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TAME (PLS)-SPACES*

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Abstract: The class of (PLS)-spaces covers most of the natural spaces of analysis, e. g. the space of real analytic functions, spaces of distributions. We characterize those (PLS)-spaces for which there exists a 'reasonable' (LFS)-topology, i. e. a topology of the inductive limit of a sequence of Fréchet-Schwartz spaces. Then we characterize - in terms of the defining sequence - power series (PLS)-type spaces which satisfy the same condition. It is known that power series (PLS)-type spaces appear naturally as kernels of convolution operators.
Keywords: Fréchet space, (PLS)-space, (LFS)-space.

1. Introduction

In the folowing paper we investigate *(PLS)-spaces*. These are (see [4], [5]) the projective limits of a sequence of duals of Fréchet-Schwartz spaces. This means that every (PLS)-space X can be viewed as

$$X = \operatorname*{proj}_{N\in\mathbb{N}} \operatorname*{ind}_{n\in\mathbb{N}} X_{N,n},$$

where all the $X_{N,n}$ are Banach spaces and all the linking maps $\iota_{N,n}^{N,n+1}\colon X_{N,n} \to X_{N,n+1}$ are compact (see Section 2 for examples). If, in addition, all these linking maps are nuclear then X is called a *(PLN)-space*. We also define $X_N := \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ and by $\iota_N\colon X \to X_N$ we denote the canonical projection.

We introduce a new linear-topological invariant in the category of (PLS)-spaces called *tameness*. The main result shows that on tame (PLS)-spaces a topology of the inductive limit of a sequence of Fréchet-Schwartz spaces can be defined – see Th. 8. We can also observe an interesting interplay between tameness and ultrabornologicity: it is proved that a power series (PLS)-type space is at the same time an (LFS)-space if and only if it is a product of an (LS)-space and an (FS)-space – see Th. 14. This shows that finding non-trivial (by trivial we mean

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a product of an (LS)-space and an (FS)-space) examples of (PLS)-spaces being at the same time (LFS)-spaces is impossible among power series (PLS)-type spaces.

The concept of a tame (PLS)-space arises from the – first defined (see [7]) – tameness in the category of Fréchet spaces. The latter notion of tameness is related to the (still open) Pełczyński's question (see [17]). Results on tame Fréchet spaces can be found in [7] and [20] (see also [16]). It is known (see [8, Th. 1.3]) that Fréchet-Schwartz spaces are also (PLS)-spaces so in this particular case we have two notions of tameness and they do not coincide (see examples in Section 2). Tameness appears also in the context of splitting of short exact sequences – see [28], [21].

For unexplained facts and notation from functional analysis we refer the reader to [15]. For informations on (PLS)-spaces see [6] and references therein.

2. Definition and examples

We start by recalling some examples of (PLS)-spaces.

Examples ([6]). 1) The space $\mathcal{A}(\Omega)$ of real analytic functions on an open subset $\Omega \subset \mathbb{R}^n$ is a (PLN)-space.
2) The space $\mathcal{D}'(\Omega)$ of distributions on an open subset $\Omega \subset \mathbb{R}^n$ is a (PLN)-space.
3) Köthe type (PLS)-sequence space. Let $A := (a_{j,N,n})_{j,N,n\in\mathbb{N}}$ be a matrix of nonnegative scalars withe the following properties:
(i) $a_{j,N,n+1} \leqslant a_{j,N,n} \leqslant a_{j,N+1,n}$,
(ii) $\lim_j \frac{a_{j,N,n+1}}{a_{j,N,n}} = 0$
for all $j, N, n \in \mathbb{N}$. We define

$$\Lambda^p(A) := \{x = (x_j)_j \mid \forall\, N \in \mathbb{N}\ \exists\, n \in \mathbb{N}\colon\ \|x\|_{N,n} < +\infty\},$$

where

$$\|x\|_{N,n} := \Big(\sum_{j=1}^{+\infty} (a_{j,N,n}|x_j|)^p\Big)^{\frac{1}{p}}.$$

For $p = +\infty$ we use the respective "sup" norms. Then

$$\Lambda^p(A) = \operatorname*{proj}_{N\in\mathbb{N}} \operatorname*{ind}_{n\in\mathbb{N}} l^p(a_{j,N,n}).$$

4) Every closed subspace and every complete quotient of a (PLS)-space is again a (PLS)-space.
5) ([14]) Let $\mathcal{E}_\omega(\mathbb{R})$ be the Roumieu class of ultradifferentiable functions which is not quasi-analytic (see [3] for the definition). If the convolution operator

$$T_\mu\colon \mathcal{E}_\omega(\mathbb{R}) \to \mathcal{E}_\omega(\mathbb{R}), \quad T_\mu f := f * \mu$$

is surjective then by [14, Th. 3.6] $\ker T_\mu$ is Köthe type (PLS)-space $\Lambda^1(A)$ with explicitly calculable matrix A.
6) ([12]) Let $F(z) := \sum_{n=0}^{+\infty} a_n z^n$ be an entire function such that

$$\sup_{\mathbb{C}} |f(z)| \exp\left(-\frac{|z|}{m}\right) < +\infty \quad \forall\, m \in \mathbb{N}.$$

Then the kernel of the partial differential operator

$$F(D)\colon \mathcal{A}(\Omega) \to \mathcal{A}(\Omega), \quad F(D)(f) := \sum_{n=1}^{+\infty} a_n (-i)^n \frac{d^n}{dx^n} f$$

is isomorphic to $\Lambda^1(A)$ with an explicitly calculable matrix A.

