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ON CONVERGENCE OF FAMILIES OF LINEAR POLYNOMIAL OPERATORS

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To the memory of Prof. Alexander Stepanets

Abstract: The convergence of families of linear polynomial operators in the scale of the L_p-spaces with $0 < p \leqslant +\infty$ is studied. The convergence conditions are formulated in terms of the Fourier transform of the generator of the kernel. The results are applied to methods generated by classical kernels.

Keywords: approximation, trigonometric polynomials, Lp-convergence, classical means,Fourier transform.

Introduction

In this paper we continue the systematical study of the methods of trigonometric approximation started in [2] and [7] - [11]. We consider Fourier means, interpolation means and families of linear polynomial operators, which are defined as follows. Let φ be a complex-valued continuous function on $\mathbb{R}^d, d \in \mathbb{N}$, with compact support satisfying $\varphi(0) = 1$ and $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$. We put

$$W_0(\varphi)(h) \equiv 1\,,\ W_\sigma(\varphi)(h) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{\sigma}\right) e^{ikh}, \qquad \sigma > 0,\ \ h \in \mathbb{R}^d. \tag{1}$$

If $f \in L_p(\mathbb{T}^d)$, $1 \leqslant p \leqslant \infty$ ($\mathbb{T}^d$ stands for the d-dimensional torus) then the Fourier means are given by

$$\mathcal{F}_\sigma^{(\varphi)}(f;x) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(h)\, W_\sigma(\varphi)(x-h)\, dh, \qquad \sigma > 0,\ \ x \in \mathbb{T}^d. \tag{2}$$

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If f belongs to the space $C(\mathbb{T}^d)$ of continuous 2π-periodic (with respect to each variable) functions then the interpolation means are defined as

$$\mathcal{I}_\sigma^{(\varphi)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu\right) \cdot W_\sigma(\varphi)\left(x - t_N^\nu\right), \qquad \sigma > 0,\ \ x \in \mathbb{T}^d. \quad (3)$$

Here

$$N = [\rho\,\sigma], \qquad \rho \geqslant \rho(\varphi) = \sup\{\,|\,\xi\,| : \ \xi \in \operatorname{supp}\varphi\,\},$$

$$t_N^\nu = \frac{2\pi\nu}{2N+1}, \quad \nu \in \mathbb{Z}^d; \qquad \sum_{\nu=0}^{2N} \equiv \sum_{\nu_1=0}^{2N} \cdots \sum_{\nu_d=0}^{2N}.$$

The functions defined in (1), (2) and (3) are trigonometric polynomials of spherical order not exceeding $\rho\,(\varphi)\sigma$. If $f \in L_p(\mathbb{T}^d)$, $0 < p < \infty$, or if $f \in C(\mathbb{T}^d)$ then we consider the functions given by

$$\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)}(f;x) = (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^\nu + \lambda\right) \cdot W_\sigma(\varphi)\left(x - t_N^\nu - \lambda\right). \quad (4)$$

In the case that $f \in L_p(\mathbb{T}^d)$, $0 < p < \infty$, formula (4) makes sense for almost all $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$. We interpret λ as a parameter and call $\{\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)}\}$ a family of linear polynomial operators. In contrast to the classical methods of trigonometric approximation the method of approximation by families (if $\sigma \to \infty$) is comparatively new (see e. g. [7], [8]). Its systematical study was continued in [2], [9] and other works. In particular, it was shown that the averaged approximation error in L_p, $0 < p < +\infty$, with respect to the parameter λ can be estimated up to a constant by the best approximation with respect to trigonometric polynomials of order $\asymp \sigma$, if the generator φ of the kernel $W_\sigma(\varphi)$ is equal to 1 in a neighborhood of 0 and if its Fourier transform belongs to $L_{\widetilde{p}}(\mathbb{R}^d)$, where $\widetilde{p} = \min(1,p)$. For applications of the method, in particular, for the algorithm of stochastic approximation (*SA-algorithm*), we refer to [9].

In this paper we give a complete picture of the behavior of families of linear polynomial operators from the point of view of their convergence including results on the comparison of approximation properties of all three methods in the scale of the L_p-spaces. More precisely, we show that

- under the assumption $1 \in \mathcal{P}_\varphi = \{p \in (0, +\infty] : \ \widehat{\varphi} \in L_p\}$ the family $\{\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)}\}$ converges in L_p if and only if $p \in \mathcal{P}_\varphi$. Here $\widehat{\varphi}$ stands for the Fourier transform of the generator φ.
- for this reason the method of approximation by a family of linear polynomial operators is *universal* in the sense that it is relevant both for $p \geqslant 1$ and $p < 1$ where the range of admissible parameters depends on the properties of its generator φ and, moreover, the approximation error is equivalent to the approximation error of the corresponding Fourier means in the case of $L_p(\mathbb{T}^d)$,

$1 \leqslant p < \infty$, or $C(\mathbb{T})$ and to the approximation error of the corresponding interpolation means in the case of $C(\mathbb{T}^d)$;

- for families with classical kernels (Fejér, de la Vallée-Poussin, Rogosinski, Bochner-Riesz) the ranges of convergence (the values of p such that the family converges in L_p) can be determined precisely.

The paper is organized as follows. Section 1 is devoted to further definitions and notations. The general approach to treat the methods defined in (2)–(4) by a unified scheme is elaborated in Section 2. The L_p-norms of families are estimated in Section 3 and in Section 4 we formulate and prove the main result of this paper - *the General Convergence Theorem (GCT)*. The sharp ranges of convergence of the families generated by the kernels of Fejér, de la Vallée-Poussin, Rogosinski and Bochner-Riesz (for parameters above the critical index) are found in Section 5.

