Functiones et Approximatio
41.1 (2009), 55–70

CONGRUENCES BETWEEN MODULAR FORMS AND RELATED MODULES

Miriam Ciavarella

Abstract: Fix a prime ℓ and let M be an integer such that $\ell \nmid M$. Let $f \in S_2(\Gamma_1(M\ell^2))$ be a newform which is supercuspidal at ℓ of a fixed type related to the nebentypus and special at a finite set of primes. Let $\mathbf{T}^\psi$ be the local quaternionic Hecke algebra associated to f. The algebra $\mathbf{T}^\psi$ acts on a module $\mathcal{M}_f^\psi$ coming from the cohomology of a Shimura curve. It follows from the Taylor-Wiles criterion and a recent Savitt's theorem, that $\mathbf{T}^\psi$ is the universal deformation ring of a global Galois deformation problem associated to $\overline{\rho}_f$. Moreover $\mathcal{M}_f^\psi$ is free of rank 2 over $\mathbf{T}^\psi$. If f occurs at minimal level, we prove a result about congruences of ideals and we obtain a raising the level result. The extension of these results to the non minimal case is still an open problem.

Keywords: modular form, deformation ring, Hecke algebra, quaternion algebra, congruences.

Introduction

The main goal of this article is to show the existence of an isomorphism of complete intersection rings between an universal deformation ring and a local quaternionic Hecke algebra and to prove the freeness of a quaternionic cohomological modules over the Hecke algebra. We deduce some consequences of these results about the congruences ideals. Our work belongs to the line of search which has its roots in the works of Wiles and Taylor-Wiles on the Shimura-Taniyama-Weil conjecture [24], [8]. In particular our results extend a work of Terracini [23] to a more general class of types and allows us to work with modular forms having a non trivial nebentypus. Our arguments are largely identical to Terracini's in many places; the debt to Terracini's work will be clear throughout the paper. An important difference is that we will work with Galois representations which are not semistable at ℓ but only potentially semistable, and therefore use a recent theorem by Savitt [22]. This result proves a conjecture of Conrad, Diamond and Taylor ([6, Conjecture 1.2.2 and Conjecture 1.2.3]), on the size of certain deformation rings parametrizing potentially Barsotti-Tate Galois representations, extending results of Breuil and

2000 Mathematics Subject Classification: 11F80.

Mézard ([1, Conjecture 2.3.1.1]) (classifying Galois lattices in semistable representations in terms of "strongly divisible modules") to the potentially crystalline case in Hodge-Tate weights $(0,1)$. Savitt's result allows to extend Conrad, Diamond and Taylor's result [6] relaxing the hypotheses on the residual representation.

Given a prime ℓ, we fix a newform $f \in S_2(\Gamma_0(N\Delta'\ell^2),\psi)$ with nebentypus ψ of order prime to ℓ, special at primes dividing Δ' and such that its local representation $\pi_{f,\ell}$ of $GL_2(\mathbf{Q}_\ell)$ is associated to a fixed regular character χ of $\mathbf{F}_{\ell^2}^\times$ satisfying $\chi|_{\mathbf{Z}_\ell^\times} = \psi_\ell|_{\mathbf{Z}_\ell^\times}$. Let $\overline{\rho} = \overline{\rho}_f$ be the residual Galois representation associated to f and denote by $\mathcal{B}$ the set of normalized newforms in $S_2(\Gamma_0(N\Delta'\ell^2),\psi)$ which are supercuspidal of type $\tau = \chi^\sigma \oplus \chi$ at ℓ, special at primes dividing Δ' and whose associated representation is a deformation of $\overline{\rho}$. For $h \in \mathcal{B}$, let $\mathcal{O}_h$ be the $\mathbf{Z}_\ell$-algebra generated in $\bar{\mathbf{Q}}_\ell$ by the Fourier coefficients of h; let $\mathbf{T}^\psi$ denote the sub-$\mathbf{Z}_\ell$-algebra of $\prod_{h\in\mathcal{B}}\mathcal{O}_h$ generated by the Fourier coefficients of the forms in $\mathcal{B}$ at primes $p \not| M\ell$.

By the Jacquet-Langlands correspondence and the Matsushima-Murakami-Shimura isomorphism, one can see such forms in a local component $\mathcal{M}^\psi$ of the ℓ-adic cohomology of a certain Shimura curve. By imposing suitable conditions on the type τ, we describe for each prime p dividing the level a local deformation condition of $\overline{\rho}_p$. By applying the Taylor-Wiles criterion in the version of Diamond [13] and Fujiwara [15], we prove that the algebra $\mathbf{T}^\psi$ is characterized as the universal deformation ring $\mathcal{R}^\psi$ of our global Galois deformation problem. We point out that in order to prove the existence of a family of sets realizing simultaneously the conditions of a Taylor-Wiles system, we make large use of Savitt's theorem [22]: assuming the existence of a newform f as above, the tangent space of the deformation functor at ℓ has dimension one over the residue field. Our first result is the following:

Theorem 1.

a) $\Phi : \mathcal{R}^\psi \to \mathbf{T}^\psi$ *is an isomorphism of complete intersections;*
b) $\mathcal{M}^\psi$ *is a free* $\mathbf{T}^\psi$*-module of rank 2.*

The problem of extending our result allowing the ramification on a set of primes S disjoint from $\Delta' N\ell$ is still an open problem. It is related to a conjectural extension of Ihara's lemma to the cohomology of Shimura curves (see [5] for details).

Since we are interested in studying congruences between modular forms, we observe that our result allows us to give a new interpretation of the congruence module of f in terms of the integer cohomology of a certain Shimura curve.

Under the hypothesis that f occurs with minimal level (i.e. the ramification at primes p dividing the Artin conductor of the Galois representation ρ_f is equal to the ramification of $\overline{\rho}_f$ at p) the module $\mathcal{M}^\psi$, used to construct the Taylor-Wiles system, can be also seen as a part of a module $\mathcal{M}^{\mathrm{mod}}$ coming from the cohomology of a modular curve, as described in [6, Section 5.3].

In Section 8, we prove that as a consequence of the generalization of Conrad, Diamond and Taylor's result, it is possible to extend the results of Terracini about the congruence ideals [23, Section 4] to the case of modular forms with non trivial nebentypus (Theorem 8.1). Moreover we interpret this result in terms of raising the level problem and obtain Theorem 8.2. We underline that these results are obtained under the assumption of minimal level. This hypothesis is fundamental since it is not available an analogue of Ihara's lemma for Shimura curves under the assumption that ℓ divides the discriminant of the indefinite quaternion algebra (for the status of art of this open problem see [5]). In [11] and in [12], Diamond and Taylor show that if ℓ does not divide the discriminant of the indefinite quaternion algebra, the quaternionic analogue of Ihara's lemma holds.

1. Notations

For a rational prime p, $\mathbf{Z}_p$ and $\mathbf{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. If A is a ring, then $A^\times$ denotes the group of invertible elements of A. We will denote by $\mathbf{A}$ the ring of rational adèles, and by $\mathbf{A}^\infty$ the finite adèles.

Let B be a quaternion algebra on $\mathbf{Q}$, we will denote by $B_\mathbf{A}$ the adelization of B, by $B_\mathbf{A}^\times$ the topological group of invertible elements in $B_\mathbf{A}$ and $B_\mathbf{A}^{\times,\infty}$ the subgroup of finite adèles.