Definition 1. (1) Let $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ be a (PLS)-space. For every $x \in X$ the function $\phi_x\colon \mathbb{N} \to \mathbb{N}$ defined by

$$\phi_x(N) := \min\{n\colon \iota_N x \in X_{N,n}\}$$

is called the characteristic of membership of the element x.
(2) If there exists $\psi\colon \mathbb{N} \to \mathbb{N}$ with the property that

$$\forall\, x \in X \ \exists\, N_0 \in \mathbb{N} \ \forall\, N \geqslant N_0\colon \ \phi_x(N) \leqslant \psi(N)$$

then X is called a tame (PLS)-space.

Remark. The definition of ϕ_x depends on the chosen spectrum of X but tameness is a topological invariant in the category of (PLS)-spaces. Moreover if we consider a space $L(X) = \operatorname{proj}_N \operatorname{ind}_n L(X_n, X_N)$ ($X = \operatorname{proj}_n X_n$ – a Fréchet space, X_n a Banach space) then the characteristic of membership of any operator $T \in L(X)$ becomes its characteristic of continuity (see [7] for the definition). The definition of tameness may as well be defined for an arbitrary *(PLB)-space* (definition like for (PLS)-spaces except the assumption on compactness of the linking maps) but since we need a completeness of the space, we restrict our considerations to the class of (PLS)-spaces.

Examples. 1) It follows from [8, Th. 1.3] that every Fréchet-Schwartz space is a (PLS)-space and by the use of the Grothendieck's Factorization Theorem [15, Th. 24.33] we can observe that it is tame as a (PLS)-space. By [7, Th. 1.3] there exist Fréchet-Schwartz spaces which are not tame as Fréchet spaces. This shows that these two definitions of tameness do not coinside for Fréchet-Schwartz spaces.
2) The space $\mathcal{A}(\Omega)$ of real analytic functions on an open subset $\Omega \subset \mathbb{R}$ is not tame. Indeed by a theorem of Martineau (see [6, Th. 1.4], compare [13]) we have

$$\mathcal{A}(\Omega) = \operatorname*{proj}_{N\in\mathbb{N}} \operatorname*{ind}_{n\in\mathbb{N}} H^\infty(U_{N,n}).$$

For an arbitrary function $\psi\colon \mathbb{N} \to \mathbb{N}$ we take a complex zero neighbourhood V of Ω and a sequence of points $(a_N)_N \subset V$ without a condensation point in V so that

$$a_N \in U_{N,\psi(N)} \setminus U_{N,\psi(N)+1} \quad \forall N \in \mathbb{N}.$$

By the Mittag-Leffler Theorem (see [22]) we construct a meromorphic function $f\colon V \to \mathbb{C}$ with poles only in the points a_N. Then $\phi_f(N) = \psi(N) + 1$.
3) The space $\mathcal{D}'(\Omega)$ of distributions on an open subset $\Omega \subset \mathbb{R}^n$ is not tame. By a theorem of Valdivia-Vogt (see [6, Th. 2.3(b)], compare [23], [25]) we have

$$\mathcal{D}'(\Omega) = [\Lambda'_\infty(\ln j)]^{\mathbb{N}}.$$

For an arbitrary $\psi\colon \mathbb{N} \to \mathbb{N}$ let x^*_N be the functional on $\Lambda_\infty(\ln j)$ which is continuous with respest to the norm $\|\cdot\|_{\psi(N)+1}$ and discontinuous with respect to the norm $\|\cdot\|_{\psi(N)}$. Then the characteristic of membership of $x^* := (x^*_N)_N$ is $\phi_{x^*}(N) = \psi(N) + 1$.