1. Definitions, notations and preliminary remarks

$\mathbf{L_p}$-spaces. As usual, $L_p \equiv L_p(\mathbb{T}^d)$, where $0 < p < +\infty$, $\mathbb{T}^d = [0, 2\pi)^d$, is the space of measurable real valued 2π-periodic on each variable functions $f(x)$ for which

$$\| f \|_p = \left(\int\limits_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p} < +\infty.$$

$C \equiv C(\mathbb{T}^d)$ is the space of real valued 2π-periodic continuous functions equipped with the Chebyshev norm

$$\| f \|_\infty = \max_{x \in \mathbb{T}^d} |f(x)|.$$

For the L_p-spaces of non-periodic functions defined on a measurable set $\Omega \subseteq \mathbb{R}^d$ we will use the notation $L_p(\Omega)$.

Often we deal with functions in $L_p(\mathbb{T}^{2d})$ which depend on both the main variable $x \in \mathbb{T}^d$ and the parameter $\lambda \in \mathbb{T}^d$. Let us denote by $\| \cdot \|_p$ or $\| \cdot \|_{p;\,x}$ the $L_p(\mathbb{T}^d)$-norm with respect to x. For the $L_p(\mathbb{T}^d)$-norm with respect to the parameter λ we use the symbol $\| \cdot \|_{p;\,\lambda}$. For shortness, $L_{\overline{p}}$ stands for the space $L_p(\mathbb{T}^{2d})$ equipped with the norm

$$\| \cdot \|_{\overline{p}} = (2\pi)^{-d/p} \Big\| \| \cdot \|_{p;\,x} \Big\|_{p;\,\lambda} . \tag{1.1}$$

Analogously, we use the symbol $\| \cdot \|_{\overline{\infty}}$ for the norm in the space $C(\mathbb{T}^{2d})$. Clearly, L_p with $0 < p < \infty$ and $C(\mathbb{T}^d)$ can be considered as a subspace of $L_{\overline{p}}$ and $C(\mathbb{T}^{2d})$, respectively, where

$$\| f \|_{\overline{p}} = \| f \|_p \ , \qquad f \in L_p \ (f \in C \text{ if } p = \infty) \ .$$

The functional $\| \cdot \|_{\overline{p}}$ is a norm if and only if $1 \leqslant p \leqslant +\infty$. For $0 < p < 1$ it is a quasi-norm, and the "triangle" inequality is valid for its p-th power. If we put $\widetilde{p} = \min(1, p)$ then the inequality

$$\| f + g \|_{\overline{p}}^{\widetilde{p}} \leqslant \| f \|_{\overline{p}}^{\widetilde{p}} + \| g \|_{\overline{p}}^{\widetilde{p}} , \qquad f, g \in L_{\overline{p}} , \tag{1.2}$$

will be valid for all $0 < p \leqslant +\infty$. This inequality is very convenient, because both cases can be treated uniformly. Moreover, for the sake of convenience we shall use the notation "norm" also in the case $0 < p < 1$.

Spaces of trigonometric polynomials. Let σ be a real non-negative number. Let us denote by $\mathcal{T}_\sigma$ the space of all real valued trigonometric polynomials of (spherical) order σ. It means

$$\mathcal{T}_\sigma = \left\{ T(x) = \sum_{k \in \mathbb{Z}^d} c_k \, e^{ikx} : c_{-k} = \overline{c_k}, \ |\, k \,| \equiv (k_1^2 + \ldots + k_d^2)^{1/2} \leqslant \sigma \right\} ,$$

where $kx = k_1 x_1 + \ldots + k_d x_d$ and $\bar{c}$ is a complex conjugate to c. Further, $\mathcal{T}$ stands for the space of all real-valued trigonometric polynomials of arbitrary order. We denote by $\mathcal{T}_{\sigma, p}$, where $0 < p \leqslant +\infty$, the space $\mathcal{T}_\sigma$, if it is equipped with the L_p-norm and we use the symbol $\mathcal{T}_{\sigma, \overline{p}}$ to denote the subspace of $L_{\overline{p}}$ which consists of functions $g(x, \lambda)$ such that $g(x, \lambda)$ as a function of x belongs to $\mathcal{T}_\sigma$ for almost all λ. Clearly, $\mathcal{T}_{\sigma, p}$ can be considered as a part of $\mathcal{T}_{\sigma, \overline{p}}$ with identity of the norms. As we can see, in our notation the line over the index p indicates that we are dealing with functions of $2d$ variables.

Generators. The conditions with respect to the generator of the method φ and the definition of the corresponding kernels $W_\sigma(\varphi)$ have been given in the Introduction. The class of all admissible generators will be denoted by $\mathcal{K}$. Recall that the set $\mathcal{K}$ consists of all complex-valued functions φ defined on $\mathbb{R}^d$ and satisfying

1) φ has a compact support ($\rho(\varphi) = \sup\{ \,|\, \xi \,|\, : \ \xi \in \operatorname{supp} \varphi \,\} < +\infty$);
2) $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$;
3) $\varphi(0) = 1$;
4) φ is continuous.

These conditions seem to be very natural. Indeed, 1)-3) guarantee that each function given by the approximation methods (2), (3) or (4) is a real-valued trigonometric polynomial and that the associated operators $\mathcal{F}_n^{(\varphi)}$, $\mathcal{I}_n^{(\varphi)}$ and $\mathcal{L}_{\sigma; \lambda}^{(\varphi)}$ map constant functions onto itself. Condition 4) can be weakened in the sense that discontinuous functions having some additional properties can be included in the class of generators. However, for the classical kernels which are considered in this paper such a modification is not needed.

Let us mention also that one usually deals with kernels produced by real-valued even generators. Practically all known constructions like, for instance, the kernels of Dirichlet, Fejér, Valle-Poussin, Rogosinski, Bochner-Riesz are of such a type.

In forthcoming papers we shall show that the proposed extension of the class of admissible generators turns out to be very useful for a constructive description of smoothness of odd orders.

