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EXPONENTIAL SUMS AND THE ABELIAN GROUP PROBLEM

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Abstract: We give new estimates for multiple exponential sums, which infers

$$A(x) = C_1x + C_2x^{\frac{1}{2}} + C_3x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}}e^{V(x)}\right), \qquad V(x) = \frac{1}{\sqrt{3}}(L\log L)^{\frac{1}{2}} + O\left((L\log L)^{\frac{1}{2}}\right),$$

where $L = \log x$, $A(x)$ is the number of non-isomorphic abelian groups of orders $\leqslant x$, and x is large.

Keywords: Abelian groups, exponential sums.

1. Introduction

For a positive integer n let $a(n)$ be the number of distinct abelian groups (up to isomorphism) of order n, and let

$$A(x) = \sum_{n\leqslant x} a(n),$$

where x is a large positive number. Recently in [RS] the authors deduced the following wonderful result:

$$A(x) = C_1x + C_2x^{\frac{1}{2}} + C_3x^{\frac{1}{3}} + \Delta(x), \qquad \Delta(x) \ll x^{\frac{1}{4}+\epsilon}, \tag{1.1}$$

where ϵ is any given small positive constants (as usual, C_1, C_2 and C_3 are the three standard constants). Previously, in 1993 in [L2] we showed the estimation:

$$\Delta(x) \ll x^{\frac{50}{199}+\epsilon}, \qquad \frac{50}{199} = 0.25125\ldots, \tag{1.2}$$

and in 2000 in [SW], the authors improved the exponent of (1.2) to $\frac{55}{219} = 0.25114\ldots$ by using an ingenious result of [BS]. Stimulated by works of Bombieri and Iwaniec [BI], and [W], the new idea of [RS] is to give up the use of the Weyl's inequality of van der Corput's method, and instead it uses only the Cauchy's

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inequality (and then it reduces the related "spacing problem" to a problem of evaluating an integration). In fact, just as we observed in [L2] (see p.296, the "Concluding remarks"), that if we use the Weyl's inequality, we can never attain the exponent $\frac{1}{4}$, because to study the corresponding "spacing problem" one must use the Taylor's expansion, which then requires that Q cannot be too large as compared with M.

In this paper, we are stimulated by the work of Vinogradov's mean-value theorem (see Chapter VI of [T]), to present a refined version of the work of [RS] for evaluating an integration, and from which we can deduce a new estimation for triple exponential sums with monomials.

Theorem 1.1. *Let* $M, M_1, N_1 \geqslant \frac{1}{2}$,

$$S = \sum_{m_1 \sim M_1} \sum_{n_1 \sim N_1} \sum_{m \in I} \phi_m \psi_{m_1 n_1} e\left(A m^\alpha m_1^\gamma n_1^\delta\right),$$

(as usual $e(\xi) = \exp(2\pi i \xi)$*), and* $t \sim T$ *means* $1 \leqslant t/T < 2$*) here* ϕ_m *is meaningful for* $m \sim M$*, and* $|\phi_m| \leqslant 1, |\psi_{m_1 n_1}| \leqslant 1, A\alpha\gamma\delta \neq 0, \alpha \neq 1, \alpha \neq 2, A, \alpha, \gamma, \delta$ *are real,* I *is an interval,* $I \subseteq [M, 2M)$*, and* I *may depend on both* m_1 *and* n_1*. Let* $F = |A| M^\alpha M_1^\gamma N_1^\delta, L = \log(9M)$*, and* $\mathcal{L} = \log(10 M_1 N_1)$*. Then, if* $F \gg 1$ *we have*

$$S \ll \left(M M_1 N_1 F^{-1/4} + \sqrt[4]{M^4 (M_1 N_1)^3} + M_1 N_1 M^{1/2} + \sqrt[4]{(M_1 N_1)^3 F M^2}\right) e^{\Phi(M)} \mathcal{L},$$

where $\Phi(M) = (L \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}} (\log L)^{-\frac{1}{2}}\right)$.

Our Theorem 1.1 is an improvement of Theorem 1 of [RS].

Let $\tau(1,2,3;n) = \left|\{(n_1, n_2, n_3) \mid n = n_1 n_2^2 n_3^3,\ n_i (1 \leqslant i \leqslant 3) \text{ be positive integers}\}\right|$. In 1968, P.G.Schmidt [S1] first got the symmetric version for the error term in the asymptotic formula:

$$\sum_{n \leqslant x} \tau(1,2,3;n) = c_1 x + c_2 x^{\frac{1}{2}} + c_3 x^{\frac{1}{3}} + \Delta_1(x), \tag{1.3}$$

where

$$c_1 = \zeta(2)\zeta(3), \qquad c_2 = \zeta\left(\frac{1}{2}\right)\zeta\left(\frac{3}{2}\right), \qquad c_3 = \zeta\left(\frac{1}{3}\right)\zeta\left(\frac{2}{3}\right).$$

Using Schmidt's formula, from Theorem 1.1 we can deduce the following:

Theorem 1.2. *For* $x > 10, L = \log x$*, one has the bound*

$$\Delta_1(x) = O\left(x^{\frac{1}{4}} e^{R(x)}\right), \quad \text{with} \quad R(x) = \frac{1}{\sqrt{3}} (L \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}} (\log L)^{-\frac{1}{2}}\right).$$

In [S1], Schmidt showed that for all $\beta > \frac{1}{4}$,

$$\Delta_1(x) \ll x^\beta \Rightarrow \Delta(x) \ll x^\beta,$$

where $\Delta(x)$ and $\Delta_1(x)$ are given by (1.1) and (1.3) respectively. Along the similar line of approach, from Theorem 1.2 we can deduce the following essentially new bound for $\Delta(x)$:

Theorem 1.3. *For the abelian group problem, we have for $x > 10$,*

$$\Delta(x) = O\left(X^{\frac{1}{4}} e^{V(x)}\right),$$

where $V(x) = \frac{1}{\sqrt{3}}(L \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}}(\log L)^{-\frac{1}{2}}\right)$, $L = \log x$.

We should add a few more words here to clarify the historical background for the study of the abelian group problem. B.R.Srinivasan's work of [Sr] was untenable, because Srinivasan's theory of the so-called "two dimensional exponent pairs" was suspected by P.G.Schmidt, see the paper [S2] of Schmidt. The result of G.Kolesnik [K] was also untenable, because although we can remedy the proof of Theorem 1 of [K] dy developing some new techniques (see [L4]), we are unable to remedy the proof of the last bound of S_0 of Lemma 6 of [K]. Our works of [L1] and [L2] are tenable, for although they used Theorem 1 of [K], for the statement and the proof of Theorem 1 of [K] have been corrected in our recent paper [L4].

