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SUMS OF ALMOST EQUAL PRIME SQUARES

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Abstract: In this paper, we prove that almost all integers N satisfying $N \equiv 3(\mathrm{mod}24)$ and $5 \nmid N$ or $N \equiv 4(\mathrm{mod}24)$ are the sum of three or four almost equal prime squares, respectively.
Keywords: quadratic equations, exponential sums, circle method.

1. Introduction

Motivated by Lagrange's theorem, it is natural to conjecture that all large integers subject to a natural congruence condition are the sum of four squares of prime numbers. Using the Hardy-Littlewood method, Hua [5] proved that an analogous result holds for sums of five squares of primes. On the other hand, he also proved that almost all integers n with $n \equiv 4\,(\mathrm{mod}\,24)$ are the sum of four squares of prime numbers. Define

$$\mathcal{A}_3 := \{N \in \mathbb{N} : N \equiv 3\,(\mathrm{mod}\,24),\ 5 \nmid N\},$$
$$\mathcal{A}_4 := \{N \in \mathbb{N} : N \equiv 4\,(\mathrm{mod}\,24)\},$$

and denote by $\mathcal{E}_k(z)$ the set of integers $N \in \mathcal{A}_k \cap [z/2, z]$ such that $N \neq p_1^2 + \cdots + p_k^2$. Hua [5] proved that $|\mathcal{E}_3(z)| \ll_A z/(\log z)^A$ for $z \geqslant 2$ and some positive constant A. The study on the size of $\mathcal{E}_k(z)$ has received attention of many authors such as Schwarz [15], Liu & Liu [7], Wooley [18], Liu [6], Liu, Wooley & Yu [9]. The best result is due to Harman & Kumchev [4]: $|\mathcal{E}_3(z)| \ll_\varepsilon z^{6/7+\varepsilon}$ and $|\mathcal{E}_4(z)| \ll_\varepsilon z^{5/14+\varepsilon}$ for any $\varepsilon > 0$.

In this paper, we will investigate this problem with localized summands:

$$\begin{cases} N = p_1^2 + \cdots + p_k^2, \\ \left|p_j - (N/k)^{1/2}\right| \leqslant N^{1/2-\delta} \quad (1 \leqslant j \leqslant k), \end{cases} \tag{1.1}$$

where $\delta > 0$ is a constant, which is hoped to be "large" as soon as possible.

In the case of $k = 3$ or 4, our result is as follows.

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Theorem 1.1. *Let $k = 3$ or 4. For any fixed $\varepsilon > 0$, the equation* (1.1) *with $\delta = \frac{9}{80} - \varepsilon$ is solvable for almost all integers $N \in \mathcal{A}_k$.*

The "almost" means that if we denote by $\mathcal{E}_k^*(z)$ the set of integers $N \in \mathcal{A}_k \cap [z/2, z]$ such that the equation (1.1) with $\delta = \frac{9}{80} - \varepsilon$ is insolvable, then we have $|\mathcal{E}_k^*(z)| \ll_\varepsilon z^{1-(4k-10)\varepsilon}$.

Following Liu & Zhan [11], we shall use the circle method to prove Theorem 1.1. Our exponent 9/80 is determined by an estimate for exponential sums over prime numbers of Liu, Lü & Zhan [8] (see Lemma 4.1 below) and a mean value theorem of Choi & Kumchev [3] (see Lemma 2.1 below). However in order to exploit these we need to introduce some new arguments in Liu & Zhan's method.

The same method allows us to consider the following variant of (1.1).