Fourier transform. The Fourier transform and its inverse are given by

$$\widehat{g}(\xi) = \int_{\mathbb{R}^d} g(x) e^{-ix\xi}\, dx \;, \quad g^{\vee}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} g(\xi) e^{ix\xi}\, d\xi, \qquad g \in L_1(\mathbb{R}^d)\,.$$

Relations up to constants. By „$A \lesssim B$" we denote the relation $A \leqslant cB$, where c is a positive constant independent of $f \in L_p$ (or $f \in C$) and $\sigma \geqslant 0$. The symbol „$\asymp$" indicates equivalence. It means that $A \lesssim B$ and $B \lesssim A$ simultaneously.

2. Approximation by general linear polynomial operators

It turns out that some properties of the methods given by (2)–(4) such as *the convergence criterion* and *the comparison principle* do not depend on their specific structure. They can be obtained applying an universal approach which will be described in this section. Let us consider linear operators

$$\mathcal{L}_\sigma \;:\; L_p \longrightarrow \mathcal{T}_{\gamma\sigma,\,\overline{p}} \subset L_{\overline{p}}\;, \quad \sigma \geqslant 0\,, \tag{2.1}$$

where $0 < p \leqslant +\infty$ and $\gamma > 0$. If $p = \infty$ the operators are defined on C. Clearly, the operators given by (2)-(4) are of such a type with $\gamma = \rho(\varphi)$. Moreover, for Fourier means (2) and the interpolation means (3) the range space $\mathcal{T}_{\gamma\sigma,\,\overline{p}}$ can be replaced by its subspace $\mathcal{T}_{\gamma\sigma,\,p}$.

As usual, a linear operator $\mathcal{L}_\sigma$ is bounded if its norm, given by

$$\|\, \mathcal{L}_\sigma \,\|_{(p)} = \sup_{\|\, f \,\|_p \leqslant 1} \|\, \mathcal{L}_\sigma(f) \,\|_{\overline{p}} \tag{2.2}$$

is finite. The family $(\mathcal{L}_\sigma)$ is called *bounded in* L_p if their norms are bounded by a constant independent of σ. That is,

$$\sup_{\sigma \geqslant 0} \|\, \mathcal{L}_\sigma \,\|_{(p)} \;<\; +\infty\;. \tag{2.3}$$

Recall that classical operators into $\mathcal{T}_{\gamma\sigma}$ as defined in (2) and (3) are not relevant for approximation in the case $0 < p < 1$. Indeed, as it was mentioned in [2], Lemma 3.2, p. 685 (see also [12], p. 37, for the case of functionals) for any $\sigma \geqslant 0$ and $0 < p < 1$ there do not exist non-trivial linear bounded operators mapping L_p into $\mathcal{T}_\nu \subset L_p$ if $0 < p < 1$. In contrast to this situation sets of operators mapping into the space $\mathcal{T}_{\gamma\sigma,\,\overline{p}}$ will be called *families of linear polynomial operators* following [7] and [8] and can be successfully applied to approximation in L_p for all $0 < p < +\infty$ as well as in C. A family $(\mathcal{L}_\sigma)$ is said to be *convergent in* L_p if for each $f \in L_p$ ($f \in C$ if $p = \infty$)

$$\lim_{\sigma \to +\infty} \|\, f - \mathcal{L}_\sigma(f) \,\|_{\overline{p}} = 0. \tag{2.4}$$

Obviously, for the approximation processes defined in (2) and (3) this concept coincides with the L_p-convergence in usual sense.

The following general statement on the convergence of families of type (2.1) is a direct consequence of the classical Banach-Steinhaus theorem for quasi-normed spaces. We mention that in contrast to the classical cases in Approximation Theory we apply this theorem to operators whose range spaces can essentially differ from the spaces where they are defined (doubled dimension).

Lemma 2.1. *Let $0 < p \leqslant +\infty$ and $\gamma > 0$. A family of linear bounded operators of type (2.1) converges in L_p if and only if the following conditions are satisfied:*

a) $\lim\limits_{\sigma \to +\infty} \| e^{ik\cdot} - \mathcal{L}_\sigma(e^{ik\cdot}) \|_{\overline{p}} = 0$ *for each* $k \in \mathbb{Z}^d$;

b) $(\mathcal{L}_\sigma)$ *is bounded in* L_p.

Now we establish *the general comparison principle* for families of type (2.1). We follow the ideas in the paper [4] on approximation by band-limited functions, where the rates of convergence of convolution integrals and generalized sampling series were compared with each other in the space of bounded uniformly continuous functions on $\mathbb{R}$. For a trigonometric version we refer to [10], where the equivalence of the rates of convergence of the Fourier means and the interpolation means generated by the same kernel was proved.

Lemma 2.2. *Let $0 < p \leqslant +\infty$, $\gamma > 0$, $\sigma \geqslant 0$, and $(\mathcal{L}_\sigma^{(j)})$, $j = 1, 2$, be families of linear bounded operators of type (2.1). If they are bounded in L_p and if $\mathcal{L}_\sigma^{(1)}(T) = \mathcal{L}_\sigma^{(2)}(T)$ for all $T \in \mathcal{T}_{\gamma\sigma}$, then*

$$\| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}} \asymp \| f - \mathcal{L}_\sigma^{(2)}(f) \|_{\overline{p}} , \qquad f \in L_p . \tag{2.5}$$