2. Preliminaries

We prove several lemmas in this section.

Lemma 2.1. *Let $\mathcal{A}$ be a set of finitely many distinct positive numbers, A_1 and A_2 are two positive constants, such that if $n \in \mathcal{A}$ then $A_1 \leqslant n/N < A_2$. Let $\omega(u)$ be a real valued function with positive arguments, let $\delta > 0$, and $V(\mathcal{A}, \delta)$ denote the number of ordered (u_1, u_2, v_1, v_2)'s of the four dimensional euclidean space such that*

$$\left| \sum_{1 \leqslant m \leqslant 2} (\omega(u_m) - \omega(v_m)) \right| \leqslant \delta, \qquad \text{and} \qquad u_1, u_2, v_1, v_2 \in \mathcal{A}.$$

Let

$$T(\mathcal{A}, x) = \sum_{u \in \mathcal{A}} u^{-\frac{1}{2}} e(\omega(u)x), \qquad W(\mathcal{A}, \delta) = \int_0^D |T(\mathcal{A}, x)|^4 \, dx,$$

where $D = (2\delta)^{-1}$.

(i) *We have*

$$8A_1^2 \pi^{-2} N^2 \delta W(\mathcal{A}, \delta) \leqslant V(\mathcal{A}, \delta) \leqslant \pi^2 A_2^2 N^2 \delta W(\mathcal{A}, \delta).$$

(ii) *If $\mathcal{A} \subseteq \mathcal{A}', \delta' \geqslant \delta > 0$, and $D' = (2\delta')^{-1}$, then (for $\mathcal{A}', n \in \mathcal{A}' \Rightarrow n/N \in [A_1, A_2)$)*

$$8A_1^2 \delta W(\mathcal{A}, \delta) \leqslant \pi^4 A_2^2 \delta' W(\mathcal{A}', \delta').$$

(iii) *Let h_u be any complex numbers with $|h_u| \leqslant 1$, then*

$$V(\mathcal{A}, \delta) \geqslant 8A_1^2 N^2 \delta \pi^{-2} \left(\int_0^D \left| \sum_{u \in \mathcal{A}} u^{-1/2} h_u e(x\omega(u)) \right|^4 dx \right).$$

Proof. (i) can be proven similarly with the proof of Lemma 2.1 of [W]. Using (i) and the obvious inequality $V(\mathcal{A},\delta) \leqslant V(\mathcal{A},\delta') \leqslant V(\mathcal{A}',\delta')$, we can get (ii). Using again the method of showing (i), but in a more careful manner, we can get (iii) (see also (2.3) on p.5 of [RS]). ■

Lemma 2.2. *Let $M > 0, c \geqslant 2, M \leqslant N_1 < N_2 \leqslant cM, z_m$ be complex numbers. Then*

$$\left|\sum_{N_1<m<N_2} z_m\right|^4 \leqslant (1+\log(cM+1))^3 \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{M\leqslant m<cM} z_m e(mt)\right|^4 \mathcal{L}(t)dt\right),$$

where $\mathcal{L}(t) = \min\left(cM, \frac{1}{2|t|}\right)$.

Proof. It can be proven similarly with Lemma 2 of [RS]. ■

Lemma 2.3. *Let $2X \geqslant X_1 > 0, X \geqslant Y > 0, M > 0, c \geqslant 2, |a_m| \leqslant 1, \phi(m)$ is a real valued function with natural number arguments, $M_i(x) = x^h K_i$, here h, K_1 and K_2 do not depend on x $(i = 1,2)$, and when $X_1 \leqslant x \leqslant 2X$ holds $M \leqslant M_1(x) < M_2(x) \leqslant cM$, then*

$$\int_{X_1}^{2X} \left|\sum_{M_1(x)<m<M_2(x)} a_m m^{-\frac{1}{2}} e(x\phi(m))\right|^4 dx$$
$$\leqslant \frac{\pi^8 c^4}{64}\cdot\frac{X}{Y}(1+\log(cM+1))^4 \left(\int_0^{2Y} \left|\sum_{M\leqslant m<cM} m^{-\frac{1}{2}} e(x\phi(m))\right|^4 dx\right).$$

Proof. It can be proven similarly with Lemma 3 of [RS], by using instead our Lemmas 1 and 2. ■

Lemma 2.4. *For a real number α with $\alpha \neq 0$, $\alpha \neq 1$, there is a positive number $c(\alpha)$, such that*

$$\int_0^{M^{3/2}} \left|\sum_{M\leqslant m<2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \leqslant c(\alpha)M^2.$$

Proof. It can be proven similarly with Lemma 5 of [RS], by adding one more step to the partial summation. ■

3. Estimation of integration

The purpose of this section is to deduce the following important theorem by using the mathematical induction.

Theorem 3.1. *Let $M \geqslant 1$, $0 < \epsilon \leqslant 1$, $\alpha \neq 0$, $\alpha < 1$, then*

$$\int_0^{M^{2-\epsilon}} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right) \right|^4 dx \leqslant M^2 (P(\alpha))^{6[1/\epsilon]} (\log(Q(M,\alpha)))^{4[1/\epsilon]}, \tag{3.1}$$

where $[1/\epsilon]$ is the largest number not exceeding $1/\epsilon$,

$$Q(M,\alpha) = 2\left(|\alpha| + \left|\frac{\alpha}{\alpha-1}\right| + 1\right) M + e^3,$$

$$P(\alpha) = \begin{cases} 8 + c(\alpha) + c\left(\dfrac{\alpha}{\alpha-1}\right) + \lambda_{10}(\alpha) \\ \quad + \widetilde{\lambda_{10}}\left(\dfrac{\alpha}{\alpha-1}\right) + \mu(\alpha) + \widetilde{\mu}\left(\dfrac{\alpha}{\alpha-1}\right), & \text{if } 0 < \alpha < 1, \\ 8 + c(\alpha) + c\left(\dfrac{\alpha}{\alpha-1}\right) + \widetilde{\lambda_{10}}(\alpha) \\ \quad + \lambda_{10}\left(\dfrac{\alpha}{\alpha-1}\right) + \widetilde{\mu}(\alpha) + \mu\left(\dfrac{\alpha}{\alpha-1}\right), & \text{if } \alpha < 0, \end{cases}$$

where $c(\alpha)$ is the positive constant of Lemma 4, $\mu(\xi)$ and $\lambda_{10}(\xi)$ are the positive functions defined for $\xi \in (0,1)$, their expressions can actually be written down explicitly, and similarly $\widetilde{\mu}(\xi)$ and $\widetilde{\lambda_{10}}(\xi)$ are the positive functions defined for $\xi < 0$, their concrete values will be clear in the context of our following arguments. (note that $P(\alpha) = P(\alpha/(\alpha-1))$, and $Q(M,\alpha) = Q(M, \alpha/(\alpha-1))$). (we think that it is unnecessary to give the precise expressions of $c(\xi)$, $\mu(\xi)$, $\widetilde{\mu}(\xi)$, $\lambda_{10}(\xi)$ and $\widetilde{\lambda_{10}}(\xi)$ here, which is possible)