Theorem 1.2. *For any fixed $\varepsilon > 0$ and $\delta = \frac{9}{80} - \varepsilon$, the equation*

$$\begin{cases} N = p_1 + p_2^2 + p_3^2 \\ |p_1 - N/3| \leqslant N^{1-\delta} \\ \left|p_j - (N/3)^{1/2}\right| \leqslant N^{1/2-\delta} \quad (j = 2, 3) \end{cases} \tag{1.2}$$

is solvable for almost all integers $N \in \mathcal{A}_2$, where

$$\mathcal{A}_2 := \{N \in \mathbb{N} : N \equiv 1 \,(\mathrm{mod}\, 2), \;\; N \not\equiv 2 \,(\mathrm{mod}\, 3)\}.$$

In [5] Hua also proved that almost all integers in $\mathcal{A}_2$ are the sum of one prime and two squares of primes. So Theorem 1.2 can be regarded as a generalization of Hua's result in short intervals. Since the proofs of Theorems 1.1 and 1.2 are very similar, we will only give the proof of Theorem 1.1.

2. Outline and preliminary lemmas

Throughout this paper, the letter p, with or without subscript, denotes a prime number and ε an arbitrarily small positive number. Let $k = 3$ or 4 and $N \in \mathcal{A}_k$ be a sufficiently large integer. Define

$$x = x_k := (N/k)^{1/2}, \qquad y := N^{1/2-9/80+4\varepsilon} \tag{2.1}$$

and

$$P := N^{24\varepsilon}, \qquad Q := N^{-24\varepsilon} y^2. \tag{2.2}$$

Without loss of generality, we can suppose that

$$\|x - y\| \asymp 1, \qquad \|x + y\| \asymp 1,$$

where $\|t\| := \min_{n\in\mathbb{Z}} |t - n|$.

The circle method begins with the observation that

$$\mathcal{R}_k(N) := \sum_{\substack{x-y\leqslant p_1,\dots,p_k\leqslant x+y\\ p_1^2+\cdots+p_k^2=N}} (\log p_1)\cdots(\log p_k) = \int_{1/Q}^{1+1/Q} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha, \tag{2.3}$$

where $e(t) := e^{2\pi it}$ and

$$S(\alpha) = S_k(\alpha) := \sum_{x-y\leqslant p\leqslant x+y} (\log p)\, e(\alpha p^2). \tag{2.4}$$

Clearly in order to prove our Theorem 1.1, it is sufficient to show that $\mathcal{R}_k(N) > 0$ for almost all integers $N \in \mathcal{A}_k$ if $k = 3, 4$.

By Dirichlet's lemma ([17], Lemma 2.1), each $\alpha \in [1/Q, 1+1/Q]$ can be written as

$$\alpha = a/q + \beta \qquad \text{with} \qquad |\beta| \leqslant 1/(qQ) \tag{2.5}$$

for some integers a and q with $1 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$. We denote by $I(a,q)$ the set of α satisfying (2.5), and define the major arcs $\mathfrak{M}$ and the minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} := \bigcup_{1\leqslant q\leqslant P} \bigcup_{\substack{1\leqslant a\leqslant q\\ (a,q)=1}} I(a,q) \qquad \text{and} \qquad \mathfrak{m} := [1/Q, 1+1/Q] \setminus \mathfrak{M}. \tag{2.6}$$

Thus we can write

$$\begin{aligned} \mathcal{R}_k(N) &= \int_{\mathfrak{M}} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha \\ &=: \mathcal{R}_k(N;\mathfrak{M}) + \mathcal{R}_k(N;\mathfrak{m}). \end{aligned} \tag{2.7}$$

We shall establish an asymptotic formula for $\mathcal{R}_k(N;\mathfrak{M})$ in Section 3 and treat $\mathcal{R}_k(N;\mathfrak{m})$ in Section 4. The next mean value theorem, due to Choi & Kumchev [3], will be useful for the major arc estimate.

Lemma 2.1. ([3], Theorem 1.1) *Let $\ell \in \mathbb{N}$, $R \geqslant 1$, $T \geqslant 1$, $X \geqslant 1$ and $\kappa := 1/\log X$. Then there is an absolute positive constant c such that*

$$\sum_{\substack{r\sim R\\ \ell\mid r}} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)} \int_{-T}^{T} \bigg|\sum_{X\leqslant n\leqslant 2X} \frac{\Lambda(n)\chi(n)}{n^{\kappa+i\tau}}\bigg|\,\mathrm{d}\tau \ll \big(\ell^{-1}R^2TX^{11/20} + X\big)(\log RTX)^c,$$

where $\sum^*_{\chi\,(\mathrm{mod}\,r)}$ means that the summation runs over the primitive characters modulo r. The implied constant is absolute.