Proof. Let $f \in L_p$ ($f \in C$ if $p = \infty$) and let $\sigma \geqslant 0$. Let us denote the function $\mathcal{L}_\sigma^{(1)}(f)$ by $\mathcal{L}_{\sigma;\lambda}^{(1)}(f)$. For almost all parameter λ it belongs to $\mathcal{T}_{\gamma\sigma}$ (as a function of x). Therefore, we get

$$\mathcal{L}_\sigma^{(1)} \left(\mathcal{L}_{\sigma;\lambda}^{(1)}(f) \right) = \mathcal{L}_\sigma^{(2)} \left(\mathcal{L}_{\sigma;\lambda}^{(1)}(f) \right) \tag{2.6}$$

in the sense of identity of two elements in $L_{\overline{p}}$. Using (1.2), (2.2) and (2.6) we obtain for almost all λ the estimate

$$\begin{aligned} \| f - \mathcal{L}_\sigma^{(2)}(f) \|_{\overline{p}}^{\widetilde{p}} &\leqslant \| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}}^{\widetilde{p}} + \| \mathcal{L}_\sigma^{(1)}(f) - \mathcal{L}_\sigma^{(1)}(\mathcal{L}_{\sigma;\lambda}^{(1)}(f)) \|_{\overline{p}}^{\widetilde{p}} \\ &\quad + \| \mathcal{L}_\sigma^{(2)}(\mathcal{L}_{\sigma;\lambda}^{(1)}(f)) - \mathcal{L}_\sigma^{(2)}(f) \|_{\overline{p}}^{\widetilde{p}} \\ &\leqslant \| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}}^{\widetilde{p}} + \left(\| \mathcal{L}_\sigma^{(1)} \|_{(p)}^{\widetilde{p}} + \| \mathcal{L}_\sigma^{(2)} \|_{(p)}^{\widetilde{p}} \right) \\ &\quad \times \| f - \mathcal{L}_{\sigma;\lambda}^{(1)}(f) \|_{p;x}^{\widetilde{p}} . \end{aligned} \tag{2.7}$$

Taking the L_p-norm with respect to λ and division by $(2\pi)^{-d/p}$ if $1 \leqslant p \leqslant +\infty$ or integrating with respect to λ and division by $(2\pi)^{-d}$ if $0 < p < 1$ lead to

$$\| f - \mathcal{L}_\sigma^{(2)}(f) \|_{\overline{p}}^{\widetilde{p}} \leqslant \big(1 + \| \mathcal{L}_\sigma^{(1)} \|_{(p)}^{\widetilde{p}} + \| \mathcal{L}_\sigma^{(2)} \|_{(p)}^{\widetilde{p}} \big) \cdot \| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}}^{\widetilde{p}}$$
$$\leqslant c \| f - \mathcal{L}_\sigma^{(1)}(f) \|_{\overline{p}}^{\widetilde{p}} .$$

in view of the boundedness of $(\mathcal{L}_\sigma^{(j)})$, $j = 1, 2$, in L_p. Changing the roles of $\mathcal{L}_\sigma^{(1)}$ and $\mathcal{L}_\sigma^{(2)}$ in the above arguments we obtain the converse estimate. ■

3. Norms of families of linear polynomial operators

In this section we estimate the norms of the family defined in (4) by means of the L_p-norm of the Fourier transform of its generator φ. Notice (4) can be interpreted as an operator of type (2.1). The norm of the family $\{ \mathcal{L}_{\sigma;\lambda}^{(\varphi)} \}$ given by (4) is, by definition, the norm of the corresponding operator mapping into the space $L_p(\mathbb{T}^{2d})$ (or $C(\mathbb{T}^{2d})$). In view of (2.2) one has

$$\| \{ \mathcal{L}_{\sigma;\lambda}^{(\varphi)} \} \|_{(p)} = \sup_{\| f \|_p \leqslant 1} \| \mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f; x) \|_{\overline{p}} , \qquad \sigma \geqslant 0 . \tag{3.1}$$

Lemma 3.1. *Let* $\varphi \in \mathcal{K}$, $0 < p \leqslant +\infty$, $\widetilde{p} = \min(1, p)$, $\widehat{p} = p$ *for* $0 < p < +\infty$ $\widehat{p} = 1$ *for* $p = +\infty$, *and* $\sigma \geqslant 0$. *Then*

$$(\sigma + 1)^{d(1/\widehat{p} - 1)} \| W_\sigma(\varphi) \|_{\widehat{p}} \lesssim \| \{ \mathcal{L}_{\sigma;\lambda}^{(\varphi)} \} \|_{(p)} \lesssim (\sigma + 1)^{d(1/\widetilde{p} - 1)} \| W_\sigma(\varphi) \|_{\widetilde{p}} . \tag{3.2}$$

Proof. *Step 1.* First we prove *the upper estimate for* $0 < p \leqslant 1$. By (4) and (1.1)–(1.2) we get for each $f \in L_p$ the estimates

$$\begin{aligned}
&(2\pi)^{d/p} \| \mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f; x) \|_{\overline{p}}^p \\
&\qquad \leqslant (2N + 1)^{-dp} \sum_{\nu=0}^{2N} \| f(t_N^k + \lambda) \| W_\sigma(\varphi)(x - t_N^k - \lambda) \|_{p;\,x} \|_{p;\,\lambda}^p \\
&\qquad \leqslant (2N + 1)^{-dp} \cdot \| W_\sigma(\varphi) \|_p^p \cdot \sum_{\nu=0}^{2N} \| f(t_N^k + \lambda) \|_p^p \\
&\qquad \leqslant (2N + 1)^{d(1-p)} \| W_\sigma(\varphi) \|_p^p \, \| f \|_p^p .
\end{aligned}$$

This implies

$$\| \{ \mathcal{L}_{\sigma;\lambda}^{(\varphi)} \} \|_{\overline{p}} \leqslant (2\pi)^{-d/p} (2N + 1)^{d(1/p - 1)} \| W_\sigma(\varphi) \|_p , \qquad 0 < p \leqslant 1 . \tag{3.3}$$

Step 2. In order to prove the lower estimate for $0 < p < +\infty$. we consider the 2π-periodic function f_* which is defined on $[-\pi, \pi)^d$ by

$$f_*(h) = \begin{cases} (\mu(D_{\tau/2}(0)))^{-1/p}, & h \in D_{\tau/2}(0) \\ 0, & \text{otherwise} \end{cases} , \qquad \Big(\tau = \frac{2\pi}{2N + 1} \Big).$$

Here $D_\delta(a)$ is the ball of the radius δ centered at the point a and μ denotes the d-dimensional Lebesgue measure. For each $\lambda \in D_{\tau/2}(0)$ and for each vector $k \in \mathbb{Z}^d \setminus \{0\}$ with components $0 \leqslant k_j \leqslant 2N$, $j = 1, \ldots, d$, we have

$$|\, t_N^k + \lambda \,| \geqslant |\, t_N^k \,| - |\, \lambda \,| \geqslant \tau - \tau/2 = \tau/2 \,.$$