Proof. For any $\epsilon \in (0,1]$, there is the unique positive integer N for which $\epsilon \in \left(\frac{1}{N+1}, \frac{1}{N}\right]$. We shall use the mathematical induction for N, to show that the estimation (3.1) is true for any $\epsilon \in (0,1]$, any $M \geqslant 1$, and any $\alpha < 1$ ($\alpha \neq 0$). For $N = 1$, $\epsilon \in \left(\frac{1}{2}, 1\right]$, by Lemma 4 we see that (3.1) is true for any $M \geqslant 1$ and $0 \neq \alpha < 1$. Assume that (3.1) is true, when $\epsilon \in \left(\frac{1}{k+1}, \frac{1}{k}\right]$ ($k \geqslant 1$), with arbitrary $M \geqslant 1$ and any $\alpha < 1$ ($\alpha \neq 0$). Then, to complete the induction, we must show that (3.1) is true also whenever $\epsilon \in \left(\frac{1}{k+2}, \frac{1}{k+1}\right]$, with any $M \geqslant 1$ and any $\alpha < 1$ ($\alpha \neq 0$). Denote the left hand side of (3.1) as $S(M,\xi,\alpha)$. Let $\delta = \epsilon/(1-\epsilon)$, then $\delta \in \left(\frac{1}{k+1}, \frac{1}{k}\right]$. We have

$$S(M,\epsilon,\alpha) = S(M,\delta,\alpha) + \int_{M^{2-\delta}}^{M^{2-\epsilon}} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right) \right|^4 dx. \tag{3.2}$$

Since $k \leqslant \delta^{-1} < k+1$, thus $\left[\delta^{-1}\right] = k$. By the inductive hypothesis we get

$$S(M,\delta,\alpha) \leqslant M^2 (P(\alpha))^{6k} (\log Q(M,\alpha))^{4k}. \tag{3.3}$$

Let $I = \left[\left(\log M^{\delta-\epsilon}\right)/\log 2\right)]$ $(\geqslant 0)$, then $2^I M^{2-\delta} \leqslant M^{2-\epsilon} < 2^{I+1}M^{2-\delta}$. Let $X_i = 2^i M^{2-\delta}$, where $0 \leqslant i \leqslant I$, then

$$\int_{M^{2-\delta}}^{M^{2-\epsilon}} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx$$
$$\leqslant \sum_{0 \leqslant i \leqslant I} \left(\int_{X_i}^{2X_i} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \right). \quad (3.4)$$

For each i $(0 \leqslant i \leqslant I)$, write $X_i = X$ for simplicity, and denote the integration inside the summation of the right hand side of (3.4) as $T(X, M, \alpha)$. For the technical reason of using Theorem 3 of [L3], we shall consider respectively the cases for $\alpha \in (-\infty, 0)$ and $\alpha \in (0,1)$. First, consider the case of $\alpha \in (0,1)$. When $X \leqslant x \leqslant 2X$, by Theorem 3 of [L3] we get

$$\sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right) = \left(-i\left(1-\alpha\right)^{\frac{1}{2}}\right)\left(\sum_{V_1 < v < V_2} v^{-\frac{1}{2}} e(g(x,v,M))\right) + E_1 + E_2 + E_3 \quad (3.5)$$

where

$$\begin{aligned} V_1 &= V_1(x,\alpha,M) = \alpha 2^{\alpha-1} x M^{-1}, \\ V_2 &= V_2(x,\alpha,M) = \alpha x M^{-1}, \\ g(x,v,M) &= \left(\alpha^{-1}-1\right)\left(\alpha x M^{-\alpha}\right)^{1/(\alpha-1))} v^{\alpha/(\alpha-1))}, \\ |E_1| + |E_2| &\leqslant c_1(\alpha)\left(MX^{-1}\right)^{\frac{1}{2}}, \\ |E_3| &\leqslant c_2(\alpha) M^{-\frac{1}{2}} \log(3+M) \end{aligned}$$

(we use $c_i(\alpha)$ to denote certain positive constants which depend on α). Let $L = X/M$, then $1 \leqslant M^{1-\delta} \leqslant L \leqslant M^{1-\epsilon}$. We have $g(x) = \tau(x)\left(\frac{v}{L}\right)^{\beta}$, here $\beta = \alpha/(\alpha-1)$, and

$$\tau(x) = \left(\alpha^{-1}-1\right)(\alpha x)^{1/(1-\alpha)} x^{\beta}. \quad (3.6)$$

By Hölder's inequality we have

$$\left(\sum_{1 \leqslant i \leqslant 4} |\Phi_i|\right)^4 \leqslant \left(\sum_{1 \leqslant i \leqslant 4} 1\right)^3 \left(\sum_{1 \leqslant i \leqslant 4} |\Phi_i|^4\right), \quad (3.7)$$

thus from (3.5) we get

$$\left|\sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 \leqslant \lambda_1(\alpha) \left|\sum_{V_1 < v < V_2} v^{-\frac{1}{2}} e\left(\tau(x)\left(\frac{v}{L}\right)^{\beta}\right)\right|^4 + \lambda_2(\alpha) L^{-2} + \lambda_3(\alpha) M^{-2} (\log(3+M))^4,$$

and thus

$$T(X,M,\alpha) \leqslant \lambda_1(\alpha) \int\limits_X^{2X} \left| \sum_{V_1<v<V_2} v^{-\frac{1}{2}} e\left(\tau(x)\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dx \\ + \lambda_2(\alpha) X L^{-2} + \lambda_3(\alpha) X M^{-2} (\log(3+M))^4 \tag{3.8}$$