In Choi & Kumchev's original statement (in a more general form), there is no factor $n^{-\kappa}$. Since $n \mapsto n^{-\kappa}$ is completely multiplicative with respect to n and $n^{-\kappa} \asymp 1$ for $X \leqslant n \leqslant 2X$, their proof covers our case as well with some trivial modification (i.e. replacing $\chi(n)$ by $\chi(n)n^{-\kappa}$ in their proof). On the other hand, it is simple to get this Lemma by partial summation and Choi & Kumchev's original result.

In order to exploit Choi & Kumchev's mean value theorem effectively, we need to prove a preliminary lemma.

Lemma 2.2. *Let χ be a Dirichlet character modulo r. Let $Q \geqslant r$, $2 \leqslant X < Y \leqslant 2X$ such that $\|X\| \asymp \|Y\| \asymp 1$, $T_0 := (\log(Y/X))^{-1}$, $T_1 := (\log(Y/X))^{-2}$, $T_2 := 8\pi X^2/(rQ)$, $T_3 := X^4$ and $\kappa := (\log X)^{-1}$. Define*

$$F(s,\chi) := \sum_{X \leqslant n \leqslant 2X} \Lambda(n)\chi(n)n^{-s}. \tag{2.8}$$

Then we have

$$\begin{aligned}\max_{|\beta|\leqslant 1/(rQ)} \Big| \sum_{X\leqslant n\leqslant Y} \Lambda(n)\chi(n)e(\beta n^2)\Big| \ll \log\left(\frac{Y}{X}\right) &\int_{|\tau|\leqslant T_1} |F(\kappa+i\tau,\chi)|\,\mathrm{d}\tau \\ &+ \int_{T_1<|\tau|\leqslant T_2} \frac{|F(\kappa+i\tau,\chi)|}{|\tau|^{1/2}}\,\mathrm{d}\tau \\ &+ \int_{T_2<|\tau|\leqslant T_3} \frac{|F(\kappa+i\tau,\chi)|}{|\tau|}\,\mathrm{d}\tau + 1\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\sum_{X\leqslant n\leqslant Y} \Lambda(n)\chi(n) \ll \log\left(\frac{Y}{X}\right) &\int_{|\tau|\leqslant T_0} |F(\kappa+i\tau,\chi)|\,\mathrm{d}\tau \\ &+ \int_{T_0<|\tau|\leqslant T_3} \frac{|F(\kappa+i\tau,\chi)|}{|\tau|}\,\mathrm{d}\tau + 1.\end{aligned} \tag{2.10}$$

The implied constants are absolute.

Proof. By Perron's formula ([16], Lemma 3.12), for any $t \in [X, 2X]$ we have

$$\begin{aligned}\sum_{X\leqslant n\leqslant t} \Lambda(n)\chi(n) &= \frac{1}{2\pi i}\int_{\kappa-iT_3}^{\kappa+iT_3} F(s,\chi)\frac{t^s-X^s}{s}\,\mathrm{d}s \\ &\quad + O\bigg(\frac{X}{T_3}(\log X)^2 + (\log X)\min\Big\{1, \frac{X}{T_3\|t\|}\Big\}\bigg).\end{aligned}$$