Let R be a maximal order in B. For a place v of $\mathbf{Q}$ we put $B_v = B \otimes_\mathbf{Q} \mathbf{Q}_v$; if p is a finite place we put $R_p = R \otimes_\mathbf{Z} \mathbf{Z}_p$.

If p is a prime not dividing the discriminant of B, including $p = \infty$, we fix an isomorphism $i_p : B_p \to M_2(\mathbf{Q}_p)$ such that if $p \neq \infty$ we have $i_p(R_p) = M_2(\mathbf{Z}_p)$.

We write $GL_2^+(\mathbf{R}) = \{g \in GL_2(\mathbf{R}) | \det g > 0\}$ and $K_\infty^+ = \mathbf{R}^\times SO_2(\mathbf{R})$. If K is a field, let $\overline{K}$ denote an algebraic closure of K; we put $G_K = \mathrm{Gal}(\overline{K}/K)$. For a local field K, K^{unr} denotes the maximal unramified extension of K in $\overline{K}$; we put $I_K = \mathrm{Gal}(\overline{K}/K^{unr})$, the inertia subgroup of G_K. For a prime p we put $G_p = G_{\mathbf{Q}_p}$, $I_p = I_{\mathbf{Q}_p}$. If ρ is a representation of $G_\mathbf{Q}$, we write ρ_p for the restriction of ρ to a decomposition group at p.

2. The local Hecke algebra $\mathbf{T}^\psi$

We fix a prime $\ell > 2$. Let $\mathbf{Z}_{\ell^2}$ denote the integer ring of $\mathbf{Q}_{\ell^2}$, the unramified quadratic extension of $\mathbf{Q}_\ell$. Let $M \neq 1$ be a square-free integer not divisible by ℓ. We fix f an eigenform in $S_2(\Gamma_1(M\ell^2))$, then $f \in S_2(\Gamma_0(M\ell^2), \psi)$ for some Dirichlet character $\psi : (\mathbf{Z}/M\ell^2\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$.

By abuse of notation, let ψ be the adelisation of the Dirichlet character ψ and we denote by ψ_p the composition of ψ with the inclusion $\mathbf{Q}_p^\times \to \mathbf{A}^\times$.

We fix a regular character $\chi : \mathbf{Z}_{\ell^2}^\times \to \overline{\mathbf{Q}}^\times$ of conductor ℓ such that $\chi|_{\mathbf{Z}_\ell^\times} = \psi_\ell|_{\mathbf{Z}_\ell^\times}$ and we extend χ to $\mathbf{Q}_{\ell^2}^\times$ by putting $\chi(\ell) = -\psi_\ell(\ell)$. We observe that χ is not uniquely determined by ψ and, if we fix an embedding of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_\ell$ and in $\mathbf{C}^\times$, we can regard the values of χ in this field.

By local classfield theory, χ can be regarded as a character of I_ℓ and we can consider the type $\tau = \chi \oplus \chi^\sigma : I_\ell \to GL_2(\overline{\mathbf{Q}}_\ell)$, where σ is the non trivial element of $\mathrm{Gal}(\mathbf{Q}_{\ell^2}/\mathbf{Q}_\ell)$.

We fix a decomposition $M = N\Delta'$ where Δ' is a product of an odd number of primes. If we choose $f \in S_2(\Gamma_1(M\ell^2))$ such that the automorphic representation $\pi_f = \otimes_v \pi_{f,v}$ of $GL_2(\mathbf{A})$ associated to f is supercuspidal of type $\tau = \chi \oplus \chi^\sigma$ at ℓ and special at every primes $p|\Delta'$, then $\pi_{f,\ell} = \pi_\ell(\chi)$, where $\pi_\ell(\chi)$ is the representation of $GL_2(\mathbf{Q}_\ell)$ associated to χ, with central character ψ_ℓ and conductor ℓ^2 (see [16, Section 2.8]). Moreover, under our hypotheses, the nebentypus ψ factors through $(\mathbf{Z}/N\ell\mathbf{Z})^\times$. As a general hypothesis, we assume that ψ has order prime to ℓ.

Let $\rho_f : G_{\mathbf{Q}} \to GL_2(\overline{\mathbf{Q}}_\ell)$ be the Galois representation associated to f and $\overline{\rho} : G_{\mathbf{Q}} \to GL_2(\overline{\mathbf{F}}_\ell)$ be its reduction modulo ℓ.
As in [23], we impose the following conditions on $\overline{\rho}$:

$$\overline{\rho} \text{ is absolutely irreducible;} \tag{1}$$

$$\text{if } p|N \text{ then } \overline{\rho}(I_p) \neq 1; \tag{2}$$

$$\text{if } p|\Delta' \text{ and } p^2 \equiv 1 \bmod \ell \text{ then } \overline{\rho}(I_p) \neq 1; \tag{3}$$

$$\mathrm{End}_{\overline{\mathbf{F}}_\ell[G_\ell]}(\overline{\rho}_\ell) = \overline{\mathbf{F}}_\ell. \tag{4}$$

$$\text{if } \ell = 3,\ \overline{\rho} \text{ is not induced from a character of } \mathbf{Q}(\sqrt{-3}). \tag{5}$$

Let $K = K(f)$ be a finite extension of $\mathbf{Q}_\ell$ containing $\mathbf{Q}_{\ell^2}$, $\mathrm{Im}(\psi)$ and the eigenvalues for f of all Hecke operators. Let $\mathcal{O}$ be the ring of integers of K, λ be a uniformizer of $\mathcal{O}$, $k = \mathcal{O}/(\lambda)$ be the residue field.

Let $\mathcal{B}$ denote the set of normalized newforms h in $S_2(\Gamma_0(M\ell^2), \psi)$ which are supercuspidal of type χ at ℓ, special at primes dividing Δ' and whose associated representation ρ_h is a deformation of $\overline{\rho}$. For $h \in \mathcal{B}$, let $h = \sum_{n=1}^\infty a_n(h)q^n$ be the q-expansion of h and let $\mathcal{O}_h$ be the $\mathcal{O}$-algebra generated in $\overline{\mathbf{Q}}_\ell$ by the Fourier coefficients of h. Let $\mathbf{T}^\psi$ denote the sub-$\mathcal{O}$-algebra of $\prod_{h\in\mathcal{B}} \mathcal{O}_h$ generated by the elements $\widetilde{T}_p = (a_p(h))_{h\in\mathcal{B}}$ for $p \nmid M\ell$.

3. Deformation problem

Our next goal is to state a global Galois deformation condition of $\overline{\rho}$ which is a good candidate for having $\mathbf{T}^\psi$ as an universal deformation ring.

3.1. The global deformation condition of type $(\mathrm{sp}, \tau, \psi)_{\mathbf{Q}}$

First of all we observe that our local Galois representation $\rho_{f,\ell} = \rho_\ell$ is of type τ [6]. We let Δ_1 be the product of primes $p|\Delta'$ such that $\overline{\rho}(I_p) \neq 1$, and Δ_2 be the product of primes $p|\Delta'$ such that $\overline{\rho}(I_p) = 1$.