($\lambda_i(\alpha)$ are indeed explicite positive constants depending at most on α). We make the variable substitution $\tau(x) = y$ in the integration of the right side of (3.8), and since we have $dy = \alpha^{-\beta}\,(X/x)^{\beta}\,dx$ by (3.8), we get

$$\int\limits_X^{2X} \left| \sum_{V_1<v<V_2} v^{-\frac{1}{2}} e\left(\tau(x)\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dx \\ \leqslant \lambda_4(\alpha) \int\limits_{Y_2(\alpha)}^{Y_1(\alpha)} \left| \sum_{W_1<v<W_2} v^{-\frac{1}{2}} e\left(y\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dy, \tag{3.9}$$

where $Y_1(\alpha) = \left(\alpha^{-1}-1\right)(2\alpha)^{1/(1-\alpha)}X$, $Y_2(\alpha) = \left(\alpha^{-1}-1\right)\alpha^{1/(1-\alpha)}X$ and $W_1 = W_1(y) = c_3(\alpha) X^{\alpha} M^{-1} y^{1-\alpha}$, $W_2 = W_2(y) = c_3(\alpha) 2^{1-\alpha} X^{\alpha} M^{-1} y^{1-\alpha}$ with $c_3(\alpha) = 2^{\alpha-1}\left(\alpha^{-1}-1\right)^{\alpha-1}$. Since

$$W_1 \geqslant c_3(\alpha) X^{\alpha} M^{-1} (Y_2(\alpha))^{1-\alpha} = \alpha 2^{\alpha-1} L,$$
$$W_2 \leqslant c_3(\alpha) 2^{1-\alpha} X^{\alpha} M^{-1} (Y_1(\alpha))^{1-\alpha} = 2\alpha L \leqslant \alpha 2^{\alpha+1} L,$$

if we set $Y_1(\alpha) = 2X$ (note that this "X" is not the one appearing in (3.8) and (3.9)), $X_1 = Y_2(\alpha)$, $c_4(\alpha) = \frac{1}{2}\min\left(\left(\alpha 2^{\alpha-1}\right)^2, \left(\alpha^{-1}-1\right)(2\alpha)^{1/(1-\alpha)}\right)$, $Y = c_4(\alpha) L^{2-\delta}$, $V = \alpha 2^{\alpha-1} L$ and $c = 4$ in Lemma 2.4, then from $X \geqslant Y$ (by $M^{2-\epsilon} \geqslant X$ we get $X \geqslant L^{2-\delta}$) we have

$$\int\limits_{Y_2(\alpha)}^{Y_1(\alpha)} \left| \sum_{W_1<v<W_2} v^{-\frac{1}{2}} e\left(y\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dy \\ \leqslant \lambda_5(\alpha)(\log(2\alpha L+3))^4 X L^{\delta-2} \left(\int\limits_0^{2Y} \left| \sum_{V\leqslant v<4V} v^{-\frac{1}{2}} e\left(y\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dy \right). \tag{3.10}$$

By the choice of the parameters we get $2Y \leqslant V^{2-\delta}$. Thus using Hölder's inequality (similarly with (3.7)) we get

$$\int\limits_0^{2Y} \left| \sum_{V\leqslant v<4V} v^{-\frac{1}{2}} e\left(y\left(\frac{v}{L}\right)^{\beta}\right) \right|^4 dy \leqslant 8\left(S\left(V,\delta,\frac{\alpha}{\alpha-1}\right) + S\left(2V,\delta,\frac{\alpha}{\alpha-1}\right)\right). \tag{3.11}$$

If $V<1$, then obviously (using the trivial estimate)

$$S\left(V,\delta,\frac{\alpha}{\alpha-1}\right)+S\left(2V,\delta,\frac{\alpha}{\alpha-1}\right)\leqslant 8\cdot 3^4. \tag{3.12}$$

If $V\geqslant 1$, then by the fact $\delta\in\left(\frac{1}{k+1},\frac{1}{k}\right]$ and inductive hypothesis we get (note that here $\frac{\alpha}{\alpha-1}<0$)

$$S\left(V,\delta,\frac{\alpha}{\alpha-1}\right)\leqslant V^2\left(P\left(\frac{\alpha}{\alpha-1}\right)\right)^{6k}\left(\log Q\left(V,\frac{\alpha}{\alpha-1}\right)\right)^{4k}, \tag{3.13}$$

$$S\left(2V,\delta,\frac{\alpha}{\alpha-1}\right)\leqslant 2V^2\left(P\left(\frac{\alpha}{\alpha-1}\right)\right)^{6k}\left(\log Q\left(2V,\frac{\alpha}{\alpha-1}\right)\right)^{4k}. \tag{3.14}$$

If $2V\geqslant M$, then from (see the arguments after (3.5))

$$M^{1-\delta}\leqslant\frac{X}{M}=L\leqslant M^{1-\epsilon},\qquad V=\alpha\cdot 2^{\alpha-1}L,$$

it follows that $\alpha\cdot 2^\alpha\cdot M^{1-\epsilon}\geqslant M$, and thus $M^\epsilon\leqslant\alpha\cdot 2^\alpha$, and the trivial estimate gives (for the definition of $T(M,\delta,\alpha)$, see (3.8) and (3.4))

$$\begin{aligned}T(M,\delta,\alpha)&\leqslant X(M+1)^4M^{-2}\leqslant\left((\alpha 2^\alpha)^{1/\epsilon}+1\right)^4\leqslant(2\alpha\cdot 2^\alpha+2)^{4/\epsilon}\\&\leqslant\left(\alpha 2^{\alpha+1}+2\right)^{6[1/\epsilon]}\leqslant(P(\alpha))^{6[1/\epsilon]}.\end{aligned} \tag{3.15}$$

(here we can show a bit more by distinguishing the cases $2^\alpha\cdot\alpha\geqslant 1$ or $2^\alpha\cdot\alpha<1$, and choose $\mu(\alpha)=\alpha\cdot 2^{\alpha+1}+2$). If $2V<M$, then from

$$Q\left(tV,\frac{\alpha}{\alpha-1}\right)=Q(tV,\alpha)\quad(t=1,2),\qquad P(\alpha)=P\left(\frac{\alpha}{\alpha-1}\right),$$

(by the definitions), we can deduce in view of (3.9) to (3.14) that (using $\log(2\alpha L+1)\leqslant\log(2\alpha M+1)\leqslant\log Q(M,\alpha)$)

$$\begin{aligned}&\int_X^{2X}\left|\sum_{V_1<v<V_2}v^{-\frac{1}{2}}e\left(\tau(x)\left(\frac{v}{L}\right)^\beta\right)\right|^4dx\\&\quad\leqslant\lambda_6(\alpha)(\log Q(M,\alpha))^4XL^{\delta-2}V^2(P(\alpha))^{6k}(\log Q(M,\alpha))^{4k}\\&\qquad+\lambda_7(\alpha)(\log Q(M,\alpha))^4XL^{\delta-2}\\&\quad\leqslant\lambda_8(\alpha)\left(X^{1+\delta}M^{-\delta}+M\right)(P(\alpha))^{6k}(\log Q(M,\alpha))^{4(k+1)}.\end{aligned} \tag{3.16}$$