Proposition 2. *Let X be a (PLB)-space and let $B \subset X$ be a Banach disc.*
(i) $\phi_{x+y} \leqslant \max\{\phi_x, \phi_y\}$, $\phi_{ax} = \phi_x$ $(x, y \in X,\ a \neq 0)$,
(ii) *there exists a function $\psi\colon \mathbb{N} \to \mathbb{N}$ such that $\phi_x \leqslant \psi$ for all $x \in \operatorname{span} B$.*

Proof. The first property is obvious. If now B is a Banach disc then by [15, Lemma 24.2] there are Banach discs $B_N \subset X_N$ such that $B \subset \prod_N B_N$. X_N is an (LB)-space so by [2, Chapter 3, Cor. 4] we find $n_N \in \mathbb{N}$ such that B_N is bounded in X_{N,n_N}. It is enough to define $\psi(N) := n_N$ to see that $\phi_x \leqslant \psi$ for all $x \in B$. By (i) the same holds for $x \in \operatorname{span} B$. ■

Let X be a (PLS)-space. Following [19] we define for an arbitrary $\phi \in \mathbb{N}^{\mathbb{N}}$ linear subspaces of X:

$$X_\phi := \{x \in X\colon\ \phi_x(N) \leqslant \phi(N) \ \text{ for all } N \in \mathbb{N}\},$$

$$X_{\phi,k} := \{x \in X\colon\ \phi_x(N) \leqslant \phi(N) \ \text{ for all } N \geqslant k\}.$$

We endow these spaces with the topologies given by sequences of seminorms

$$\|\cdot\|^\phi_m = \max_{1 \leqslant i \leqslant m} \|\cdot\|_{i,\phi(i)} \quad \text{and} \quad \|\cdot\|^{\phi,k}_m = \max_{k \leqslant i \leqslant m} \|\cdot\|_{i,\phi(i)},$$

respectively. Now we need some lemmas that will lead us to the main theorem in the next Section.

Lemma 3. *The spaces $X_\phi, X_{\phi,k}$ become Fréchet spaces.*

Proof. We will show the completeness of X_ϕ. Let $B_{N,n}$ be a closed unit ball in $X_{N,n}$ and $\iota^N_{N-1}\colon X_N \to X_{N-1}$ a linking map. The set

$$C_N := B_{N,\phi(N)} \cap (\iota^N_{N-1})^{-1}(B_{N-1,\phi(N-1)}) \cap \ldots \cap (\iota^N_1)^{-1}(B_{1,\phi(1)})$$

is compact in X_N thus, by [15, Cor. 23.14], a Banach disc. Denote by μ_{C_N} the Minkowski functional of C_N and let

$$Y_{\phi,N} := (\text{span } C_N, \mu_{C_N})$$

be a Banach space. Obviously $\mu_{C_N}(\iota_N x) = \|x\|_N^\phi$. Let now $(x_p)_p$ be a Cauchy sequence in X_ϕ. Then $(\iota_N x_p)_p$ is a Cauchy sequence in $Y_{\phi,N}$ for all $N \in \mathbb{N}$ and so we may write

$$\lim_p \iota_N x_p = y_N \in Y_{\phi,N}.$$

Obviously $x := (y_N)_N \in \text{proj}_N Y_{\phi,N} \subset X$. Moreover $\|x - x_p\|_m^\phi \to 0$ for every $m \in \mathbb{N}$ which shows the completeness of X_ϕ. ■

Proposition 4. *(1) The inductive topology of the system* $(\iota_\phi : X_\phi \to X)_{\phi\in\mathbb{N}^\mathbb{N}}$ *exists.*
(2) $X^{ub} = \text{ind}_{\phi\in\mathbb{N}^\mathbb{N}} X_\phi =: X^i$ *topologically.*

Proof. (1): Obviously $X = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} X_\phi$. For every function $\phi : \mathbb{N} \to \mathbb{N}$ we have $X_\phi = \text{proj}_N X_{N,\phi(N)}$ therefore the map $\iota_N|_{X_\phi} : X_\phi \to X_N$ is continuous for all N. By [10, Chapter 4, 19.6(6)] the inclusion $X_\phi \hookrightarrow X$ is continuous. This holds for every ϕ thus by [15, Lemma 24.6] the result follows.

(2) $X^i \hookrightarrow X^{ub}$: By the continuity of the map $X_\phi \hookrightarrow X$ and [11, Chapter 7, 35.8(1)] we get the continuity of the map $X_\phi^{ub} \hookrightarrow X^{ub}$. But X_ϕ is a Fréchet space therefore ultrabornological by [15, 24.15(c)]. The arbitrarieness of ϕ together with [15, Prop. 24.7] gives the continuity of the desired inclusion.