We denote by $\mathcal{C}_{\mathcal{O}}$ the category of local complete noetherian $\mathcal{O}$-algebras with residue field k. Let $\epsilon : G_p :\to \mathbf{Z}_\ell^\times$ be the cyclotomic character and $\omega : G_p \to \mathbf{F}_\ell^\times$ be its reduction mod ℓ. By analogy with [23], we define the global deformation condition of type $(\mathrm{sp}, \tau, \psi)_{\mathrm{Q}}$:

Definition 3.1. *Let Q be a square-free integer, prime to $M\ell$. We consider the functor $\mathcal{F}_Q$ from $\mathcal{C}_\mathcal{O}$ to the category of sets which associate to an object $A \in \mathcal{C}_\mathcal{O}$ the set of strict equivalence classes of continuous homomorphisms $\rho : G_\mathbf{Q} \to GL_2(A)$ lifting $\overline{\rho}$ and satisfying the following conditions:*

- a$_Q$) *ρ is unramified outside $MQ\ell$;*
- b) *if $p|\Delta_1 N$ then $\rho(I_p) \simeq \overline{\rho}(I_p)$;*
- c) *if $p|\Delta_2$ then ρ_p satisfies the sp-condition, that is $\mathrm{tr}(\rho(F))^2 = \psi_p(p)(p+1)^2$ for a lift F of Frob_p in G_p;*
- d) *ρ_ℓ is weakly of type τ;*
- e) *$\det(\rho) = \epsilon\psi$, where $\epsilon : G_\mathbf{Q} \to \mathbf{Z}_\ell^\times$ is the cyclotomic character.*

It is easy to prove that the functor $\mathcal{F}_Q$ is representable and characterizes a global Galois deformation problem with fixed determinant [20].

Let $\mathcal{R}_Q^\psi$ be the universal ring associated to the functor $\mathcal{F}_Q$. We put $\mathcal{F} = \mathcal{F}_0$, $\mathcal{R}^\psi = \mathcal{R}_0^\psi$.

We observe that if $\overline{\rho}(I_p) = 1$, by the Ramanujan-Petersson conjecture proved by Deligne, the sp-condition rules out those deformations of $\overline{\rho}$ arising from modular forms which are not special at p. This space includes the restrictions to G_p of representations coming from forms in $S_2(\Gamma_0(N\Delta'\ell^2), \psi)$ which are special at p, but it does not contain those coming from principal forms in $S_2(\Gamma_0(N\Delta'\ell^2), \psi)$.

4. Cohomological modules coming from the Shimura curves

Let B be the indefinite quaternion algebra over $\mathbf{Q}$ of discriminant $\Delta = \ell\Delta'$. Let R be a maximal order in B. Let N be an integer prime to Δ. We put

$$K_p^0(N) = i_p^{-1}\left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in GL_2(\mathbf{Z}_p) \mid c \equiv 0 \mod N\right\}$$

$$K_p^1(N) = i_p^{-1}\left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in GL_2(\mathbf{Z}_p) \mid c \equiv 0 \mod N,\ a \equiv 1 \mod N\right\}.$$

We define

$$V_0(N) = \prod_{p \nmid N} R_p^\times \times \prod_{p|N} K_p^0(N)$$

and

$$V_1(N) = \prod_{p \nmid N\ell} R_p^\times \times \prod_{p|N} K_p^1(N) \times (1 + u_\ell R_\ell).$$

We observe that there is an isomorphism

$$\Omega = V_0(N)/V_1(N) \simeq (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{F}_{\ell^2}^\times.$$

Let $\widehat{\psi}$ be the character of $V_0(N)$ with kernel $V_1(N)$ defined as follow:

$$\widehat{\psi} := \prod_{p|N} \psi_p \times \chi : (\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{F}_{\ell^2}^\times \to \mathbf{C}^\times.$$

For $i = 0, 1$ we put $\Phi_i(N) = (GL_2^+(\mathbf{R}) \times V_i(N)) \cap B^\times$ and we define the Shimura curves:

$$\mathbf{X}_i(N) = B_{\mathbf{Q}}^\times \setminus B_{\mathbf{A}}^\times / K_\infty^+ \times V_i(N).$$

The finite commutative group Ω naturally acts on the $\mathcal{O}$-module $H^*(\mathbf{X}_1(N), \mathcal{O})$ via its action on $\mathbf{X}_1(N)$. Since there is an injection of $H^*(\mathbf{X}_1(N), \mathcal{O})$ in $H^*(\mathbf{X}_1(N), K)$, by [17, Section 7] the cohomology group $H^1(\mathbf{X}_1(N), \mathcal{O})$ is also equipped with the action of Hecke operator T_p, for $p \neq \ell$ and diamond operators $\langle n \rangle$ for $n \in (\mathbf{Z}/N\mathbf{Z})^\times$. The Hecke action commutes with the action of Ω, since we do not have a T_ℓ operator. The two actions are $\mathcal{O}$-linear.

Proposition 4.1. *Let $H^*(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$ be the sub-Hecke-module of $H^*(\mathbf{X}_1(N), \mathcal{O})$ on which Ω acts by the character $\widehat{\psi}$. Then $H^*(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$ is a direct summand of $H^*(\mathbf{X}_1(N), \mathcal{O})$.*

Proof. We write $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 is the ℓ-Sylow subgroup of Ω and Ω_2 is the subgroup of Ω with order prime to ℓ. Then $H^*(\mathbf{X}_1(N), \mathcal{O}) = \bigoplus_\varphi H^*(\mathbf{X}_1(N), \mathcal{O})^\varphi$ where φ runs over the characters of Ω_2 and $H^*(\mathbf{X}_1(N), \mathcal{O})^\varphi$ is the sub-Hecke-module of $H^*(\mathbf{X}_1(N), \mathcal{O})$ on which Ω_2 acts by the character φ. Since, by hypothesis, ψ has order prime to ℓ, $H^*(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} = H^*(\mathbf{X}_1(N), \mathcal{O})^\varphi$ for some character φ of Ω_2. So $H^*(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$ is a direct summand of $H^*(\mathbf{X}_1(N), \mathcal{O})$. ■

It follows easily from the Hochschild-Serre spectral sequence that

$$H^*(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} \simeq H^*(\mathbf{X}_0(N), \mathcal{O}(\widehat{\psi}))$$

where $\mathcal{O}(\widehat{\psi})$ is the sheaf $B^\times \setminus B_{\mathbf{A}}^\times \times \mathcal{O}/K_\infty^+ \times V_0(N)$, $B^\times$ acts on $B_{\mathbf{A}}^\times \times \mathcal{O}$ on the left by $\alpha \cdot (g, m) = (\alpha g, m)$ and $K_\infty^+ \times V_0(N)$ acts on the right by $(g, m) \cdot v = (g, m) \cdot (v_\infty, v^\infty) = (gv, \widehat{\psi}(v^\infty)m)$ where v_∞ and v^∞ are respectively the infinite and finite part of v . By translating to the cohomology of groups we obtain (see [10, Appendix]) $H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} \simeq H^1(\Phi_0(N), \mathcal{O}(\widetilde{\psi}))$, where $\widetilde{\psi}$ is the restriction of $\widehat{\psi}$ to $\Phi_0(N)/\Phi_1(N)$ and $\mathcal{O}(\widetilde{\psi})$ is $\mathcal{O}$ with the action of $\Phi_0(N)$ given by $a \mapsto \widetilde{\psi}^{-1}(\gamma)a$. The Hecke action on $H^1(\Phi_0(N), \mathcal{O}(\widetilde{\psi}))$ and the structure of the module $H^1(\mathbf{X}_1(N), K)^{\widehat{\psi}}$ over the Hecke algebra are well known (see for example [23]). Let $\mathbf{T}_0^{\widehat{\psi}}(N)$ be the $\mathcal{O}$-algebra generated by the Hecke operators T_p, $p \neq \ell$ acting on $H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}$.