Since $M^{2-\delta}\leqslant X\leqslant M^{2-\epsilon}, L=X/M$, it is easy to verify that $X^{1+\delta}M^{-\delta}\geqslant XL^{-2}$ and $M\geqslant XM^{-2}$, and thus from (3.16) and (3.8) we get

$$T(X,M,\alpha)\leqslant\lambda_9(\alpha)\left(X^{1+\delta}M^{-\delta}+M\right)(P(\alpha))^{6k}(\log Q(M,\alpha))^{4(k+1)}. \tag{3.17}$$

From (3.15) and $X^{1+\delta}M^{-\delta} = XL^{\delta} \geqslant 1$, we see that (3.17) holds also if $2V \geqslant M$ (obviously, we can take $\lambda_9(\alpha) > 1$). Using $X = X_i = M^{2-\delta}2^i$ $(0 \leqslant i \leqslant I)$ to make summation for i on both sides of (3.17), and noting that

$$\begin{aligned}\sum_i X^{1+\delta} &= \left(2^{(1+\delta)(1+I)} - 1\right)\left(2^{1+\delta} - 1\right)^{-1}\left(M^{2-\delta}\right)^{1+\delta} \\ &\leqslant M^{(2-\delta)(1+\delta)}2^{(1+\delta)(1+I)} \\ &\leqslant M^{(2-\delta)(1+\delta)}2^{(1+\delta)\left(1+\left(\log M^{\delta-\epsilon}\right)/(\log 2)\right)} \\ &\leqslant 4M^{(\delta-\epsilon)(1+\delta)+(2-\delta)(1+\delta)} = 4M^{(2-\epsilon)(1+\delta)},\end{aligned}$$

and

$$M^{(2-\epsilon)(1+\delta)}M^{-\delta} = M^2,$$

we deduce from (3.17) and (3.4) that

$$\begin{aligned}\int_{M^{2-\delta}}^{M^{2-\epsilon}} \left|\sum_{M\leqslant m<2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \\ \leqslant \lambda_{10}(\alpha)M^2(P(\alpha))^{6k}(\log Q(M,\alpha))^{4(k+1)}.\end{aligned} \tag{3.18}$$

From (3.2), (3.3) and (3.18), and the value of $P(\alpha)$, we obtain

$$S(M,\epsilon,\alpha) \leqslant M^2(P(\alpha))^{6(k+1)}(\log Q(M,\alpha))^{4(k+1)}. \tag{3.19}$$

Thus the estimation (3.1) is true also for $0 < \alpha < 1$ and $\epsilon \in \left(\frac{1}{k+2}, \frac{1}{k+1}\right]$.

When $\alpha < 0$ (then $\frac{\alpha}{\alpha-1} \in (0,1)$), we can proceed by the same manner, but replace the constants $c_i(\alpha)$ and $\lambda_j(\alpha)$ of (3.5) to (3.18) by the similar constants $\widetilde{c_i}(\alpha)$ and $\widetilde{\lambda_j}(\alpha)$ respectively (in fact all these constants can be chosen explicitly), and we can also obtain (3.19). Thus, the induction can always be completed. The proof of Theorem 3.1 is thus finished. ■

Theorem 3.2. *Let α be a real number, and $\alpha \neq 0, 1$. Then for all $M \geqslant 1$ one has*

$$\int_0^{M^2} \left|\sum_{M\leqslant m<2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \ll M^2 e^{F(M)},$$

where $F(m) = 4(\log 3M)^{\frac{1}{2}}(\log\log 3M)^{\frac{1}{2}}\left(1 + O\left(\frac{1}{\log\log 3M}\right)\right)$, and the constant implied by the "$\ll$" and the "O" symbols depending at most on α.

Proof. At first we assume that $\alpha \neq 0$ and $\alpha < 1$. For any $\epsilon \in \left(0, \frac{1}{2}\right)$, by (ii) of Lemma 2.1 and Theorem 3.1, we get

$$\begin{aligned}\int_0^{M^2} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \\ &\ll M^{\epsilon} \int_0^{M^{2-\epsilon}} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x\left(\frac{m}{M}\right)^{\alpha}\right)\right|^4 dx \\ &\leqslant M^{2+\epsilon} \left(\log\left(C(\alpha)M + e^3\right)\right)^{4/\epsilon} (P(\alpha))^{6/\epsilon},\end{aligned} \tag{3.20}$$

where "$\ll$" constant depends neither on α nor on $\epsilon, C(\alpha) = 2\left(|\alpha| + \left|\frac{\alpha}{\alpha-1}\right| + 1\right)$, and $P(\alpha)$ is a positive number depending only on α. Let M be large enough, such that

$$C(\alpha)M + e^3 \leqslant M^2, \tag{3.21}$$

thus we find out that the quantity of (3.20) does not exceed

$$(3M)^{2+\epsilon}(\log 3M)^{4/\epsilon}(2P(\alpha))^{6/\epsilon}. \tag{3.22}$$

For sufficiently large M, we have

$$\epsilon := 2\left(\frac{\log\log(3M)}{\log(3M)}\right)^{\frac{1}{2}} < \frac{1}{2},$$

and we find that out the quantity of (3.22) is

$$(3M)^2 \exp\left(4(L \cdot \log L)^{\frac{1}{2}} + O\left((L/\log L)^{\frac{1}{2}}\right)\right),$$

where $L = \log 3M$. Thus there is a number $M_0(\alpha)$ such that for $M \geqslant M_0(\alpha)$ the required bound holds. Of course, in case $M < M_0(\alpha)$ the required estimate holds trivially. Hence, the result is true when $\alpha < 1 \quad (\alpha \neq 0)$.

For $\alpha < 0$, we deduce similarly and obtain the result.