Hence, by definition of f_* we get that

$$f_*(t_N^k + \lambda) = 0 \quad \text{for } \lambda \in D_{\tau/2}(0),\ \ k \in \mathbb{Z}^d,\ \ k \neq 0\,,\ 0 \leqslant k_j \leqslant 2N,\ \ j = 1, \ldots, d\,.$$

Therefore, in view of (4)

$$\mathcal{L}_{n;\lambda}^{(\varphi)}(f_*; x) = (2N+1)^{-d}\, f_*(\lambda)\, W_n(\varphi)(x - \lambda), \qquad x \in \mathbb{T}^d,\ \lambda \in D_{\tau/2}(0)\,. \quad (3.4)$$

Moreover, it holds

$$\begin{aligned}
\mathcal{L}_{n;\lambda+\tau}^{(\varphi)}(f_*; x) &= (2N+1)^{-d} \cdot \sum_{\nu=0}^{2N} f\left(t_N^{\nu+1} + \lambda\right) \cdot W_\sigma(\varphi)\left(x - t_N^{\nu+1} - \lambda\right) \\
&= (2N+1)^{-d} \cdot \sum_{\nu=1}^{2N+1} f\left(t_N^{\nu} + \lambda\right) \cdot W_\sigma(\varphi)\left(x - t_N^{\nu} - \lambda\right) \\
&= \mathcal{L}_{n;\lambda}^{(\varphi)}(f_*; x)
\end{aligned}$$

because of the 2π-periodicity of the functions f_* and $W_\sigma(\varphi)$. Taking also into account that $\|f_*\|_p = 1$ we obtain the estimates

$$\begin{aligned}
\|\, \{\mathcal{L}_{\sigma;\lambda}^{(\varphi)}\}\,\|_{(p)}^p &\geqslant \|\, \mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f_*; x)\,\|_p^p \\
&= \tau^{-d} \int\limits_{[-\tau/2,\tau/2]^d} \Big(\int\limits_{\mathbb{T}^d} |\mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f_*; x)|^p\, dx\Big) d\lambda \\
&\geqslant \tau^{-d} \int\limits_{D_{\tau/2}(0)} \Big(\int\limits_{\mathbb{T}^d} |\, \mathcal{L}_{\sigma;\lambda}^{(\varphi)}(f_*; x)|^p\, dx\Big) d\lambda \\
&= (2\pi)^{-d}\, (2N+1)^{d(1-p)} \int\limits_{D_{\tau/2}(0)} |\, f_*(\lambda)|^p \Big(\int\limits_{\mathbb{T}^d} |\, W_\sigma(\varphi)(x-\lambda)|^p dx\Big) d\lambda \\
&= (2\pi)^{-d}\, (2N+1)^{d(1-p)}\, \|\, W_\sigma(\varphi)\,\|_p^p
\end{aligned}$$

from (3.4). Finally,

$$\|\, \{\mathcal{L}_{\sigma;\lambda}^{(\varphi)}\}\|_{(p)} \ \geqslant\ (2\pi)^{-d/p}\, (2N+1)^{d(1/p-1)}\, \|\, W_\sigma(\varphi)\,\|_p\,, \qquad 1 < p < +\infty\,. \quad (3.5)$$

Step 3. Now let $p = +\infty$. Using the composition

$$\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)} = S_{-\lambda} \circ \mathcal{I}_\sigma^{(\varphi)} \circ S_\lambda \,, \quad (3.6)$$

where $S_t f(\cdot) = f(\cdot + t)$ is the translation operator, we obtain

$$\begin{aligned}\| \mathcal{I}_\sigma^{(\varphi)} \|_{(\infty)} \leqslant \|\{\mathcal{L}_{\sigma;\lambda}^{(\varphi)}\}\|_{(\infty)} &= \sup_{f\in C\,,\, \| f \|_\infty \leqslant 1} \max_\lambda \| S_{-\lambda} \circ \mathcal{I}_\sigma^{(\varphi)} \circ S_\lambda(f) \|_\infty \\ &= \sup_{f\in C\,,\, \| f \|_\infty \leqslant 1} \max_\lambda \| \mathcal{I}_\sigma^{(\varphi)} \circ S_\lambda(f) \|_\infty \\ &\leqslant \sup_{f\in C\,,\, \| f \|_\infty \leqslant 1} \max_\lambda \| \mathcal{I}_\sigma^{(\varphi)} \|_{(\infty)} \cdot \| S_\lambda(f) \|_\infty = \| \mathcal{I}_\sigma^{(\varphi)} \|_{(\infty)} \,.\end{aligned}$$

Hence,

$$\|\{\mathcal{L}_{\sigma;\lambda}^{(\varphi)}\}\|_{(\infty)} = \| \mathcal{I}_\sigma^{(\varphi)} \|_{(\infty)} \,.$$

Applying the known estimate for the norm of $\mathcal{I}_\sigma^{(\varphi)}$ (see, for instance, [10]) we obtain (3.2) for $p = +\infty$.