$X^{ub} \hookrightarrow X^i$: Let $B \subset X$ be a Banach disc. By Prop. 2(ii) and [15, Remark 23.2(a)] the Banach space $X_B := (\text{span } B, \|\cdot\|_B)$ is a subset of X_ϕ for some $\phi : \mathbb{N} \to \mathbb{N}$. Again by Prop. 2(ii) we see that B is bounded in X_ϕ therefore the map $X_B \hookrightarrow X^i$ is continuous. Again by [15, Prop. 24.7] we get the desired inclusion. ■

Lemma 5. *Let X be a (PLS)-space and let $\phi, (\psi_k)_{k\in\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ be nondecreasing functions. The following conditions are equivalent:*
i) $X_\phi \subset \bigcup_{k\in\mathbb{N}} X_{\psi_k}$,
ii) $\exists\, k\ \forall\, m\ \exists\, n, C_m\ \forall\, x \in X_\phi : \max_{1\leqslant l\leqslant m} \|x\|_{l,\psi_k(l)} \leqslant C_m \max_{1\leqslant p\leqslant n} \|x\|_{p,\phi(p)}$. (1)

Proof. $(i) \Rightarrow (ii)$: Consider the diagram

$$\begin{array}{ccc} \bigcup_k X_{\psi_k} & \xrightarrow{id} & X \\ & & \uparrow{\scriptstyle id} \\ & & X_\phi \end{array}$$

For every $k \in \mathbb{N}$ the horizontal identity $X_{\psi_k} \xrightarrow{id} X$ is continuous as well as the vertical one. Therefore by the Grothendieck's Factorization Theorem [15, Th. 24.33] we find a natural number k such that the inclusion $X_\phi \hookrightarrow X_{\psi_k}$ is continuous but this is exactly the condition (1). The other implication is obvious. ■

Lemma 6. *Let X be a (PLS)-space and Z a Fréchet space. Let τ be a locally convex topology on X such that the τ-bounded sets are bounded in the original* *topology of X. If $S: Z \to (X,\tau)$ is a linear and continuous operator then the set $S(Z)$ is contained in X_ϕ for some function $\phi: \mathbb{N} \to \mathbb{N}$ and the operator $S: Z \to X_\phi$ is continuous.*

Proof. Suppose there exists $N \in \mathbb{N}$ such that

$$\sup\{\phi_x(N):\ x \in S(Z)\} = +\infty.$$

This means that there is a sequence $(z_n)_n \subset Z$ with

$$\phi_{Sz_n}(N) \geqslant n. \tag{2}$$

By [9, Prop. 6.5.2] there exists a bounded set $B \subset Z$ such that $(z_n)_n \subset \operatorname{span} B$. By continuity of S the set $S(B)$ is τ-bounded and by assumption bounded. (PLS)-spaces are complete ([18, Prop. 8.5.26(iii)] and [15, Prop. 24.4]) and the set $\overline{S(B)}$ is bounded ([15, 23.2(b)]) therefore a Banach disc. By 2(ii) we find a function $\phi: \mathbb{N} \to \mathbb{N}$ such that

$$\phi_x \leqslant \phi \quad \text{for all } x \in \operatorname{span} \overline{S(B)}$$

which contradicts the condition (2). This shows that $S(Z) \subset X_\phi$ for some function ϕ. The continuity of S is obtained as follows. The operator S maps Banach discs into Banach discs therefore it is continuous from Z^{ub} into X^{ub}. By assumption Z is a Fréchet space so by [15, 24.15(c)] $Z^{ub} = Z$ and the operator $S: Z \to X^{ub}$ is continuous. Moreover the inclusion $X_\phi \hookrightarrow X^{ub}$ is also continuous thus the graph of the operator

$$S: Z \to X_\phi$$

is closed in the product topology $Z \times X^{ub}$ coarser then the product topology $Z \times X_\phi$. Therefore it is closed in the latter topology $Z \times X_\phi$ and by the Closed Graph Theorem [15, Th. 24.31] and Lemma 3 the operator $S: Z \to X_\phi$ is continuous. ■

Lemma 7. *Let $(X_{M,m})_{M,m\in\mathbb{N}}$ be Banach spaces and let $X = \operatorname{ind}_{M\in\mathbb{N}} \operatorname{proj}_{m\in\mathbb{N}} X_{M,m}$ be an (LF)-space. The following conditions are equivalent:*

(i) *X is an (LFS)-space,*

(ii) *$\forall\, M\ \exists\, L = L_M\ \forall\, l\ \exists\, m = m_l$: $X_{M,m} \to X_{L,l}$ is a compact operator.*

Proof. Only the implication $(ii) \Rightarrow (i)$ needs to be proved. Notice that we may assume $L_M := M+1$ and $m_l := l$. Consider the commutative diagram

$$\begin{array}{ccc} X_{1,1} & \xrightarrow{i_1} & X_{2,1} \\ {\scriptstyle \iota_1^2}\uparrow & & \uparrow{\scriptstyle \kappa_1^2} \\ X_{1,2} & \xrightarrow{i_2} & X_{2,2}, \end{array} \tag{3}$$

where ι_1^2, κ_1^2 are the respective linking maps and the operators i_1, i_2 are compact. Let $q_1: X_{1,1} \to X_{1,1}/\ker i_1$ be the quotient map and $j_1: X_{1,1}/\ker i_1 \to X_{2,1}$ the induced injective continuous operator. Let also q_2, j_2 be defined analogously. If $Z_k := X_{1,k}/\ker i_k$ $(k = 1, 2)$ and

$$\phi\colon Z_2 \to Z_1, \qquad \phi(q_2 x) := q_1(\iota_1^2 x)$$

then by (3) we obtain the commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{j_1} & X_{2,1} \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\kappa_1^2} \\ Z_2 & \xrightarrow{j_2} & X_{2,2}. \end{array}$$