For $\alpha > 1$, we note that actually we can first prove a theorem which is similar to Theorem 3.1, and since $\frac{\alpha}{\alpha-1} > 1$ when $\alpha > 1$, the corresponding constant $P(\alpha)$ of (3.1) should be

$$P(\alpha) = 8 + c(\alpha) + c\left(\frac{\alpha}{\alpha-1}\right) + \mu(\alpha) + \mu\left(\frac{\alpha}{\alpha-1}\right) + \lambda_{10}(\alpha) + \lambda_{10}\left(\frac{\alpha}{\alpha-1}\right),$$

where the values of $c(\alpha), \mu(\alpha)$ and $\lambda_{10}(\alpha)$ can all be determined similarly as those appearing in the proof of Theorem 3.1; then, using such a theorem, we can deduce the conclusion of Theorem 3.2 similarly. ∎

4. Proof of Theorem 1.1

Let $\alpha \neq 0, 1$ (α is real), $\Delta > 0$, $t(M, \Delta)$ be the number of ordered lattice points (m_1, m_2, m_3, m_4) satisfying

$$|m_1^\alpha + m_2^\alpha - m_3^\alpha - m_4^\alpha| \leqslant \Delta M^\alpha, \qquad M \leqslant m_i < 2M \qquad (1 \leqslant i \leqslant 4).$$

In [RS] the authors showed that

$$t(M, \Delta) \ll \left(M^2 + \Delta M^4\right) M^\epsilon,$$

for any given small positive constant ϵ. Using our Theorem 3.2, we can improve their estimation.

Theorem 4.1. *For $\alpha \neq 0, 1, M \geqslant 1, \Delta > 0,$ we have*

$$t(M, \Delta) \ll \left(M^2 + \Delta M^4\right) e^{T(M)}, \tag{4.1}$$

where $T(M) = 4(L \cdot \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}} (\log L)^{-\frac{1}{2}}\right), L = \log 6M.$

Proof. By (i) of Lemma 2.1 we have

$$t\left(M, M^{-2}\right) \ll \int_0^{M^2} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x \left(\frac{m}{M}\right)^\alpha\right) \right|^4 dx. \tag{4.2}$$

If $\Delta \leqslant M^{-2}$, then from $t(M, \Delta) \leqslant t\left(M, M^{-2}\right)$, (4.2) and Theorem 3.2 we get (4.1). If $\Delta > M^{-2}$, then by (i) of Lemma 2.1 we have

$$\begin{aligned} t(M, \Delta) &\ll M^2 \Delta \int_0^{(2\Delta)^{-1}} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x \left(\frac{m}{M}\right)^\alpha\right) \right|^4 dx \\ &\leqslant M^2 \Delta \int_0^{M^2} \left| \sum_{M \leqslant m < 2M} m^{-\frac{1}{2}} e\left(x \left(\frac{m}{M}\right)^\alpha\right) \right|^4 dx. \end{aligned} \tag{4.3}$$

Using Theorem 3.2 and (4.3) we also get (4.1). Thus Theorem 4.1 is true. ■

Proof of Theorem 1.1. Suppose that $\min(M, M_1 N_1) \geqslant 100$. Let $L = \log 9M$. At first, similarly with the work of p.264 of [L1], by using Lemma 2.1 of [L1] we can get

$$L^{-1}|S| \ll \sum_{m_1 \sim M_1} \sum_{n_1 \sim N_1} \left| \sum_{m \sim M} \widetilde{\phi_m} e\left(A m_1^\gamma n_1^\delta m^\alpha\right) \right|,$$

where $\left|\widetilde{\phi_m}\right| \leqslant 1$. By Cauchy's inequality we have

$$L^{-2}|S|^2 \ll M_1N_1 \sum_{m_1\sim M_1} \sum_{n_1\sim N_1} \left| \sum_{m\sim M} \widetilde{\phi_m} e\left(Am_1^\gamma n_1^\delta m^\alpha\right) \right|^2$$
$$\ll M_1N_1\left(M_1N_1M + S_1\right),$$
$$S_1 = \sum_{m_1\sim M_1} \sum_{n_1\sim N_1} \sum_{\substack{m,\widetilde{m}\sim M\\ m>\widetilde{m}}} \widetilde{\phi}_m \overline{\widetilde{\phi}_{\widetilde{m}}} e\left(Am_1^\gamma n_1^\delta\left(m^\alpha - \widetilde{m}^\alpha\right)\right).$$

Let $X = C|A|M_1^\gamma N_1^\delta, Y = CM^\alpha, C$ is a sufficiently large constant (it may depend on γ, δ and α), such that $\left|Am_1^\gamma n_1^\delta\right| \leqslant X$ and $|m^\alpha - \widetilde{m}^\alpha| \leqslant Y$ always hold. By Lemma 4 of [L1] (which is the new type of the large sieve inequality) we get

$$|S_1|^2 \ll (XY+1)B_1B_2,$$

where B_1 is the number of lattice points $(m_1, \widetilde{m}_1, n_1, \widetilde{n}_1)$ such that

$$\left|Am_1^\gamma n_1^\delta - A\widetilde{m}_1^\gamma \widetilde{n}_1\right| \leqslant Y^{-1}, \qquad m_1, \widetilde{m}_1 \sim M_1, \qquad n_1, \widetilde{n}_1 \sim N_1,$$

and B_2 is the number of lattice points $(m, \widetilde{m}, n, \widetilde{n})$ satisfying

$$|m^\alpha - \widetilde{m}^\alpha - n^\alpha + \widetilde{n}^\alpha| \leqslant X^{-1}, \qquad m, \widetilde{m}, n, \widetilde{n} \sim M.$$

By Lemma 5 of [L1] we have

$$B_1 \ll \left(M_1N_1 + F^{-1}M_1^2N_1^2\right)\left(\log\left(10M_1N_1\right)\right)^2.$$

By our Theorem 4.1 (choosing $\Delta \approx F^{-1}$ in it) we have

$$B_2 \ll \left(M^2 + F^{-1}M^4\right)e^{T(M)}.$$

Since $XY \approx F \gg 1$, combining the above inequalities we get

$$|S_1|^2 \ll \left(M_1N_1M^2F + M_1N_1M^4 + \left(MM_1N_1\right)^2 + F^{-1}M_1^2N_1^2M^4\right)e^{T(M)}\mathcal{L}^2,$$

here $\mathcal{L} = \log\left(10M_1N_1\right)$. Put this bound into the inequality for $|S|^2$, we find that the required estimate for $|S|$ follows. The proof is finished. ∎

5. Proof of Theorem 1.2

By the preliminary work on p.266 of [L1], to prove our Theorem 1.2 it suffice to show that for each $H \in \left[\frac{1}{2}, x^2\right]$ we always have the estimate

$$\Phi(H,M,N) = H^{-1} \sum_{h\sim H} \left| \sum_{(m,n)\in D} e(hf(m,n)) \right|$$
$$\ll x^\theta \exp\left(\frac{1}{\sqrt{3}}\left(L\cdot\log L\right)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}}\left(\log L\right)^{-\frac{1}{2}}\right)\right), \tag{5.1}$$

where $\theta = \frac{1}{4}$, $f(m,n) = \left(xm^{-b}n^{-c}\right)^{1/a}$, (a,b,c) is an arbitrary permutation of $(1,2,3)$, and we have chosen $K = x$ in the inequality between (3.14) and (3.15) on p.266 of [L1], and M and N satisfy

$$MN > x^{\theta}, \qquad 2M \geqslant N \geqslant \frac{1}{2}, \qquad M^{a+b}N^{c} \leqslant x. \tag{5.2}$$