Step 4. By the classical Riesz-Thorin interpolation theorem we get

$$\| \{\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)} \|_{(p)} \;\leqslant\; \| \{\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)}\}\|_{(1)}^{1/p} \cdot \| \{\mathcal{L}_{\sigma;\,\lambda}^{(\varphi)}\}\|_{(\infty)}^{1-1/p}$$

for $1 < p < +\infty$. This leads to the upper estimate for $1 < p < +\infty$ by Step 1–Step 3. The proof is complete. ■

Lemma 3.2. *Let* $0 < p \leqslant 1$. *The set* $\{ \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p \,:\, \sigma \geqslant 0 \}$ *is bounded if and only if* $\widehat{\varphi} \in L_p(\mathbb{R}^d)$. *Here,* $\varphi \in \mathcal{K}$. *Moreover, in this case*

$$\lim_{\sigma \to +\infty} \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p = \sup_{\sigma\geqslant 0} \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p = \| \widehat{\varphi} \|_{L_p(\mathbb{R}^d)} \,. \tag{3.7}$$

Proof. *Step 1.* Using the Poisson summation formula one easily obtains (see [2], Lemma 2.2, p. 681, for details)

$$\left\| \sum_{k \in \mathbb{Z}^d} g(k) \cdot e^{ikx} \right\|_p \leqslant \| \widehat{g} \|_{L_p(\mathbb{R}^d)} \tag{3.8}$$

for all continuous function g with compact support satisfying $\widehat{g} \in L_p(\mathbb{R}^d)$. Applying (3.8) to $\varphi(\cdot/\sigma)$ and using the equalities $\widehat{\varphi(\sigma^{-1}\cdot)}(x) = \sigma^d \widehat{\varphi}(\sigma x)$ we get

$$\sup_{\sigma\geqslant 0} \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p \leqslant \| \widehat{\varphi} \|_{L_p(\mathbb{R}^d)} \,. \tag{3.9}$$

Step 2. Now we prove: *I*f there exists a sequence $(\nu_n)_{n\in\mathbb{N}}$ of strictly increasing natural numbers such that the sequence $\nu_n^{d(1/p-1)} \|W_{\nu_n}(\varphi)\|_p$ is bounded, then $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ and

$$\| \widehat{\varphi} \|_{L_p(\mathbb{R}^d)} \leqslant \sup_{n\in\mathbb{N}} \nu_n^{d(1/p-1)} \| W_{\nu_n}(\varphi)\|_p \,. \tag{3.10}$$

Consider the sequence of functions $F_n(x)$, $n \in \mathbb{N}$, given by

$$F_n(x) = \begin{cases} \nu_n^{-dp} \left| W_{\nu_n}(\varphi) \left(\frac{x}{\nu_n}\right)\right|^p, & x \in [-\pi\nu_n, \pi\nu_n]^d \\ 0, & \text{otherwise} \end{cases} . \tag{3.11}$$

Clearly, the functions F_n, $n \in \mathbb{N}$, are non-negative and measurable. Let $x_0 \in \mathbb{R}^d$. Then there exists $n_0 \in \mathbb{N}$ such that $x_0 \in [-\pi\nu_n, \pi\nu_n]^d$ for $n \geqslant n_0$. The function $\varphi(\cdot)e^{ix_0\cdot}$ is integrable in the Riemannian sense on the cube $\Omega \subset \mathbb{R}^d$ containing its support. By definition of the Riemann-integral we get

$$\lim_{n\to+\infty} \nu_n^{-d} \sum_{k\,\in\,\mathbb{Z}^d} \varphi\left(\frac{k}{\nu_n}\right) \cdot e^{(ikx_0)/\nu_n} = \int_{\Omega} \varphi(\xi) \cdot e^{i\xi x_0} d\xi = \widehat{\varphi}(-x_0) \ .$$

Hence,

$$\lim_{n\to+\infty} F_n(x_0) = |\,\widehat{\varphi}(-x_0)\,|^p \ . \tag{3.12}$$

By definition of F_n in (3.11) it follows

$$\begin{aligned} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} F_n(x)dx = {} & \sup_{n\in\mathbb{N}} \nu_n^{d(1-p)} \int_{[-\pi\nu_n,\pi\nu_n]^d} \left| W_{\nu_n}(\varphi)\Big(\frac{x}{\nu_n}\Big)\right|^p \nu_n^{-d}\, dx \\ = {} & \sup_{n\in\mathbb{N}} \nu_n^{d(1-p)} \, \|\, W_{\nu_n}(\varphi)\,\|_p^p \ < \ +\infty \ . \end{aligned} \tag{3.13}$$

Thus, we have proved that the sequence $F_n(x)$, $n \in \mathbb{N}$, satisfies all conditions of the Fatou lemma. Combining it with (3.12) and (3.13) we obtain $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ and (3.10).

Step 3. Now, the criterion and the second relation in (3.7) follow immediately from the statements above. Suppose that the set $\sigma^{d(1/p-1)}\,\|\, W_\sigma(\varphi)\,\|_p$ is bounded and let a be one of its accumulation points. By (3.9) a does not exceed the norm of $\widehat{\varphi}$ in $L_p(\mathbb{R}^d)$. The inverse estimate follows from (3.10). As a consequence we get (3.7). ∎

Let us mention that the idea of using the Poisson summation formula for estimates of type (3.9) can be found in [6], where the case of infinitely differentiable generators φ has been considered. For $p = 1$ the second relation in (3.7) can be also derived from the results on the connections between periodic and non-periodic multipliers [15], Theorem 3.8, p. 260; Theorem 3.18, p.264; Corollary 3.28, p. 267, in combination with the criterion for non-periodic multipliers in $L_1(\mathbb{R}^d)$ [13], pp. 28, 95. The existence of the limit and formula (3.7) in this case were established in [10].

4. General Convergence Theorem

Before we formulate and prove the main result of this paper - *the General Convergence theorem (GCT)* - we introduce two sets of parameters p which are associated with the generator φ. For $\varphi \in \mathcal{K}$ we put

$$\mathcal{P}_\varphi = \{\, p \in (0,+\infty] : \widehat{\varphi} \in L_p(\mathbb{R}^d)\,\} \ .$$

Since $\lim_{|x|\to+\infty} |\,\widehat{\varphi}(x)\,| = 0$ we have $\widehat{\varphi} \in L_q(\mathbb{R}^d)$ for $p \leqslant q \leqslant \infty$ if $\widehat{\varphi} \in L_p(\mathbb{R}^d)$. Hence, $\mathcal{P}_\varphi$ is $(p_0, +\infty]$ or $[p_0, +\infty]$, where $p_0 = \inf \mathcal{P}_\varphi$. The set

$$\mathcal{C}_\varphi \;=\; \{\, p \in (0, +\infty] : \{\, \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \} \text{ converges in } L_p \,\}$$

is called *range of convergence* of the family $\{\, \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \}$.