From this, a simple partial summation gives

$$\begin{aligned}\sum_{X\leqslant n\leqslant Y} \Lambda(n)\chi(n)e(\beta n^2) &= \int_X^Y e(\beta t^2)\,\mathrm{d}\Big(\sum_{X\leqslant n\leqslant t}\Lambda(n)\chi(n)\Big) \\ &= \frac{1}{2\pi i}\int_{\kappa-iT_3}^{\kappa+iT_3} F(s,\chi)V(s,\beta)\,\mathrm{d}s + R,\end{aligned} \tag{2.11}$$

where

$$V(s,\beta) := \int_X^Y t^{s-1}e(\beta t^2)\,\mathrm{d}t$$

and

$$R := \int_X^Y e(\beta t^2)\,\mathrm{d}O\biggl(\frac{X}{T_3}(\log X)^2 + (\log X)\min\biggl\{1, \frac{X}{T_3\|t\|}\biggr\}\biggr).$$

First we estimate R. By an integration by parts, it follows that

$$R \ll \biggl(\frac{X}{T_3} + |\beta|\frac{X^2}{T_3}(Y-X) + \frac{|\beta|X}{\log X}\int_X^Y \min\biggl\{1, \frac{X}{T_3\|t\|}\biggr\}\,\mathrm{d}t\biggr)(\log X)^2.$$

Spliting the last integral, we deduce

$$\begin{aligned} R &\ll \biggl(\frac{X}{T_3} + |\beta|\frac{X^2}{T_3}(Y-X) + \frac{|\beta|X}{\log X}\sum_{X\leqslant n\leqslant Y}\int_{n-1/2}^{n+1/2}\min\biggl\{1, \frac{X}{T_3|t-n|}\biggr\}\,\mathrm{d}t\biggr)(\log X)^2 \\ &\ll \biggl(\frac{X}{T_3} + |\beta|\frac{X^2}{T_3}(Y-X) + \frac{|\beta|X}{\log X}\sum_{X\leqslant n\leqslant Y}\int_0^{1/2}\min\biggl\{1, \frac{X}{T_3 u}\biggr\}\,\mathrm{d}u\biggr)(\log X)^2 \\ &\ll \biggl(\frac{X}{T_3} + |\beta|\frac{X^2}{T_3}(Y-X)\biggr)(\log X)^2 \\ &\ll 1. \end{aligned}$$

In order to treat the first term on the right-hand side of (2.11), we notice, for all $\beta\in\mathbb{R}$,

$$|V(\kappa+i\tau,\beta)| \leqslant \int_X^Y t^{\kappa-1}\,\mathrm{d}t \ll \log(Y/X). \tag{2.12}$$

On the other hand, the change of variables $u = t^2$ and the second mean value formula allow us to write

$$\begin{aligned} V(\kappa+i\tau,\beta) &= \frac{1}{2}\int_{X^2}^{Y^2} u^{\kappa/2-1}e(\beta u + (\tau/4\pi)\log u)\,\mathrm{d}u \\ &= \frac{X^{\kappa-2}}{2}\int_{X^2}^{\xi} e(\beta u + (\tau/4\pi)\log u)\mathrm{d}u + \frac{Y^{\kappa-2}}{2}\int_{\xi}^{Y^2} e(\beta u + (\tau/4\pi)\log u)\,\mathrm{d}u \end{aligned}$$

for some $\xi\in[X^2, Y^2]$. We estimate the last two integrals by using Lemma 4.3 of [16] if $T_2 < |\tau| \leqslant T_3$ and Lemma 4.4 of [16] if $T_1 < |\tau| \leqslant T_2$ and use (2.12) for $|\tau|\leqslant T_1$. We obtain

$$\max_{|\beta|\leqslant 1/(rQ)} |V(\kappa+i\tau,\beta)| \ll \begin{cases} \log(Y/X) & \text{if } |\tau|\leqslant T_1, \\ |\tau|^{-1/2} & \text{if } T_1 < |\tau| \leqslant T_2, \\ |\tau|^{-1} & \text{if } T_2 < |\tau| \leqslant T_3. \end{cases}$$

Now the inequality (2.9) follows from (2.11) by splitting the integral into three parts according to $|\tau|\leqslant T_1$ or $T_1\leqslant|\tau|\leqslant T_2$ or $T_2\leqslant|\tau|\leqslant T_3$ and by using the preceding estimates.

