{i\theta})\right)e^{i\alpha} + \left(p(Re^{i\theta}) - p(e^{i\theta})\right)\right|^r d\alpha\right\}d\theta \\
&\quad = \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left(R^n p\left(\frac{e^{i\theta}}{R}\right) - p(e^{i\theta})\right)e^{i\alpha} + \left(p(Re^{i\theta}) - p(e^{i\theta})\right)\right|^r d\theta\right\}d\alpha \\
&\quad \leqslant (R^n - 1)^r \int_0^{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^r d\theta d\alpha \\
&\quad = 2\pi(R^n - 1)^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta\,. \qquad (33)
\end{aligned}$$

Now for every real α and $t_1 \geqslant t_2 \geqslant 1$, we have

$$|t_1 + e^{i\alpha}| \geqslant |t_2 + e^{i\alpha}|,$$

which implies for every $r > 0$

$$\int_0^{2\pi} |t_1 + e^{i\alpha}|^r d\alpha \geqslant \int_0^{2\pi} |t_2 + e^{i\alpha}|^r d\alpha\,.$$

If $F(\theta) \neq 0$, we take $t_1 = \left|\dfrac{G(\theta)}{F(\theta)}\right|$ and $t_2 = \psi_0(R)$, then from (31) and noting by Lemma 6 that $\psi_0(R) \geqslant 1$, we have $t_1 \geqslant t_2 \geqslant 1$, hence

$$\begin{aligned}
\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^r d\alpha &= |F(\theta)|^r \int_0^{2\pi}\left|1 + \frac{G(\theta)}{F(\theta)}e^{i\alpha}\right|^r d\alpha \\
&= |F(\theta)|^r \int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)} + e^{i\alpha}\right|^r d\alpha \\
&= |F(\theta)|^r \int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right| + e^{i\alpha}\right|^r d\alpha \\
&\geqslant |F(\theta)|^r \int_0^{2\pi} |\psi_0(R) + e^{i\alpha}|^r d\alpha \\
&= \left\{|p(Re^{i\theta}) - p(e^{i\theta})| + \left(\frac{R^n - 1}{1 + \psi_0(R)}\right)m\right\}^r \\
&\qquad \times \int_0^{2\pi} |\psi_0(R) + e^{i\alpha}|^r d\alpha\,.
\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in (33), we conclude that for each $r>0$, $R>1$,

$$\int_0^{2\pi} |\psi_0(R)+e^{i\alpha}|^r d\alpha \int_0^{2\pi} \left\{|p(Re^{i\theta})-p(e^{i\theta})|+\left(\frac{R^n-1}{1+\psi_0(R)}\right)m\right\}^r d\theta \leqslant 2\pi(R^n-1)^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta\,. \qquad (34)$$

Now using the fact that for every complex number β with $|\beta|\leqslant 1$,

$$\left|p(Re^{i\theta})-p(e^{i\theta})+\beta m\left(\frac{R^n-1}{1+\psi_0(R)}\right)\right| \leqslant |p(Re^{i\theta})-p(e^{i\theta})|+m\left(\frac{R^n-1}{1+\psi_0(R)}\right),$$

the desired result follows from (34). ■

Remark 1. If we divide both sides of (24) by $R-1$ and let $R\to 1$, we get

$$k^{t+1}\left\{\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right\}|p'(z)| \leqslant |q'(z)|-mn\,. \qquad (35)$$

This inequality was also recently proved by Gardner, Govil and Weems [8, Lemma 8].

Remark 2. The proof of Theorem 1 follows along the lines of the proof of Theorem 2, by applying inequality (35) instead of Lemma 5.

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