Proposition 4.2. *$H^1(\mathbf{X}_1(N), K)^{\widehat{\psi}}$ is free of rank 2 over $\mathbf{T}_0^{\widehat{\psi}}(N) \otimes K$.*

The proof of Proposition 4.2, follows from:

Proposition 4.3. *Let $\mathbf{T}_0^{\widehat{\psi}}(N, \mathbf{C})$ denote the algebra generated over $\mathbf{C}$ by the operators T_p for $p \neq \ell$ acting on $H^1(\mathbf{X}_1(N), \mathbf{C})^{\widehat{\psi}}$. Then $H^1(\mathbf{X}_1(N), \mathbf{C})^{\widehat{\psi}}$ is free of rank 2 over $\mathbf{T}_0^{\widehat{\psi}}(N, \mathbf{C})$.*

The two key ingredients to prove the last proposition are:

- the Matsushima-Shimura isomorphism

$$H^1(\mathbf{X}_1(N), \mathbf{C}) \cong S_2(V_1(N)) \oplus \overline{S_2(V_1(N))}$$

 as Hecke modules, where $S_2(V_1(N))$ is the space of weight 2 automorphic forms on $B_{\mathbf{A}}^{\times}$ which are right invariant for $V_1(N)$, [19];
- the homomorphism

$$JL : S_2(V_1(N)) \to S_2(\Gamma_0(\Delta') \cap \Gamma_1(N\ell^2))$$

 which is injective when restricted to the subspace $S_2(V_0(N), \widehat{\psi})$ of $S_2(V_1(N))$ consisting of forms φ such that $\varphi(gk) = \psi(k)\varphi(g)$ for all $g \in B_{\mathbf{A}}^{\times}$, $k \in V_0(N)$ and are equivariant for the action of the Hecke operators.

For details see [23, Proposition 1.2].

5. The $\mathcal{O}$-module $\mathcal{M}^{\psi}$

This section follows closely Section 3 of [23], and formulates a result that generalizes Theorem 3.1 of Terracini to the case of non trivial nebentypus.

By the Jacquet-Langlands correspondence, the form f determines a character $\mathbf{T}_0^{\widehat{\psi}}(N) \to k$ sending the operator t to the class mod λ of the eigenvalue of t acting on f. The kernel of this character is a maximal ideal $\mathfrak{m}$ in $\mathbf{T}_0^{\widehat{\psi}}(N)$. We define

$$\mathcal{M}^{\psi} = H^1(\mathbf{X}_1(N), \mathcal{O})_{\mathfrak{m}}^{\widehat{\psi}}.$$

By combining Proposition 4.7 of [7] with the Jacquet-Langlands correspondence we see that there is a natural isomorphism $\mathbf{T}^{\psi} \simeq \mathbf{T}_0^{\widehat{\psi}}(N)_{\mathfrak{m}}$. Therefore, by Proposition 4.2, $\mathcal{M}^{\psi} \otimes_{\mathcal{O}} K$ is free of rank 2 over $\mathbf{T}^{\psi} \otimes_{\mathcal{O}} K$.

For a newform $h \in \mathcal{B}$, let A_h denote the subring of $\mathcal{O}_h$ consisting of those elements whose reduction mod λ is in k. We know that with respect to some basis, we have a deformation $\rho_h : G_{\mathbf{Q}} \to GL_2(A_h)$ of $\overline{\rho}$ satisfying our global deformation problem.

The universal property of $\mathcal{R}^{\psi}$ furnishes a unique homomorphism $\pi_h : \mathcal{R}^{\psi} \to A_h$ such that the composite $G_{\mathbf{Q}} \to GL_2(\mathcal{R}^{\psi}) \to GL_2(A_h)$ is equivalent to ρ_h. Since $\mathcal{R}^{\psi}$ is topologically generated by the traces of $\rho^{\mathrm{univ}}(\mathrm{Frob}_p)$ for $p \neq \ell$, (see [21, Section 1.8]), we conclude that the map $\mathcal{R}^{\psi} \to \prod_{h \in \mathcal{B}} \mathcal{O}_h$ such that $r \mapsto (\pi_h(r))_{h \in \mathcal{B}}$ has image $\mathbf{T}^{\psi}$. Thus there is a surjective homomorphism of $\mathcal{O}$-algebras $\Phi : \mathcal{R}^{\psi} \to \mathbf{T}^{\psi}$. As in [23], our goal is to prove the following

Theorem 5.1.

a) *$\mathcal{R}^{\psi}$ is complete intersection of dimension 1;*
b) *$\Phi : \mathcal{R}^{\psi} \to \mathbf{T}^{\psi}$ is an isomorphism;*
c) *$\mathcal{M}^{\psi}$ is a free $\mathbf{T}^{\psi}$-module of rank 2.*

5.1. Proof of Theorem 5.1

To prove Theorem 5.1, we apply the Taylor-Wiles criterion in the version of Diamond [13] and Fujiwara [15].
The criterion consists in proving the existence of a family $\mathcal{Q}$ of finite sets Q of prime numbers, not dividing $M\ell$ and of a $\mathcal{R}_Q^\psi$-module $\mathcal{M}_Q^\psi$ for each $Q \in \mathcal{Q}$ such that the system $(\mathcal{R}_Q^\psi, \mathcal{M}_Q^\psi)_{Q\in\mathcal{Q}}$ satisfies the following conditions:

(TWS1) For every $Q \in \mathcal{Q}$ and every $q \in Q$, $q \equiv 1 \mod \ell$; for such a q, let Δ_q be the ℓ-Sylow of $(\mathbf{Z}/q\mathbf{Z})^\times$ and define $\Delta_Q = \prod_{q\in Q}\Delta_q$. Let I_Q be the augmentation ideal of $\mathcal{O}[\Delta_Q]$. Then $\mathcal{R}_Q^\psi$ is a local complete $\mathcal{O}[\Delta_Q]$-algebra and $\mathcal{R}_Q^\psi/I_Q\mathcal{R}_Q^\psi \simeq \mathcal{R}^\psi$;

(TWS2) $\mathcal{M}_Q^\psi$ is $\mathcal{O}[\Delta_Q]$-free of finite rank α independent of Q;

(TWS3) for every positive integer m there exists $Q_m \in \mathcal{Q}$ such that $q \equiv 1 \mod \ell^m$ for any prime q in Q_m;

(TWS4) $r = |Q|$ does not depend on $Q \in \mathcal{Q}$;

(TWS5) for any $Q \in \mathcal{Q}$, $\mathcal{R}_Q^\psi$ is generated by at most r elements as a local complete $\mathcal{O}$-algebra;

(TWS6) $\mathcal{M}_Q^\psi/I_Q\mathcal{M}_Q^\psi$ is isomorphic to $\mathcal{M}^\psi$ as $\mathcal{R}^\psi$ modules, for every $Q \in \mathcal{Q}$.