Let $G = \left(xM^{-b}N^{-c}\right)^{1/a}$. First, using our Theorem 1.1 directly to the triple sum of $\Phi(H,M,N)$ (recall that $D = D(M,N) = \{(m,n)|m \sim M, n \sim N, m^{a+b}n^{c} \leqslant x, m > n\}$), we get

$$L^{-1}\Phi(H,M,N) \ll \left(MN(HG)^{-1/4} + \sqrt[4]{M^4N^3H^{-1}} + M^{\frac{1}{2}} + \sqrt[4]{GM^2N^3}\right)e^{\Phi(M)}, \tag{5.3}$$

where $L = \log x$. From (5.2) we have $M \ll x^{1/3}$, thus

$$\Phi(M) \leqslant \frac{1}{\sqrt{3}}(L\cdot\log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}}(\log L)^{-\frac{1}{2}}\right). \tag{5.4}$$

Since $a+b+c = 6$, $a+b \geqslant 3$, from (5.2) we have $MN \ll \left(M^{a+b}N^{c}\right)^{1/3} \ll x^{1/3}$, and thus

$$M^{\frac{1}{2}}N \ll (MN)^{3/4} \ll x^{\theta}, \quad M^{2a-b}N^{3a-c} \ll (MN)^{\frac{1}{2}(5a-b-c)} \ll x^{a-1}, \tag{5.5}$$

and hence

$$\sqrt[4]{GM^2N^3} = \sqrt[4a]{xM^{2a-b}N^{3a-c}} \ll x^{\theta}. \tag{5.6}$$

When $H \geqslant M^4N^3x^{-1}$, we also have bounds (using (5.2))

$$\sqrt[4]{M^4N^3H^{-1}} \ll x^{\theta}, \tag{5.7}$$

$$MN(HG)^{-\frac{1}{4}} \ll \sqrt[4a]{x^{a-1}M^{b}N^{a+c}} \ll \sqrt[4a]{x^{a-1}M^{a+b}N^{c}} \ll x^{\theta}. \tag{5.8}$$

Hence, from (5.3) to (5.8) we see that (5.1) is true in case of $H \geqslant M^4N^3x^{-1}$. In the following we assume that

$$H < M^4N^3x^{-1}. \tag{5.9}$$

Similarly to (27) on p.269 of [L1], we get

$$\Phi(H,M,N) \ll H^{-1}\left(HGM^{-2}\right)^{-\frac{1}{2}}\Phi_1(H,M,N) + MN(HG)^{-\frac{1}{2}} + NL, \tag{5.10}$$

where

$$\Phi_1(H,M,N) = \sum_{h\sim H}\sum_{n\sim N}\left|\sum_{u_1<u<u_2} G_1(u)e\left(G_2(h,n,u)\right)\right|,$$

$$u_i \approx U = \frac{HG}{M} \quad (i = 1,2),$$

where $|G_1(u)| \leqslant 1$ and $G_2(h,n,u) = B_2\left(xh^a u^b n^{-c}\right)^{1/(a+b)}$ with

$$B_2 = -\left(\frac{a}{b}\right)^{b/(a+b)} - \left(\frac{b}{a}\right)^{a/(a+b)} < 0.$$

By Theorem 1.1 we get the estimate

$$\begin{aligned} L^{-1}\Phi_1(H,M,N) \ll \Big(& H(HG)^{3/4}M^{-1}N + \sqrt[4]{H^7G^4M^{-4}N^3} \\ &+\sqrt{H^3GM^{-1}N^2} + \sqrt[4]{H^6G^3M^{-2}N^3}\Big) e^{\Phi\left(c_1 \frac{HG}{M}\right)}, \end{aligned} \tag{5.11}$$

where c_1 is an absolute positive constant. By (5.9) and (5.2) (see also (5.5)) we find that

$$HG/M < M^3N^3x^{-1}G = \left(\sqrt[a]{x^{1-a}M^{2a-b}N^{3a-c}}\right) M \ll M \ll x^{1/3},$$

and thus

$$\Phi\left(c_1\frac{HG}{M}\right) \leqslant \frac{1}{\sqrt{3}}(L\cdot \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}}(\log L)^{-\frac{1}{2}}\right). \tag{5.12}$$

From (5.10) and (5.11) we get

$$\begin{aligned} L^{-2}\Phi(H,M,N) \ll \left(\sqrt[4]{HGN^4} + \sqrt[4]{HG^2N^3} + NM^{\frac{1}{2}} + \sqrt[4]{GM^2N^3}\right) e^{\Phi(c_1HG/M)} \\ + MNG^{-\frac{1}{2}} + N. \end{aligned} \tag{5.13}$$

From (5.2) we have $G \gg M \gg N$, and hence $\sqrt[4]{HGN^4} \ll \sqrt[4]{HG^2N^3}$, $MNG^{-\frac{1}{2}} \ll NM^{\frac{1}{2}}$, and from (5.13) we get the estimate

$$L^{-2}\Phi(H,M,N) \ll \left(\sqrt[4]{HG^2N^3} + NM^{\frac{1}{2}} + \sqrt[4]{GM^2N^3}\right) e^{\Phi(c_1HG/M)}. \tag{5.14}$$

In view of (5.9), by (5.5) and (5.6) we find that

$$\sqrt[4]{HG^2N^3} \ll \sqrt[4]{M^4N^6x^{-1}G^2} \ll \sqrt[4a]{x^{2-a}M^{4a-2b}N^{6a-2c}} \ll x^\theta,$$

$$M^{\frac{1}{2}} + \sqrt[4]{GM^2N^3} \ll x^\theta,$$

and thus from (5.12) and (5.14) we also get (5.1). The proof is finished.