Applying the real interpolation method with parameters $\theta_1, 1\,(0 < \theta_1 < 1)$ to the Banach couples $\overline{Y_1} := (Z_1, X_{2,1}), \overline{Y_2} := (Z_2, X_{2,2})$ we obtain by [1, Th. 3.11.8] a continuous map

$$\mathfrak{J}_{\theta_1,1}(\overline{Y_2}) \to \mathfrak{J}_{\theta_1,1}(\overline{Y_1}).$$

between the interpolation spaces $\mathfrak{J}_{\theta_1,1}(\overline{Y_2})$ and $\mathfrak{J}_{\theta_1,1}(\overline{Y_1})$. The operator j_2 is compact and injective so by the proof of [8, Th. 1.3] we can see that for $0 < \theta_1 < \theta_2 < 1$ one can find compact injective operators

$$u_2\colon Z_2 \to \mathfrak{J}_{\theta_2,1}(\overline{Y_2}), \quad v_2\colon \mathfrak{J}_{\theta_2,1}(\overline{Y_2}) \to X_{2,2}$$

with $j_2 = v_2 \circ u_2$. By [1, Cor. 3.8.2] the inclusion

$$\mathfrak{J}_{\theta_2,1}(\overline{Y_2}) \hookrightarrow \mathfrak{J}_{\theta_1,1}(\overline{Y_2})$$

is compact therefore the map

$$i_{1,2}\colon \mathfrak{J}_{\theta_2,1}(\overline{Y_2}) \to \mathfrak{J}_{\theta_1,1}(\overline{Y_1})$$

is also compact. We apply the same procedure to the commutative diagram

$$\begin{array}{ccc} X_{1,2} & \xrightarrow{i_2} & X_{2,2} \\ {\scriptstyle\iota_2^3}\uparrow & & \uparrow{\scriptstyle\kappa_2^3} \\ X_{1,3} & \xrightarrow{i_3} & X_{2,3}. \end{array}$$

For $\overline{Y_3} := (Z_3, X_{2,3})$ and $0 < \theta_2 < \theta_3 < 1$ we obtain the compact operator

$$i_{2,3}\colon \mathfrak{J}_{\theta_3,1}(\overline{Y_3}) \to \mathfrak{J}_{\theta_2,1}(\overline{Y_2}).$$

Proceeding this way we obtain a Fréchet-Schwartz space

$$Y_1 := \operatorname*{proj}_{n\in\mathbb{N}} \mathfrak{J}_{\theta_n,1}(\overline{Y_n})$$

with the linking maps $\rho_m^n = i_{m,n}$ together with the operators $(u_n)_n, (v_n)_n$. The proof will be complete if we show the continuity and injectivity of the maps

$$X_1 \xrightarrow{j} Y_1 \xrightarrow{v} X_2$$

where $X_1 = \operatorname{proj}_n X_{1,n}$, $X_2 = \operatorname{proj}_n X_{2,n}$. Recall that these maps are defined by

$$j\colon (x_n)_n \mapsto (u_n q_n x_n)_n \quad \text{and} \quad v\colon (y_n)_n \mapsto (v_n y_n)_n.$$

Since $i = v \circ j$ is injective j must be injective too. Moreover all v_n's are injective therefore also v is injective. The above arguments work for all Fréchet spaces X_M and X_{M+1} therefore

$$X = \operatorname*{ind}_{M\in\mathbb{N}} Y_M$$

where all Y_M are (FS)-spaces. ∎

3. Main results

We are now going to give the main theorem of the paper.

Theorem 8. *Let X be a (PLS)-space. The following conditions are equivalent:*

(i) *X is tame,*

(ii) *there exists a function $\psi\colon\mathbb{N}\nearrow\mathbb{N}$ such that for any $\phi\colon\mathbb{N}\to\mathbb{N}$ the condition below holds:*

$$\exists\, k\ \forall\, m\geqslant k\ \exists\, n, C_m\ \forall\, x\in X_\phi :\ \max_{k\leqslant l\leqslant m}\|x\|_{l,\psi(l)}\leqslant C_m \max_{1\leqslant p\leqslant n}\|x\|_{p,\phi(p)}. \quad (4)$$

(iii) *the ultrabornological space X^{ub} associated to X is an (LFS)-space,*

(iv) *there exists a locally convex topology τ on X satysfying two conditions:*
  (a) *τ-bounded sets are bounded in the oryginal topology of X,*
  (b) *τ is an (LF)-topology.*