If these conditions are satisfied, the family $(\mathcal{R}_Q^\psi, \mathcal{M}_Q^\psi)_{Q\in\mathcal{Q}}$ will be called a **Taylor–Wiles system** for $(\mathcal{R}^\psi, \mathcal{M}^\psi)$. Then Theorem 5.1 will follow from the isomorphism criterion developed by Wiles, Taylor–Wiles [13, Theorem 2.1].

Let Q be a finite set of prime numbers not dividing $N\Delta$ and such that

(A) $q \equiv 1 \mod \ell, \quad \forall q \in Q$;

(B) if $q \in Q$, $\overline{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues $\alpha_{1,q}$ and $\alpha_{2,q}$ contained in k.

Let Δ_q, Δ_Q, I_Q as in condition (TWS1) above; the ring $\mathcal{R}_Q^\psi$ defined in Section 3.1 is naturally equipped with a structure of $\mathcal{O}[\Delta_q]$-module. Then the condition (TWS1) holds and the proof of it is exactly the same of Proposition 3.2 of [23].

We will define the modules $\mathcal{M}_Q^\psi$. If $q \in Q$ we put

$$K_q' = \left\{ \alpha \in R_q^\times \mid i_q(\alpha) \in \begin{pmatrix} H_q & * \\ q\mathbf{Z}_q & * \end{pmatrix} \right\}$$

where H_q is the subgroup of $(\mathbf{Z}/q\mathbf{Z})^\times$ consisting of elements of order prime to ℓ.
We define

$$V_Q'(N) = \prod_{p\nmid NQ} R_p^\times \times \prod_{p|N} K_p^0(N) \times \prod_{q|Q} K_q'$$

$$V_Q(N) = \prod_{p\nmid NQ} R_p^\times \times \prod_{p|NQ} K_p^0(NQ)$$

$$\Phi_Q = (GL_2(\mathbf{R}^+) \times V_Q(N)) \cap B^\times, \qquad \Phi_Q' = (GL_2(\mathbf{R}^+) \times V_Q'(N)) \cap B^\times.$$

Then $\Phi_Q/\Phi_Q' \simeq \Delta_Q$ acts on $H^1(\Phi_Q', \mathcal{O}(\widetilde{\psi}))$. Let $\mathbf{T}_Q^{'\widehat{\psi}}(N)$ (resp. $\mathbf{T}_Q^{\widehat{\psi}}(N)$) be the Hecke $\mathcal{O}$-algebra generated by the Hecke operators T_p, $p \neq \ell$ and the diamond operators (that are those Hecke operators coming from Δ_Q) acting on

$H^1(\mathbf{X}'_Q(N),\mathcal{O})^{\widehat{\psi}}$ (resp. $H^1(\mathbf{X}_Q(N),\mathcal{O})^{\widehat{\psi}}$) where $\mathbf{X}'_Q(N)$ (resp. $\mathbf{X}_Q(N)$) is the Shimura curve associated to $V'_Q(N)$ (resp. $V_Q(N)$).

There is a natural surjection $\sigma_Q : \mathbf{T}_Q^{'\widehat{\psi}}(N) \to \mathbf{T}_Q^{\widehat{\psi}}(N)$. As shown in [23, Section 3.3], there exists a character $\theta_Q : \mathbf{T}_Q^{\widehat{\psi}}(N) \to k$. We define $\widetilde{\mathfrak{m}}_Q = \ker(\theta_Q)$, $\mathfrak{m}_Q = \sigma_Q^{-1}(\widetilde{\mathfrak{m}}_Q)$, and $\mathcal{M}_Q^{\psi} = H^1(\Phi'_Q, \mathcal{O}(\widetilde{\psi}))_{\mathfrak{m}_Q}$. Then conditions (TWS2) and (TWS6) are satisfied and the proof is the same as Proposition 3.5 of [23]. The existence of a family Q realizing conditions (TWS3), (TWS4), (TWS5) is proved by the same methods as in [8, Section 6 and Theorem 2.49] or [9, Sections 4, 5]; we only show that under our hypothesis the dimensions of the cohomological subgroups defining the local conditions allows one to apply that technique.

We let $ad^0\overline{\rho}$ denote the sub-representation of the adjoint representation of $\overline{\rho}$ over the space of the trace-0-endomorphisms and we let $ad^0\overline{\rho}(1) = \mathrm{Hom}(ad^0\overline{\rho}, \mu_p) \simeq Symm^2(\overline{\rho})$, with the action of G_p given by $(g\varphi)(v) = g\varphi(g^{-1}v)$. Local deformation conditions a$_Q$), b), c), d) allows one to define for each place v of $\mathbf{Q}$, a subgroup L_v of $H^1(G_v, ad^0\overline{\rho})$, the tangent space of the deformation functor, see [20, Section 23]. The calculations on Selmer groups are the same as in [23, Section 3.4] , except for the computation of $\dim_k L_\ell$. Let $\mathbf{R}^D_{\mathcal{O},\ell}$ be the local universal deformation ring associated to a local deformation problem which is weakly of type τ [6]. Since, in dimension 2, potentially Barsotti-Tate is equivalent to potentially crystalline (hence potentially semi stable) of Hodge-Tate weight $(0,1)$ [14, Theorem C2], this allows us to apply Savitt's result [22, Theorem 6.22]. Since under our hypothesis $\mathbf{R}^D_{\mathcal{O},\ell} \neq 0$, we deduce that there is an isomorphism $\mathcal{O}[[X]] \simeq \mathbf{R}^D_{\mathcal{O},\ell}$.

The dimension formula allows us to obtain the following identity:

$$\dim_k Sel_Q(ad^0\overline{\rho}) - \dim_k Sel_Q^*(ad^0\overline{\rho}(1)) = |Q| \tag{6}$$

and, since the minimal number of topological generators of $\mathcal{R}_Q^{\psi}$ is equal to $\dim_k Sel_Q(ad^0\overline{\rho})$, we obtain that the $\mathcal{O}$-algebra $\mathcal{R}_Q^{\psi}$ can be generated topologically by $|Q| + \dim_k Sel_Q^*(ad^0\overline{\rho}(1))$ elements. Applying the same arguments as in [7] the proof of Theorem 5.1 follows.

6. Quaternionic congruence module

Consider the injective homomorphism

$$JL : S_2(V_0(N), \widehat{\psi}) \to S_2(\Gamma_0(\Delta') \cap \Gamma_1(N\ell^2)).$$

Let $S = JL(V_0(N), \widehat{\psi})$ the subspace of $S_2(\Gamma_0(\Delta') \cap \Gamma_1(N\ell^2))$ generated by the new eigenforms with nebentypus ψ which are supercuspidal of type τ at ℓ and special at primes dividing Δ'.

We fix a $f \in S$ and let K be a finite extension of $\mathbf{Q}_\ell$ containing $\mathbf{Q}_{\ell^2}$, $\mathrm{Im}(\psi)$ and all the Hecke-eigenvalues of f of all the Hecke operators. Let $\mathcal{O}$ be the ring of integers of K.

Let X be the subspace of S spanned by f, together with all of its conjugates (in this way we can identify X with K) and let Y be the orthogonal complement to X under the Petersson inner product on S. We consider the decomposition

$$S(K) = X \oplus Y$$

as a direct sum of two subspaces, stable under the Hecke action.