Theorem 4.1. *Let $\varphi \in \mathcal{K}$ and assume $1 \in \mathcal{P}_\varphi$. Then, $\mathcal{C}_\varphi = \mathcal{P}_\varphi$. Moreover, for $1 \leqslant p \leqslant +\infty$*

$$\|\, f - \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}(f)\,\|_{\overline{p}} \asymp \|\, f - \mathcal{F}^{(\varphi)}_{\sigma}(f)\,\|_p\,, \;\; f \in L_p \;\, (f \in C \;\, if \;\, p = \infty), \;\; \sigma \geqslant 0\,. \qquad (4.1)$$

If $p = +\infty$ then (4.1) *holds with $\mathcal{I}^{(\varphi)}_\sigma$ in place of $\mathcal{F}^{(\varphi)}_\sigma$ for all $f \in C$ and $\sigma \geqslant 0$.*

Proof. *Step 1.* Recall $\rho \geqslant \rho(\varphi)$. First we claim that for all trigonometric polynomials $T(x) = \sum\limits_{k\in\mathbb{Z}^d} c_k\, e^{ikx} \in \mathcal{T}_{\rho\sigma}$ and for all $\sigma \geqslant 0$

$$\mathcal{F}^{\varphi}_{\sigma}(T;x) \;=\; \sum_{k\,\in\,\mathbb{Z}^d} \varphi\Big(\frac{k}{\sigma}\Big) c_k\, e^{ikx} \qquad (4.2)$$

$$\mathcal{I}^{\varphi}_{\sigma}(T;x) \;=\; \sum_{k\,\in\,\mathbb{Z}^d} \varphi\Big(\frac{k}{\sigma}\Big)\, c_k\, e^{ikx} \;=\; \mathcal{F}^{\varphi}_{\sigma}(T;x) \qquad (4.3)$$

$$\mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}(T;x) \;=\; \sum_{k\in\mathbb{Z}^d} \varphi\Big(\frac{k}{\sigma}\Big) c_k e^{ikx} \;=\; \mathcal{F}^{\varphi}_{\sigma}(T;x)\;. \qquad (4.4)$$

The proofs of (4.2) and (4.3) are straightforward and can be found, for instance, in [10]. Equality (4.4) immediately follows from (4.3) and (3.6). Hence, the operators defined in (2) - (4) coincide on the space of trigonometric polynomials $\mathcal{T}_{\rho\sigma}$.

Step 2. Let $0 < p \leqslant +\infty$. Suppose $\nu \in \mathbb{Z}^d$, $\lambda \in \mathbb{R}^d$ and $\sigma \geqslant |\nu|/\rho$. Using (4.4) we get

$$\|\, e^{i\nu\cdot} - \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}(e^{i\nu\cdot})\,\|_{\overline{p}} \;=\; \left|\, 1 - \varphi\left(\frac{\nu}{\sigma}\right)\right| \cdot \|\, e^{i\nu\cdot}\,\|_p \;=\; (2\pi)^{d/p} \left|\, 1 - \varphi\left(\frac{\nu}{\sigma}\right)\right|\;.$$

Taking into account that φ is continuous we obtain therefrom

$$\lim_{\sigma\to+\infty} \|\, e^{i\nu\cdot} - \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda}(e^{i\nu\cdot})\,\|_{\overline{p}} \;=\; 0, \qquad \nu \in \mathbb{Z}^d\,. \qquad (4.5)$$

Step 3. By assumption $\mathcal{P}_\varphi$ contains the interval $[1, +\infty]$. We show that $[1, +\infty] \subset \mathcal{C}_\varphi$ as well. Indeed, by Lemmas 3.1 and 3.2 we get

$$\|\, \{\, \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \}\,\|_{(p)} \;\lesssim\; \|\, W_\sigma(\varphi)\,\|_1 \;\leqslant\; \|\, \widehat{\varphi}\,\|_{L_1(\mathbb{R}^d)} \;<\; +\infty$$

for $1 \leqslant p \leqslant +\infty$ which is the boundedness of $\{\, \mathcal{L}^{(\varphi)}_{\sigma;\,\lambda} \}$ in L_p. Combining this and (4.5) we obtain by Lemma 2.1 the convergence in L_p. Hence, $[1, +\infty] \subset \mathcal{C}_\varphi$.

In order to complete the proof of the coincidence of $\mathcal{P}_\varphi$ and $\mathcal{C}_\varphi$ it is enough to show that $p \in (0,1)$ belongs or does not belong to $\mathcal{P}_\varphi$ and $\mathcal{C}_\varphi$ simultaneously. Indeed, the statement "$p \in \mathcal{P}_\varphi$" is equivalent to the boundedness of the set $\{ \sigma^{d(1/p-1)} \| W_\sigma(\varphi) \|_p , \sigma \geqslant 0 \}$ by Lemma 3.2 and to the boundedness of the family $\{ \mathcal{L}^{(\varphi)}_{\sigma;\lambda} \}$ in L_p by Lemma 3.1. In view of (4.5) and Lemma 2.1 this statement is equivalent to the convergence of $\{ \mathcal{L}^{(\varphi)}_{\sigma;\lambda} \}$ in L_p, that is, to the statement "$p \in \mathcal{C}_\varphi$". Taking into account that the condition "$1 \in \mathcal{P}_\varphi$" implies the boundedness of $\mathcal{F}^{(\varphi)}_\sigma$ in L_p with $1 \leqslant p \leqslant +\infty$ and $\mathcal{I}^{(\varphi)}_\sigma$ in $C(\mathbb{T}^d)$ (see, for instance, [10] for references) we conclude the equivalence of the approximation errors of $\{ \mathcal{L}^{(\varphi)}_{\sigma;\lambda} \}$ and $\mathcal{F}^{(\varphi)}_\sigma$ for $1 \leqslant p \leqslant +\infty$ as well as of $\{ \mathcal{L}^{(\varphi)}_{\sigma;\lambda} \}$ and $\mathcal{I}^{(\varphi)}_\sigma$ in $C(\mathbb{T}^d)$ from *Step 1* and Lemma 2.2. The proof is complete. ∎

5. Families generated by classical kernels

In this section we apply the General Convergence Theorem to families of linear polynomial operators generated by the classical kernels in order to find their sharp ranges of convergence. In general, the classical kernels are of type (1), where the set $\{ \sigma \geqslant 0 \}$ is replaced by a certain sequence $(\nu_n)_{n \in \mathbb{N}_0}$ satisfying $\lim\limits_{n \to +\infty} \nu_n = +\infty$. It should be noticed in this respect that such a replacement does not affect the results above.