Proof. (i) $\Rightarrow$ (ii): Let ψ be a function making X tame (see Def. (2)). We use Lemma 5 to the functions $(\psi_k)_k$ defined by

$$\psi_k(m)=\begin{cases}\psi(k), & \text{for}\quad m<k\\ \psi(m), & \text{for}\quad m\geqslant k.\end{cases}$$

(ii) $\Rightarrow$ (iii): Let us define

$$\psi_k(N)=\begin{cases}k\psi(k), & \text{dla}\quad N<k\\ k\psi(N), & \text{dla}\quad N\geqslant k\end{cases}.$$

By (4), we can see that for every function $\phi\colon\mathbb{N}\to\mathbb{N}$ there exists $k\in\mathbb{N}$ such that the inclusion $X_\phi\hookrightarrow X_{\psi_k}$ is continuous. By Prop. 4(2), we obtain $X^{ub}=\operatorname{ind}_k X_{\psi_k}$. Let $Y_{\psi_k,m}$ denote the completion of the quotient space $X/_{\ker\|\cdot\|_m^{\psi_k}}$ in the quotient norm of $\|\cdot\|_m^{\psi_k}:=\max_{1\leqslant N\leqslant m}\|\cdot\|_{N,\psi_k(N)}$. Under this notation we obviously have $X_{\psi_k}=\operatorname{proj}_m Y_{\psi_k,m}$. Fix for a moment $k=1, m=2$. By the assumption and the definition of the functions $(\psi_k)_k$ the maps

$$j_l\colon X_{l,\psi_1(l)}\to X_{l,\psi_2(l)}\quad (l=1,2)$$

are compact and we need to prove the compactness of

$$j\colon Y_{\psi_1,2}\to Y_{\psi_2,2}.$$

Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence in $Y_{\psi_1,2}$. Without loss of generality we may assume that $y_n=x_n+\ker\|\cdot\|_2^{\psi_1}, x_n\in X$ and $\|x_n\|_2^{\psi_1}\leqslant 1$ for all $n\in\mathbb{N}$. Applying compactness of j_l $(l=1,2)$ to the sequence $(x_n)_n$ we obtain a subsequence (denoted again by $(x_n)_n$) whose image under j_l is Cauchy in $X_{l,\psi_2(l)}$. Therefore

$$(jy_n)_n=(x_n+\ker\|\cdot\|_2^{\psi_2})_n\subset X/_{\ker\|\cdot\|_2^{\psi_2}}$$

is a Cauchy sequence, which is convergent in $Y_{\psi_2,2}$. The above argument is true for all $k, m \in \mathbb{N}$ so by Theorem 7(ii) the conclusion follows.

(iii) $\Rightarrow$ (iv): Take the topology of X^{ub}.

(iv) $\Rightarrow$ (i): By (b) we have

$$(X, \tau) = \bigcup_{m \in \mathbb{N}} Z_m$$

where all Z_m are Fréchet spaces. By (a) and Lemma 6 we get functions $(\phi_m)_m : \mathbb{N} \to \mathbb{N}$ with the property that the inclusion $Z_m \hookrightarrow X_{\phi_m}$ is continuous for every $m \in \mathbb{N}$. This gives

$$X = \bigcup_{m \in \mathbb{N}} X_{\phi_m}$$

together with the tameness if we define

$$\psi(n) := \max_{1 \leqslant m \leqslant n} \phi_m(n).$$

■

Corollary 9. *Let $1 \leqslant q \leqslant +\infty$. A Köthe (PLS)-space $\Lambda^q(A)$ is tame if and only if there exists an increasing function $\psi : \mathbb{N} \nearrow \mathbb{N}$ such that for any $\phi : \mathbb{N} \to \mathbb{N}$ we have:*

$$\exists\, k\ \forall\, m \geqslant k\ \exists\, n, D_m\ \forall\, j \in \mathbb{N} : a_{j,m,\psi(m)} \leqslant D_m \max_{1 \leqslant p \leqslant n} a_{j,p,\phi(p)}. \tag{5}$$

4. Power series (PLS)-type spaces

In this section we evaluate our characterization for especially useful class of (PLS)-spaces: power series (PLS)-type spaces. This will allow us to describe power series (PLS)-spaces which are (LFS)-spaces.

Definition 10 [[27]]. Let $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ and $\beta = (\beta_j)_{j \in \mathbb{N}}$ be nonnegative sequences of scalars with

$$\lim_{j \to +\infty} \alpha_j = \lim_{j \to +\infty} \beta_j = +\infty.$$

For $r, s \in \mathbb{R} \cup \{+\infty\}$ we choose sequences $r_n \nearrow r, s_n \nearrow s$. For a matrix $A = (a_{j,N,n})_{j,N,n \in \mathbb{N}}$ defined by

$$a_{j,N,n} := e^{r_N \alpha_j - s_n \beta_j}$$

the Köthe type (PLS)-space $\Lambda^p(A)$ is called a power series (PLS)-space and we denote it by $\Lambda^p_{r,s}(\alpha, \beta)$.