6. Proof of Theorem 1.3

Let $x \geqslant 100, L = \log x$, and $M = [L/\log 2] + 1$. Then, it is well-known that

$$A(x) = \sum_{n_1 n_2^2 \cdots n_r^r \cdots \leqslant x} 1 = \sum_{n_1 n_2^2 \cdots n_M^M \leqslant x} 1, \tag{6.1}$$

(see p.261 of [L1], for instance), where the two summations of (6.1) are taken over the lattice points $(n_1, n_2, \cdots, n_r, \cdots)$ and $(n_1, n_2, \cdots, n_M)$ respectively (the latter is an M-dimensional lattice point). Let

$$b(m) = \begin{cases} 1, & \text{if } m = g_1 g_2^2 g_3^3, \quad g_1, g_2 \text{ and } g_3 \text{ are positive integers,} \\ 0, & \text{otherwise,} \end{cases}$$

$$B(n) = \begin{cases} 1, & \text{if } n = g_4^4 \cdots g_M^M, \text{ with positive integers } g_i \quad (4 \leqslant i \leqslant M), \\ 0, & \text{otherwise.} \end{cases}$$

Then from (6.1) and our Theorem 1.2 we get

$$\begin{aligned} A(x) &= \sum_{mn \leqslant x} b(m) B(n) = \sum_{n \leqslant x} B(n) \left(\sum_{m \leqslant x/n} b(m) \right) \\ &= \sum_{n \leqslant x} B(x) \left(c_1 \left(\frac{x}{n}\right) + c_2 \left(\frac{x}{n}\right)^{\frac{1}{2}} + c_3 \left(\frac{x}{n}\right)^{\frac{1}{3}} + O\left(\left(\frac{x}{n}\right)^{\theta} \cdot e^{\xi(x)} \right) \right) \\ &= c_1 x \sum_{n \leqslant x} \frac{B(n)}{n} + c_2 x^{\frac{1}{2}} \sum_{n \leqslant x} \frac{B(n)}{n^{\frac{1}{2}}} + c_3 x^{\frac{1}{3}} \sum_{n \leqslant x} \frac{B(n)}{n^{\frac{1}{3}}} \\ &\quad + O\left(x^{\theta} e^{\xi(x)} \sum_{n \leqslant x} \frac{B(n)}{n^{\theta}} \right), \end{aligned} \tag{6.2}$$

where $\theta = \frac{1}{4}, \quad \xi(x) = \frac{1}{\sqrt{3}} (L \cdot \log L)^{\frac{1}{2}} + O\left(L^{\frac{1}{2}} (\log L)^{-\frac{1}{2}} \right)$, and

$$c_i = \prod_{\substack{1 \leqslant j \leqslant 3 \\ j \neq i}} \zeta\left(\frac{j}{i}\right), \qquad \text{for } 1 \leqslant i \leqslant 3.$$

Using the inequalities

$$(1 - 1/y)^{-1} \leqslant 1 + 2/y \quad (\text{for } y \geqslant 2), \quad z \geqslant \log(1+z) \quad (\text{for } z \geqslant 0)$$

we deduce that

$$\begin{aligned}\sum_{n\leqslant x} B(n)n^{-\theta} &= \sum_{g_4^4\cdots g_M^M\leqslant x}\left(g_4^4\cdots g_M^M\right)^{-\theta} < \left(\sum_{g_4\leqslant x} g_4^{-1}\right)\zeta(5\theta)\cdots\zeta(M\theta)\\ &\ll L\left(\zeta(5\theta)\cdots\zeta(M\theta)\right) = L\prod_{5\leqslant k\leqslant M}\prod_p\left(1-p^{-k\theta}\right)^{-1}\\ &\leqslant L\prod_{5\leqslant k\leqslant M}\prod_p\left(1+2p^{-k\theta}\right) = L\prod_{5\leqslant k\leqslant M}\prod_p\exp\left(\log\left(1+2p^{-k\theta}\right)\right)\\ &\leqslant L\prod_{5\leqslant k\leqslant M}\prod_p\exp\left(2p^{-k}\right) = L\prod_p\exp\left(2\sum_{5\leqslant k\leqslant M} p^{-k\theta}\right)\\ &< L\left(\prod_p\exp\left(2p^{-5\theta}\left(1-p^{-\theta}\right)^{-1}\right)\right) = O(L). \qquad (6.3)\end{aligned}$$

Note that although the definition of $B(n)$ depends on x (since $M = M(x)$), from the derivation of (6.3) we see that we can already show that for $\lambda > \theta$ there holds

$$\sum_{n=1}^{\infty} B(n)n^{-\lambda} = \sum_{g_4=1}^{\infty}\cdots\sum_{g_M=1}^{\infty}\left(g_4^4\ldots g_M^M\right)^{-\lambda} = O(1),$$

and the constant implied by O-symbol is absolute. Thus, if $\phi > \theta$, letting $\delta = \frac{1}{2}(\phi-\theta)$, by (6.3) we get

$$\begin{aligned}\sum_{n\leqslant x} B(n)n^{-\phi} &= \sum_{n=1}^{\infty} B(n)n^{-\phi} - \sum_{x<n\leqslant x^2}\frac{B(n)}{n^{\theta}}\cdot\frac{1}{n^{\phi-\theta}} - \sum_{n>x^2}\frac{B(n)}{n^{\theta+\delta}}\cdot\frac{1}{n^{\phi-\theta-\delta}}\\ &= \sum_{n=1}^{\infty} B(n)n^{-\phi} + O\left(Lx^{\theta-\phi}\right) = \prod_{4\leqslant k\leqslant M}\zeta(k\phi) + O\left(Lx^{\theta-\phi}\right). \qquad (6.4)\end{aligned}$$

Since (for $k \geqslant 4$)

$$\sum_{p\geqslant 2}\log\left(\left(1-p^{-k\phi}\right)^{-1}\right) \leqslant 2\sum_{p\geqslant 2} p^{-k\phi} \leqslant C(\phi)2^{-k\phi},$$

it follows that

$$\begin{aligned}\prod_{4\leqslant k\leqslant M}\zeta(k\phi) &= \exp\left(\sum_{4\leqslant k\leqslant M}\sum_p\log\left(1-p^{-k\theta}\right)^{-1}\right)\\ &= \exp\left(\sum_{k\geqslant 4}\sum_p\log\left(1-p^{-k\theta}\right)^{-1} + O\left(x^{-\phi}\right)\right) \qquad (6.5)\\ &= \left(\prod_{k\geqslant 4}\zeta(k\phi)\right)\left(1+O\left(x^{-\phi}\right)\right).\end{aligned}$$

From (6.4) and (6.5) we get, for any $\phi > \theta$, that

$$\sum_{n\leqslant x} B(n)n^{-\phi} = \prod_{k\leqslant 4} \zeta(k\phi) + O\left(Lx^{\theta-\phi}\right). \tag{6.6}$$

Theorem 1.3 follows from (6.2), (6.3) and (6.6).

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