Similarly there is a real number $\xi \in [X, Y]$ such that

$$V(\kappa + i\tau, 0) = X^{\kappa-1} \int_X^\xi t^{i\tau}\,\mathrm{d}t + Y^{\kappa-1} \int_\xi^Y t^{i\tau}\,\mathrm{d}t \ll (|\tau| + 1)^{-1}. \tag{2.13}$$

Now the inequality (2.10) follows from (2.11) with $\beta = 0$ by splitting the integral into two parts according to $|\tau| \leqslant T_0$ or $T_0 \leqslant |\tau| \leqslant T_3$ and by using (2.13) and (2.12) with $\beta = 0$. This completes the proof. ■

Next we shall prove three estimates (see (2.17), (2.18) and (2.19) below), which play an important role in Liu's iterative procedure [6]. Define

$$S^0(\beta) := \sum_{x-y\leqslant n\leqslant x+y} e(\beta n^2), \tag{2.14}$$

$$W_\chi(\beta) := \sum_{x-y\leqslant p\leqslant x+y} (\log p)\chi(p)e(\beta p^2) - \delta_\chi S^0(\beta) \tag{2.15}$$

and $\delta_\chi = 1$ or 0 according as χ is principal or not. We also set

$$W_\chi^\sharp := \max_{|\beta|\leqslant 1/(rQ)} |W_\chi(\beta)| \quad \text{and} \quad \|W_\chi\|_2 := \left(\int_{-1/(rQ)}^{1/(rQ)} |W_\chi(\beta)|^2\,\mathrm{d}\beta\right)^{1/2}. \tag{2.16}$$

Proposition 2.1. *Let $d \geqslant 1$ and $k = 3, 4$. Let x, y and P, Q be defined as in (2.1) and (2.2), respectively. Then there is an absolute positive constant c such that for any $\varepsilon > 0$ we have*

$$\sum_{r\leqslant P} [d, r]^{-(k-2)/2+\varepsilon} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)} W_\chi^\sharp \ll_\varepsilon d^{-(k-2)/2+\varepsilon} y\mathcal{L}^c, \tag{2.17}$$

$$\sum_{r\leqslant P} [d, r]^{-(k-2)/2+\varepsilon} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)} \|W_\chi\|_2 \ll_\varepsilon d^{-(k-2)/2+\varepsilon} N^{-1/4} y^{1/2}\mathcal{L}^c, \tag{2.18}$$

where $\mathcal{L} := \log N$ and $\sum^*$ means that the summation runs over primitive character. Further if $d = 1$, the first estimate can be improved to

$$\sum_{r\leqslant P} r^{-(k-2)/2+\varepsilon} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)} W_\chi^\sharp \ll_A y\mathcal{L}^{-A} \tag{2.19}$$

for any fixed $A > 0$.

Proof. Introducing

$$\widetilde{W}_\chi(\beta) := \sum_{x-y\leqslant n\leqslant x+y} \Lambda(n)\chi(n)e(\beta n^2) - \delta_\chi S^0(\beta), \tag{2.20}$$

we have, for all $\beta \in \mathbb{R}$,

$$\big|\widetilde{W}_\chi(\beta) - W_\chi(\beta)\big| \leqslant 2 \sum_{\substack{x-y\leqslant p^\nu\leqslant x+y\\ \nu\geqslant 2}} \log p \ll x^{-1/2}y \ll N^{-1/4}y. \tag{2.21}$$

Thus

$$W_\chi^\sharp \leqslant \widetilde{W}_\chi^\sharp + O\big(N^{-1/4}y\big),$$

where

$$\widetilde{W}_\chi^\sharp := \max_{|\beta|\leqslant 1/(rQ)} \big|\widetilde{W}_\chi(\beta)\big|.$$

The contribution of $O\big(N^{-1/4}y\big)$ to (2.17) is , writing $[d,r]=dr/\ell$ and $\ell=(d,r)$,

$$\begin{aligned}&\ll N^{-1/4}y \sum_{\ell|d,\,\ell\leqslant P}\ \sum_{r\leqslant P,\,\ell|r} (dr/\ell)^{-(k-2)/2+\varepsilon} r\\ &\ll d^{-(k-2)/2+\varepsilon} y N^{-1/4} P^{(9-k)/4+\varepsilon}\\ &\ll d^{-(k-2)/2+\varepsilon} y,\end{aligned}$$

since $P^{9-k+4\varepsilon} \ll_\varepsilon N$ in view of our choice of P (see (2.2)).