Remark. One can easily show that it is enough to work with $r, s \in \{0, +\infty\}$ and sequences $\left(-\frac{1}{n}\right)_n$, $(n)_n$. In other words there are four non-isomorphic cases. The most important examples are kernels of convolution operators. The Köthe type

(PLS)-space described in the Example 5) in Section 2 is in fact a power series (PLS)-space $\Lambda^1_{\infty,0}(\alpha,\beta)$ with explicitly calculable sequences α and β.

Theorem 11. *Let* $r \in \{0,+\infty\}$. *Every (PLS)-space* $\Lambda^p_{r,0}(\alpha,\beta)$ *is tame.*

Proof. As easily seen

$$\Lambda^p_{r,s}(\alpha,\beta) = \operatorname*{proj}_N \operatorname*{ind}_n l^p(e^{r_N\alpha_j - s_n\beta_j}) = \operatorname*{proj}_{t<r} \operatorname*{ind}_{u<s} l^p(e^{t\alpha_j - u\beta_j}).$$

For every $x \in \Lambda^p_{r,s}(\alpha,\beta)$ we consider the function $s_x\colon (-\infty,r) \to (-\infty,s)$ given by

$$s_x(t) := \inf\{u < s\colon \|x\|_{t,u} < +\infty\}$$

where $\|\cdot\|_{t,u}$ is the norm in the space $l^p(e^{t\alpha_j - u\beta_j})$. For every element x the function s_x is convex (compare [24, Lemma 5.1]). Indeed, let $t = \lambda t_1 + \mu t_2$ be the convex combination and take $s_i > s_x(t_i)$, $i = 1,2$. Using Hölder's inequality we have:

$$\begin{aligned}\|x\|^p_{t,\lambda s_1+\mu s_2} &= \sum_{j=1}^{+\infty} |x_j|^p e^{p(\lambda t_1+\mu t_2)\alpha_j - p(\lambda s_1+\mu s_2)\beta_j} \\ &\leqslant \Big(\sum_{j=1}^{+\infty} |x_j|^p e^{p(t_1\alpha_j - s_1\beta_j)}\Big)^{\lambda} \Big(\sum_{j=1}^{+\infty} |x_j|^p e^{p(t_2\alpha_j - s_2\beta_j)}\Big)^{\mu} < +\infty.\end{aligned}$$

By the choice of numbers s_i we obtain

$$s_x(\lambda t_1 + \mu t_2) \leqslant \lambda s_x(t_1) + \mu s_x(t_3).$$

Obviously $N = \frac{1}{N}\cdot 1 + \frac{N-1}{N}(N+1)$ therefore, if $s = 0, r = +\infty$, we get

$$s_x(N) \leqslant \frac{1}{N} s_x(1) + \frac{N-1}{N} s_x(N+1) < \frac{1}{N} s_x(1).$$

The function $\psi(N) := N^2$ realizes the tameness of $\Lambda^p_{\infty,0}(\alpha,\beta)$. A similar argument holds in the case $s = r = 0$. ∎

Theorem 12. *Let* $r \in \{0,+\infty\}$. *For a (PLS)-space* $\Lambda^p_{r,\infty}(\alpha,\beta)$ *the following conditions are equivalent:*

(1) $\Lambda^p_{r,\infty}(\alpha,\beta)$ *is tame,*
(2) *the set of finite limit points of the sequence* $\big(\frac{\alpha_j}{\beta_j}\big)_{j\in\mathbb{N}}$ *is bounded,*
(3) $\Lambda^p_{r,\infty}(\alpha,\beta)$ *is a product of an (FS)-space and an (LS)-space.*

Proof. We show the case $r = +\infty$. The other one is analogous.

(1) $\Rightarrow$ (2): We follow the idea of [7, Th. 1.3]. Let ψ denote the function giving tameness. We construct a function $f\colon (0,+\infty) \to (0,+\infty)$, continuously differentiable, with the following properties:

(i) f is non-decreasing,
(ii) $f^{'}$ is increasing, $\lim_{t\to+\infty} f^{'}(t) = +\infty$,
(iii) $f(m) \geqslant m\psi(m+1) \ \ \forall\, m \in \mathbb{N}$.