Let π_f be the character $\pi_f : \mathbf{T}^{\psi} \to \mathcal{O}$ corresponding to f and let η denote the ideal $\pi_f(Ann_{\mathbf{T}^{\psi}} \ker \pi_f)$. Then the congruence module of f is $\mathcal{O}/\eta$.

By the Jacquet-Langlands correspondence and by the Matsushima-Shimura isomorphism, there is an isomorphism between the Hecke algebra $\mathbf{T}_0^{\widehat{\psi}}(N, K) = \mathbf{T}_0^{\widehat{\psi}}(N) \otimes K$ generated over K by the operators T_p for $p \neq \ell$ acting on $H^1(\mathbf{X}_1(N), K)^{\widehat{\psi}}$ and the algebra generated over K by all the Hecke operators acting on $S(K)$ (we recall that the forms occurring in S are supercuspidal at ℓ and so $T_\ell = 0$ on S).

By dimension considerations, there is an isomorphism:

$$\mathbf{T}_0^{\widehat{\psi}}(N, K) \cong \mathbf{T}_0^{\widehat{\psi}}(N, K)|_X \oplus \mathbf{T}_0^{\widehat{\psi}}(N, K)|_Y$$

where $\mathbf{T}_0^{\widehat{\psi}}(N, K)|_X$, $\mathbf{T}_0^{\widehat{\psi}}(N, K)|_Y$ are the algebras generated over K by the restrictions of the operators to the subspaces X and Y respectively.

Let H be the lattice in $S(K)$ consisting of modular forms with integral Fourier coefficients. We consider the isomorphism of $\mathbf{T}_0^{\widehat{\psi}}(N)$-modules $\varphi : \mathbf{T}_0^{\widehat{\psi}}(N) \to \mathrm{Hom}_{\mathcal{O}}(H, \mathcal{O})$ defined by $\varphi(T) : h \mapsto a_1(h|T)$ where $h|T$ denotes the image of h under T and a_1 is the first Fourier coefficient. Then φ induces an isomorphism of $\mathcal{O}$-modules (for details see [2, Sections 1, 3.6]):

$$\mathcal{O}/\eta \cong \frac{\mathbf{T}_0^{\widehat{\psi}}(N)|_X \oplus \mathbf{T}_0^{\widehat{\psi}}(N)|_Y}{\mathbf{T}_0^{\widehat{\psi}}(N)}.$$

Moreover the projector map e_f onto the component corresponding to f gives the isomorphism [2, Sections 1, 3.6]:

$$\frac{\mathbf{T}_0^{\widehat{\psi}}(N)|_X \oplus \mathbf{T}_0^{\widehat{\psi}}(N)|_Y}{\mathbf{T}_0^{\widehat{\psi}}(N)} \cong \frac{e_f \mathbf{T}_0^{\widehat{\psi}}(N)}{e_f \mathbf{T}_0^{\widehat{\psi}}(N) \cap \mathbf{T}_0^{\widehat{\psi}}(N)}.$$

Let $\mathfrak{m}$ be the ideal of $\mathbf{T}_0^{\widehat{\psi}}(N)$ defined is Section 5; since $e_f \mathbf{T}_0^{\widehat{\psi}}(N) = e_f \mathbf{T}_0^{\widehat{\psi}}(N)_{\mathfrak{m}}$ then

$$\mathcal{O}/\eta \cong \frac{e_f \mathbf{T}^{\psi}}{e_f \mathbf{T}^{\psi} \cap \mathbf{T}^{\psi}}.$$

We observe that this module does not depend on the discriminant Δ of the fixed quaternion algebra B, so that it is at the same time quaternionic and classical.

Define the module

$$C(f) = \frac{e_f H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}}{e_f H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}} \cap H^1(\mathbf{X}_1(N), \mathcal{O})^{\widehat{\psi}}}.$$

Then by Theorem 5.1:

Proposition 6.1. *There is an isomorphism of $\mathbf{T}_0(N)^{\widehat{\psi}}$-modules*

$$(\mathcal{O}/\eta)^2 \cong C(f).$$

7. A generalization of the Conrad, Diamond and Taylor's result using Savitt's theorem

In [6], Conrad, Diamond and Taylor fix a continuous global two dimensional Galois representation $\overline{\rho}$, absolutely irreducible and modular; they consider the global Galois deformation problem of type (S, τ) where S is a finite set of primes not containing ℓ and τ is an ℓ-type strongly acceptable for $\overline{\rho}|_{G_\ell}$. We write $\mathcal{R}_S^{\mathrm{mod}}$ for the universal deformation ring of type (S, τ), $\mathbf{T}_S^{\mathrm{mod}}$ for the classical Hecke algebra acting on the space of the modular forms of type (S, τ) and $\mathcal{M}_S^{\mathrm{mod}}$ for the cohomological module defined in [6, Section 5.3], which is essentially the τ-part of the first cohomology group of a modular curve of level depending on S. Their main result, Theorem 5.4.2 of [6], consists of proving that there is an isomorphism of complete intersections between $\mathcal{R}_S^{\mathrm{mod}}$ and $\mathbf{T}_S^{\mathrm{mod}}$ and $\mathcal{M}_S^{\mathrm{mod}}$ is free over $\mathbf{T}_S^{\mathrm{mod}}$.

In [22], Savitt proves the Conjectures 1.2.2 and 1.2.3 of Conrad, Diamond and Taylor [6] on the size of certain deformation rings parametrizing potentially Barsotti-Tate Galois representations. In particular Savitt's main result, [22, Theorem 1.2], allows to suppress the assumption of acceptability of the ℓ-type τ in the global Galois deformation problem and, as conjectured, Theorem 5.4.3 of [6] still holds.

As a particular case we can introduce a non trivial nebentypus, considering deformations ρ of type (S, τ) of $\overline{\rho}$ such that $\det(\rho) = \epsilon\psi$ where ϵ is the cyclotomic character and ψ is the Dirichlet character defined in Section 2. We will call this global Galois deformation problem of type (S, τ, ψ). Then, Savitt's result assures that the tangent space of the deformation functor at ℓ is still unidimensional and so it is possible to go on with the same construction of [6] obtaining the following:

Theorem 7.1. *There is an isomorphism of complete intersections between $\mathcal{R}_S^{\mathrm{mod},\psi}$ and $\mathbf{T}_S^{\mathrm{mod},\psi}$ and $\mathcal{M}_S^{\mathrm{mod},\psi}$ is free over $\mathbf{T}_S^{\mathrm{mod},\psi}$.*

Let f be a new form in $S_2(\Gamma_0(\Delta' N\ell^2), \psi)$ as in Section 2; if we assume that the representation $\overline{\rho}_f$ occurs with type τ and minimal level (that is $\overline{\rho}_f$ is ramified at every prime in Δ') then

$$\mathcal{R}_\emptyset^{\mathrm{mod},\psi} \simeq \mathcal{R}^\psi \tag{7}$$

where $\mathcal{R}^\psi$ is the universal ring associated to the functor $\mathcal{F}_\emptyset$ defined in Section 3.1. Then the following result holds:

Theorem 7.2. *Under the assumption of minimal level, there is an isomorphism of $\mathbf{T}^\psi$-modules between $H^1(\mathbf{X}_1(N), \mathcal{O})_{\mathfrak{m}}^{\widehat{\psi}}$ and $\mathcal{M}_\emptyset^{\mathrm{mod},\psi}$.*

Proof. We observe that if f occurs with type τ and minimal level, then by (7), Theorem 5.1 and Theorem 7.1, there is an isomorphism of Hecke algebras $\mathbf{T}^{\psi} \simeq \mathbf{T}^{\mathrm{mod},\psi}_{\emptyset}$. Thus, in particular $\mathcal{M}^{\psi} \simeq \mathcal{M}^{\mathrm{mod},\psi}_{\emptyset}$ as $\mathbf{T}^{\psi}$-modules. ■

We will describe some consequences of this results.