Therefore in order to prove (2.17), it is enough to show

$$\sum_{r\sim R} [d,r]^{-(k-2)/2+\varepsilon} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\,r)} \widetilde{W}_\chi^\sharp \ll d^{-(k-2)/2+\varepsilon} y\mathcal{L}^c \tag{2.22}$$

for any $R\leqslant P$, where $r\sim R$ means that $R\leqslant r<2R$.

If $R=1$ and $r\sim R$, we have $\chi=\chi_0^*\,(\mathrm{mod}\,1)$ (the primitive character modulo 1). Thus

$$\widetilde{W}_\chi^\sharp \leqslant \sum_{x-y\leqslant n\leqslant x+y} 2\mathcal{L} \ll y\mathcal{L}.$$

This will contribute $O(d^{-(k-2)/2+\varepsilon}y\mathcal{L})$, which is acceptable.

For $2\leqslant R\leqslant P$ and $r\sim R$, we have $\delta_\chi=0$. Since $\|x-y\|\asymp 1$ and $\|x+y\|\asymp 1$, we can apply (2.9) to write

$$\begin{aligned}\widetilde{W}_\chi^\sharp \ll \frac{y}{x}\int_{|\tau|\leqslant T_1} |F(\kappa+i\tau,\chi)|\,\mathrm{d}\tau + \int_{T_1<|\tau|\leqslant T_2} \frac{|F(\kappa+i\tau,\chi)|}{|\tau|^{1/2}}\,\mathrm{d}\tau\\ + \int_{T_2<|\tau|\leqslant T} \frac{|F(\kappa+i\tau,\chi)|}{|\tau|}\,\mathrm{d}\tau + 1,\end{aligned} \tag{2.23}$$

where $T_1\asymp (x/y)^2$, $T_2\asymp x^2/(RQ)$ and $T\asymp x^4$.

By Lemma 2.1, the contribution of the first term on the right-hand side of (2.23) to (2.22) is

$$\begin{aligned}&\ll d^{-(k-2)/2+\varepsilon} x^{-1}y \sum_{\ell|d,\,\ell\leqslant 2R} (R/\ell)^{-(k-2)/2+\varepsilon}\big(\ell^{-1}R^2T_1x^{11/20}+x\big)\\ &\ll d^{-(k-2)/2+\varepsilon} y\big(P^{(9-k)/4+\varepsilon}N^{31/40}y^{-2}+1\big)\mathcal{L}^c\\ &\ll d^{-(k-2)/2+\varepsilon} y\mathcal{L}^c\end{aligned} \tag{2.24}$$

in view of our choice of (P,y) (see (2.1) and (2.2)).

Introducing

$$M(\ell, R, T', x) := \sum_{r\sim R,\, \ell|r} \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, r)} \int_{T'}^{2T'} |F(\kappa + i\tau, \chi)|\, \mathrm{d}\tau,$$

the contribution of the second term on the right-hand side of (2.23) to (2.22) is

$$\begin{aligned}
&\ll d^{-(k-2)/2+\varepsilon}\mathcal{L}^c \sum_{\ell|d,\, \ell\leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \max_{T_1\leqslant T'\leqslant T_2} \big(T'^{-1/2} M(\ell, R, T', x)\big)\\
&\ll d^{-(k-2)/2+\varepsilon}\mathcal{L}^c \sum_{\ell|d,\, \ell\leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \big(\ell^{-1}R^2 T_2^{1/2} x^{11/20} + T_1^{-1/2} x\big)\mathcal{L}^c\\
&\ll d^{-(k-2)/2+\varepsilon} y\big(P^{(7-k)/4+\varepsilon} Q^{-1/2} N^{31/40} y^{-1} + 1\big)\mathcal{L}^c\\
&\ll d^{-(k-2)/2+\varepsilon} y\mathcal{L}^c,
\end{aligned} \tag{2.25}$$

in view of our choice of (P, Q, y) (see (2.1) and (2.2)).