By (iii) we get

$$\lim_{m\to+\infty} \frac{\psi(m+1)}{f(m)} = 0. \tag{6}$$

Define $\phi(m) := [f(m)]+1$ and choose numbers k, m, n, D_m according to (5). Then for every $j \in \mathbb{N}$ we have

$$e^{m\alpha_j - \psi(m)\beta_j} \leqslant D_m \max_{1\leqslant p\leqslant n} \{e^{p\alpha_j - \phi(p)\beta_j}\}.$$

If now $A = \lim_{k_j} \frac{\alpha_{k_j}}{\beta_{k_j}}$ is a finite limit point then – using the fact that $\frac{\ln D_m}{\beta_{k_j}} \to 0$ – we obtain

$$mA - \psi(m) \leqslant \max_{1\leqslant p\leqslant n} \{pA - \phi(p)\}.$$

But $\phi(p) \geqslant f(p)$ so we immediately get

$$mA - \psi(m) \leqslant \max_{1\leqslant p\leqslant n} \{pA - f(p)\} \leqslant \sup_{t>0}\{tA - f(t)\}. \tag{7}$$

Using calculus and denoting $h := (f')^{-1}$ we have

$$\sup_{t>0}(At - f(t)) = Ah(A) - f(h(A)).$$

Together with (7) this gives

$$Am + f(h(A)) \leqslant Ah(A) + \psi(m).$$

Suppose that A is such that $k \leqslant m := [h(A)] + 1$. It is easy to show that

$$f([h(A)]) \leqslant f(h(A)) \leqslant \psi([h(A)] + 1)$$

or

$$1 \leqslant \frac{\psi([h(A)] + 1)}{f([h(A)])}.$$

Suppose that a sequence $(A_n)_{n\in\mathbb{N}}$ of finite limit points tends to infinity. By (6) we obtain

$$1 \leqslant \frac{\psi([h(A_n)] + 1)}{f([h(A_n)])} \to 0;$$

a contradiction.

(2) $\Rightarrow$ (3): By assumption we may choose a partition of $\mathbb{N}$ into two disjoint infinite subsets N_1, N_2 such that

$$c := \sup_{j\in N_1} \frac{\alpha_j}{\beta_j} < +\infty, \qquad \lim_{j\in N_2} \frac{\alpha_j}{\beta_j} = +\infty.$$

Obviously $\Lambda^p_{\infty,\infty}(\alpha,\beta) = \Lambda^p(A|_{N_1}) \operatorname{ind} lus \Lambda^p(A|_{N_2})$.

The space $\Lambda^p(A|_{N_1})$. For arbitrary $N, n \in \mathbb{N}$ we find $l \in \mathbb{N}$ so that

$$e^{-l\beta_j} \leqslant e^{N\alpha_j - l\beta_j} \leqslant e^{-n\beta_j} \quad \forall\, j.$$

This shows that

$$\operatorname*{ind}_n l^p(e^{N\alpha_j - n\beta_j}) = \operatorname*{ind}_n l^p(e^{-n\beta_j}).$$

Consequently

$$\Lambda^p(A|_{N_1}) = \operatorname*{proj}_N \operatorname*{ind}_n l^p(e^{N\alpha_j - n\beta_j}) = \operatorname*{ind}_n l^p(e^{-n\beta_j})$$

and the latter space is an (LS)-space because $e^{-\beta_j} \to 0$.

The space $\Lambda^p(A|_{N_2})$. By the choice of the set N_2 we can find - for every $n \in \mathbb{N}$ - a constant $C_n > 0$ such that

$$e^{-\alpha_j + (n-1)\beta_j} \leqslant C_n.$$

Therefore the embedding

$$l^p(e^{(N+1)\alpha_j - n\beta_j}) \hookrightarrow l^p(e^{N\alpha_j - \beta_j})$$

is continuous for every $N, n \in \mathbb{N}$. By [15, Prop. 24.7] this gives a continuity of

$$X_N := \operatorname*{ind}_{n\in\mathbb{N}} l^p(e^{(N+1)\alpha_j - n\beta_j}) \hookrightarrow l^p(e^{N\alpha_j - \beta_j}).$$

Obviously $l^p(e^{(N+2)\alpha_j - \beta_j})$ embedds continously into X_{N+1} which gives

$$\Lambda^p(A|_{N_2}) = \operatorname*{proj}_N X_N = \operatorname*{proj}_N l^p(e^{N\alpha_j - \beta_j})$$

and the latter space is an (FS)-space because $e^{-\alpha_j} \to 0$.

(3) $\Rightarrow$ (1): Both (LS)-spaces and (FS)-spaces are (LFS)-spaces (see the proof of [8, Th. 1.3]) and so are their products. Using Theorem 8 we get the conclusion. ∎

Let us compare these theorems with the following result.

Theorem 13 ([27, Th. 4.3]). *Let $1 \leqslant p \leqslant +\infty$, $r \in \{0, +\infty\}$. The space $\Lambda^p_{r,\infty}(\alpha,\beta)$ is always ultrabornological. The space $\Lambda^p_{r,0}(\alpha,\beta)$ is ultrabornological if and only if it is a product of an (LS)-space and an (FS)-space.*

As a consequence we obtain the following characterization.

Corollary 14. *Let* $1 \leqslant p \leqslant +\infty$, $r, s \in \{0, +\infty\}$. *The space* $\Lambda^p_{r,s}(\alpha, \beta)$ *is topologically an (LFS)-space if and only if it is a product of an (LS)-space and an (FS)-space.*

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