For more information concerning classical trigonometric kernels and approximation processes of types (2) and (3) and results for the spaces $L_p(\mathbb{T}^d)$ with $1 \leqslant p < +\infty$ and $C(\mathbb{T}^d)$ we refer to [5] and [3]. The convergence of the methods generated by Bochner-Riesz kernels has been investigated in [11]. For properties of the families related to Fejér and de la Vallée-Poussin kernels we refer to [1]. Nevertheless, in order to demonstrate the power of our general approach and for the sake of completeness we consider here these kernels as well.

The **Fejér kernel** is defined by (1) with $\varphi(\xi) = (1 - |\xi|)_+$ as generator ($a_+ = \max(a,0)$), where $\{ \sigma \geqslant 0 \}$ is replaced by the set of non-negative natural numbers $\mathbb{N}_0$. The corresponding family $\{\mathcal{F}_{n;\lambda}\}$ is given by (4) with $n \in \mathbb{N}_0$ in place of σ. Straightforward calculation gives

$$\widehat{\varphi}(x) = \frac{4\sin^2(x/2)}{x^2} .$$

Hence, $\mathcal{P}_\varphi = (1/2, +\infty]$. By the GCT *the family* $\{\mathcal{F}_{n;\lambda}\}$ *generated by the Fejér kernel converges in* L_p *if and only if* $1/2 < p \leqslant +\infty$.

The **de la Vallée-Poussin kernel** is defined by (1) with

$$\varphi(\xi) = \begin{cases} 1, & \xi\,| \leqslant 1 \\ 2 - |\xi|, & 1 < |\xi| \leqslant 2 , \\ 0, & |\xi| > 2 \end{cases}$$

where the set $\{\sigma \geqslant 0\}$ is replaced by $\mathbb{N}_0$. The corresponding family $\{\mathcal{V}_{n;\,\lambda}\}$ is given by (4) with $n \in \mathbb{N}_0$ in place of σ. Again, by straightforward calculation one has

$$\widehat{\varphi}(x) = \frac{4\sin(x/2)\sin((3x)/2)}{x^2} \ .$$

Hence, we have the same range of convergence $\mathcal{P}_\varphi = (1/2, +\infty]$ as in the case of Fejér and by our GCT *the family* $\{\mathcal{V}_{n;\,\lambda}\}$ *generated by the de la Vallée-Poussin kernels converges in* L_p *if and only if* $1/2 < p \leqslant +\infty$.

The **Rogosinski kernel** is of type (1) with

$$\varphi(\xi) = \begin{cases} \cos\frac{\pi\xi}{2}, & |\,\xi\,| \leqslant 1 \\ 0, & |\,\xi\,| > 1 \end{cases}$$

as generator and $\sigma = n + 1/2$, $n \in \mathbb{N}_0$. In a straightforward manner we get

$$\widehat{\varphi}(x) = \frac{\pi\cos x}{(\pi/2)^2 - x^2} \ , \qquad \mathcal{P}_\varphi = (1/2, +\infty] \ .$$

By the GCT *the family* $\{\mathcal{R}_{n;\,\lambda}\}$ *generated by the Rogosinski kernel converges in* L_p *if and only if* $1/2 < p \leqslant +\infty$.

The **Bochner-Riesz kernels** with the index α in the d-dimensional case are given by (1), where $\varphi(x) = \varphi_\alpha(\xi) = \big(1 - |\,\xi\,|^2\big)_+^\alpha$ and $\sigma = n$, $n \in \mathbb{N}_0$. As it is known ([14], Ch. 9, pp. 389-390),

$$\widehat{\varphi}_\alpha(x) = \pi^{-\alpha}\,\Gamma(\alpha+1)\,|\,x\,|^{-\alpha-d/2}\,J_{\alpha+d/2}(|\,x\,|) \ ,$$

where $J_s(x)$, $s > -1/2$, denotes the Bessel function of order s. Using the properties of Bessel functions, in particular, their asymptotic formula (see, for example, [14], Ch. 8, pp. 356-357) we see that $J_s(x)$ can be estimated from above and from below by $|\,x\,|^{-1/2}$ on a certain set of infinite measure. In particular, we find

$$\|\,\widehat{\varphi}_\alpha\,\|^p_{L_p(\mathbb{R}^d)} \asymp \int\limits_{|x|\,\geqslant\,1} |\,x\,|^{-p(\alpha+d/2+1/2)}\,dx \asymp \int\limits_1^{+\infty} r^\sigma\,dr \ ,$$

where $\sigma = -p(\alpha + d/2 + 1/2) + d - 1$. Hence,

$$\mathcal{P}_{\varphi_\alpha} = (\,2d/(d + 2\alpha + 1),\, +\infty\,] .$$

Thus, $1 \in \mathcal{P}_{\varphi_\alpha}$ if and only if $\alpha > (d-1)/2$ (Bochner's critical index). By the GCT we obtain the *convergence of the family* $\{\mathcal{B}^{(\alpha)}_{n;\,\lambda}\}$ *generated by a kernel* φ_α *with* $\alpha > (d-1)/2$ *in* L_p *if and only if* $2d/(d + 2\alpha + 1) < p \leqslant +\infty$.

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