8. Congruence ideals

In this section we will use the results obtained in the previous sections in order to generalize a result about congruence ideals of Terracini [23], considering modular forms with nontrivial nebentypus.

We will just formulate the results extending Terracini's work, without technical proofs but we will underline where the introduction of a non trivial nebentypus changes the details of them.

Let Δ_1 be a set of primes, disjoint from ℓ. By an abuse of notation, we shall sometimes denote by Δ_1 also the product of the prime in this set.

Let f be a newform in $S_2(\Gamma_1(\Delta_1\ell^2))$ of nebentypus ψ and supercuspidal of type $\tau = \chi \oplus \chi^{\sigma}$ at ℓ, where the character ψ and the ℓ-type τ are defined as in Section 2. As a general hypothesis we will suppose that the residual representation $\overline{\rho}_f$ is absolutely irreducible and it occurs with type τ and minimal level (we are assuming the ramification at every prime in Δ_1).

Let Δ_2 be a finite set of primes p, not dividing $\Delta_1\ell$ such that $p^2 \not\equiv 1 \mod \ell$ and $\mathrm{tr}(\overline{\rho}(\mathrm{Frob}_p))^2 \equiv \psi(p)(p+1)^2 \mod \ell$. We let $\mathcal{B}^{\psi}_{\Delta_2}$ denote the set of newforms h of weight 2, nebentypus ψ and level dividing $\Delta_1\Delta_2\ell$ which are special at Δ_1, supercuspidal of type τ at ℓ and such that $\overline{\rho}_h = \overline{\rho}_f$. We choose an ℓ-adic ring $\mathcal{O}$ with residue field k, sufficiently large, so that every representation ρ_h for $h \in \mathcal{B}^{\psi}_{\Delta_2}$ is defined over $\mathcal{O}$ and $Im(\psi) \subseteq \mathcal{O}$. For every pair of disjoint subsets S_1, S_2 of Δ_2 we denote by $\mathcal{R}^{\psi}_{S_1,S_2}$ the universal solution over $\mathcal{O}$ for the deformation problem of $\overline{\rho}_f$ consisting of the deformations ρ satisfying:

a) ρ is unramified outside $\Delta_1 S_1 S_2 \ell$;
b) if $p|\Delta_1$ then $\rho(I_p) = \overline{\rho}(I_p)$;
c) if $p|S_2$ then ρ_p satisfies the sp-condition;
d) ρ_ℓ is weakly of type τ;
e) $\det(\rho) = \epsilon\psi$ where $\epsilon : G_{\mathbf{Q}} \to \mathbf{Z}_\ell^{\times}$ is the cyclotomic character.

Let $\mathcal{B}^{\psi}_{S_1,S_2}$ be the set of newforms in $\mathcal{B}^{\psi}_{\Delta_2}$ of level dividing $\Delta_1 S_1 S_2 \ell$ which are special at S_2 and let $\mathbf{T}^{\psi}_{S_1,S_2}$ be the sub-$\mathcal{O}$-algebra of $\prod_{h \in \mathcal{B}^{\psi}_{S_1,S_2}} \mathcal{O}$ generated by the elements $\widetilde{T}_p = (a(h))_{h \in \mathcal{B}^{\psi}_{S_1,S_2}}$ for p not in $\Delta_1 \cup S_1 \cup S_2 \cup \{\ell\}$. Since $\mathcal{R}^{\psi}_{S_1,S_2}$ is generated by traces, we know that there exists a surjective homomorphism of $\mathcal{O}$-algebras $\mathcal{R}^{\psi}_{S_1,S_2} \to \mathbf{T}^{\psi}_{S_1,S_2}$. Moreover by the results obtained in Section 7, we have that

$$\mathcal{R}^{\psi}_{S_1,\emptyset} \to \mathbf{T}^{\psi}_{S_1,\emptyset}$$

is an isomorphism of complete intersections, for any subset S_1 of Δ_2.

If $\Delta_1 \neq 1$ then each $\mathbf{T}^{\psi}_{\emptyset,S_2}$ acts on a local component of the cohomology of a suitable Shimura curve, associated with an indefinite quaternion algebra of discriminant $S_2\ell$ or $S_2\ell p$ for a prime p in Δ_1. Therefore, Theorem 5.1 gives the following:

Corollary 8.1. *Suppose that $\Delta_1 \neq 1$ and that $\mathcal{B}^{\psi}_{\emptyset,S_2} \neq \emptyset$; then the map*

$$\mathcal{R}^{\psi}_{\emptyset,S_2} \to \mathbf{T}^{\psi}_{\emptyset,S_2}$$

is an isomorphism of complete intersections.

If $h \in \mathcal{B}^{\psi}_{S_1,S_2}$ let $\theta_{h,S_1,S_2} : \mathbf{T}^{\psi}_{S_1,S_2} \to \mathcal{O}$ be the character corresponding to h.

We consider the congruence ideal of h relatively to $\mathcal{B}^{\psi}_{S_1,S_2}$:

$$\eta_{h,S_1,S_2} = \theta_{h,S_1,S_2}(Ann_{\mathbf{T}^{\psi}_{S_1,S_2}}(\ker\ \theta_{h,S_1,S_2})).$$

It is know that η_{h,S_1,S_2} controls congruences between h and linear combinations of forms different from h in $\mathcal{B}^{\psi}_{S_1,S_2}$.

For every $p|\Delta_2$ the deformation over $\mathcal{R}^{\psi}_{\Delta_2,\emptyset}$ restricted to G_p gives maps

$$\mathcal{R}^{\psi}_{p} = \mathcal{O}[[X,Y]]/(XY) \to \mathcal{R}^{\psi}_{\Delta_2,\emptyset}.$$

The image x_p of X and the ideal (y_p) generated by the image y_p of Y in $\mathcal{R}^{\psi}_{\Delta_2,\emptyset}$ do not depend on the choice of the map. By an abuse of notation, we shall call x_p, y_p also the image of x_p, y_p in every quotient of $\mathcal{R}^{\psi}_{\Delta_2,\emptyset}$. If h is a form in $\mathcal{B}^{\psi}_{\Delta_2,\emptyset}$, we denote by $x_p(h), y_p(h) \in \mathcal{O}$ the images of x_p, y_p by the map $\mathcal{R}^{\psi}_{\Delta_2,\emptyset} \to \mathcal{O}$ corresponding to ρ_h.
Following the same construction as in [23], we obtain the following:

Theorem 8.1. *Suppose $\Delta_1 \neq 1$ and Δ_2 as above. Then*

a) $\mathcal{B}^{\psi}_{\emptyset,\Delta_2} \neq 0$;
b) *for every subset $S \subseteq \Delta_2$, the map $\mathcal{R}^{\psi}_{S,\Delta_2/S} \to \mathbf{T}^{\psi}_{S,\Delta_2/S}$ is an isomorphism of complete intersections;*
c) *for every $h \in \mathcal{B}^{\psi}_{\emptyset,\Delta_2}$, $\eta_{h,S,\Delta_2/S} = (\prod_{p|S} y_p(h))\eta_{h,\emptyset,\Delta_2}$.*

If we combine point c) of Theorem 8.1 with the results in Section 5.5 of [6], we obtain:

Corollary 8.2. *If $h \in \mathcal{B}^{\psi}_{S_1,S_2}$ then*

$$\eta_{h,\Delta_2,\emptyset} = \prod_{p|\frac{\Delta_2}{S_1S_2}} x_p(h) \prod_{p|S_2} y_p(h)\eta_{h,S_1,S_2}.$$

We observe that as a consequence of the sp-condition if $h \in \mathcal{B}^{\psi}_{\Delta_2}$ and $p|\Delta_2$, then: $x_p(h) = 0$ if and only if h is special at p; moreover, since

$$a_p(h) = \mathrm{tr}(\rho_h(\mathrm{Frob}_p))$$

and

$$\rho_h(\mathrm{Frob}_h) = \begin{pmatrix} \pm p\sqrt{\psi(p)} + x_p(h) & 0 \\ 0 & p\psi(p)/(\pm p\sqrt{\psi(p)} + x_p(h)) \end{pmatrix}$$

then if h is unramified at p then $(x_p(h)) = (a_p(h)^2 - \psi(p)(p+1)^2)$ and $y_p(h) = 0$. We can reinterpret this result in terms of congruences between modular forms:

Theorem 8.2. *We fix $\ell > 2$ a prime number, Δ_1 a set of primes disjoint from ℓ and S_1, S_2 two sets of primes p not dividing $\Delta_1\ell$ such that $p^2 \not\equiv 1 \mod \ell$. Let $f = \sum a_n q^n$ be a normalized newform in $S_2(\Gamma_1(S_1S_2\Delta_1\ell^2))$ whit nebentypus ψ, supercuspidal of type τ at ℓ, special at primes dividing Δ_1S_2 and such that $\overline{\rho}_f$ is ramified at every prime in Δ_1 and*

$$trace(\overline{\rho}_f(Frob_p))^2 \equiv (p+1)^2 \mod \ell.$$

Then there exists $g \in S_2(\Gamma_1(qS_1S_2\Delta_1\ell^2))$ with nebentypus ψ, supercuspidal of type τ at ℓ, special at primes dividing Δ_1S_2 such that $f \equiv g \mod \lambda$ if and only if

$$a_q^2 \equiv \psi(q)(1+q)^2 \mod \lambda$$

where q is a prime such that $(q, S_1S_2\Delta_1\ell) = 1$, $q^2 \not\equiv 1 \mod \ell$.

Proof. The theorem follows from corollary 8.2, considering the relation between the congruence ideals:

$$\eta_{f,qS_1,S_2} = x_q(f)\eta_{f,S_1,S_2}. \qquad \blacksquare$$

References

[1] C. Breuil, A. Mézard, *Multiplicités modulaires et représentations de $GL_2(\mathbf{Z}_p)$ et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en $\ell = p$*, with an appendix by Guy Henniart, Duke Math. J. **115**, no. 2, (2002), 205–310.

[2] M. Ciavarella, *Congruences Between Quaternionic Modular Forms and Related Modules*, Tesi di Dottorato di Ricerca in Matematica, Università degli Studi di Torino, Italy, A.A.2003/2004.

[3] M. Ciavarella, *Congruences Between Modular Forms and Related Modules*, Nota Preventiva, Bollettino U.M.I. VIII-Vol.IX-B-2, 2006, 507–514.

[4] M. Ciavarella, *Eisenstein ideal and reducible λ-adic representations unramified outside a finite number of primes*, Bollettino U.M.I., Serie VIII-Vol.IX-B-3, 2006, 711–721.

[5] M. Ciavarella, L. Terracini, *About an analogue of Ihara's lemma for Shimura curves*, preprint 2009.

[6] B. Conrad, F. Diamond, R. Taylor, *Modularity of certain Potentially Barsotti-Tate Galois Representations*, Journal of the American Mathematical Society **12** (2) (1999), 521–567.

[7] H. Darmon, F. Diamond, R. Taylor, *Fermat's Last Theorem*, Current Developments in Mathematics, 1995, International Press, 1–154.

[8] H. Darmon, F. Diamond, R. Taylor, *Fermat's Last Theorem. Elliptic Curves, Modular Forms and Fermat's Last Theorem* (Hong Kong, 1993), 2nd edn, International Press, Cambridge, MA, 1997, 2–140.

[9] E. de Shalit, *Hecke Rings and Universal Deformation Rings. Modular Forms and Fermat's Last Theorem*, G. Cornell, H. Silverman, and G. Stevens, Eds. Springer, 1997, 421–445.

[10] B. de Smit, H.W. Lenstra, *Explicit construction of universal deformation rings. Modular Forms and Fermat's Last Theorem*, G. Cornell, H. Silverman et G. Stevens, Eds. Springer, 1997, 313–326.

[11] F. Diamond, R. Taylor, *Lifting modular* mod ℓ *representations*, Duke Math. J. **74** (1994), 253–269.

[12] F. Diamond, R. Taylor, *Non-optimal levels of mod ℓ modular representations*, Invent. Math. **115** (1994), 435–462.

[13] F. Diamond, *The Taylor-Wiles construction and multiplicity one*, Invent. Math. **128** (1997), 379–391.

[14] J.-M. Fontaine, B. Mazur, *Geometric Galois representation*, Conference on Elliptic Curves and Modular Forms, Hong Kong, 1993, International Press, 41–78.

[15] K. Fujiwara, *Deformation rings and Hecke algebras in the totally real case*, preprint, Nagoya University, 1996.

[16] P. Gerardin. *Facteur locaux des algebres simples de rang 4.I. Groupes Réductif et Formes Automorphes I*, Publications Mathématiques Univ Paris VII, 1978, 37–77.

[17] H. Hida, *On p-adic Hecke algebras for GL_2 over totally reals fields*, Ann. of Math. **128** (1988), 295–384.

[18] H. Jacquet, R. LANGLANDS, *Automorphic forms on GL_2*, vol. 114, Lecture Notes Math. Springer, 1970.

[19] Y. Matsushima, G. Shimura, *On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes*, Ann. of Math. **78** (1963), 417–449.

[20] B. Mazur, *An introduction to the deformation theory of Galois representations. Modular Forms and Fermat's Last Theorem*, G. Cornell, H. Silverman, and G. Stevens, Eds. Springer, 1997, 243–311.

[21] B. Mazur, *Deforming Galois Representations. In Galois Groups over* $\mathbf{Q}$, Ed. Ihara Ribet Serre, Springer, 1989.

[22] D. Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, preprint, 2004.

[23] L. Terracini, *A Taylor-Wiles system for quaternionic Hecke algebras*, Compositio Mathematica **137** (2003), 23–47.

[24] A. Wiles, *Modular elliptic curves and Fermat last Theorem*, Ann. of Math. **141** (1995), 443–551.

Address: Miriam Ciavarella: Università degli Studi di Torino, Dipartimento di Matematica, Via Carlo Alberto,10, 10123 Torino, Italy;

E-mail: miriam.ciavarella@unito.it

Received: 10 September 2008; **revised:** 15 June 2009