Similarly the contribution of the third term on the right-hand side of (2.23) to (2.22) is

$$\begin{aligned}
&\ll d^{-(k-2)/2+\varepsilon}\mathcal{L}^c \sum_{\ell|d,\, \ell\leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \max_{T_2\leqslant T'\leqslant T} \big(T'^{-1} M(\ell, R, T', x)\big)\\
&\ll d^{-(k-2)/2+\varepsilon}\mathcal{L}^c \sum_{\ell|d,\, \ell\leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \big(\ell^{-1}R^2 x^{11/20} + T_2^{-1} x\big)\mathcal{L}^c\\
&\ll d^{-(k-2)/2+\varepsilon} y\big(P^{(9-k)/4+\varepsilon} N^{11/40} y^{-1} + PQ(xy)^{-1}\big)\mathcal{L}^c\\
&\ll d^{-(k-2)/2+\varepsilon} y\mathcal{L}^c,
\end{aligned} \tag{2.26}$$

in view of our choice of (P, Q, y) (see (2.1) and (2.2)).

Finally the contribution of the last term on the right-hand side of (2.23) to (2.22) is

$$\ll d^{-(k-2)/2+\varepsilon} \sum_{\ell|d,\, \ell\leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \ll d^{-(k-2)/2+\varepsilon} \ll d^{-(k-2)/2+\varepsilon} y. \tag{2.27}$$

Now the inequality (2.22) follows from (2.24), (2.25), (2.26) and (2.27). This proves (2.17).

The proof of (2.18) is rather similar. Therefore we shall only point out the differences. First the inequality (2.21) implies

$$\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, r)} \|W_\chi\|_2 \ll \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}\, r)} \|\widetilde{W}_\chi\|_2 + N^{-1/4} y (r/Q)^{1/2}.$$

The contribution of $O(N^{-1/4}y(r/Q)^{1/2})$ to (2.18) is

$$\begin{aligned}&\ll N^{-1/4}yQ^{-1/2}\sum_{\ell|d,\,\ell\leqslant P}\sum_{r\leqslant P,\,\ell|r}(dr/\ell)^{-(k-2)/2+\varepsilon}r^{1/2}\\&\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}yP^{1/2+\varepsilon}Q^{-1/2}\\&\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2},\end{aligned}$$

since $P^{1+2\varepsilon}y \ll_\varepsilon Q$ in view of our choice of (P, Q, y) (see (2.1) and (2.2)). Thus in order to prove (2.18), it suffices to show that

$$\sum_{r\sim R}[d,r]^{-(k-2)/2+\varepsilon}\sum_{\chi\,(\mathrm{mod}\,r)}{}^{*}\,\|\widetilde{W}_\chi\|_2 \ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\mathcal{L}^c \tag{2.28}$$

for any $R \leqslant P$. For this, by Lemma 1.9 of [14] we write, for $r \sim R$,

$$\begin{aligned}\|\widetilde{W}_\chi\|_2 &\ll \frac{1}{RQ}\bigg(\int_{-\infty}^{\infty}\Big|\sum_{\substack{v-RQ/3<n^2\leqslant v+RQ/3\\ x-y\leqslant n\leqslant x+y}}(\Lambda(n)\chi(n)-\delta_\chi)\Big|^2\,\mathrm{d}v\bigg)^{1/2}\\&\ll \frac{1}{RQ}\bigg(\int_{(x-y)^2-RQ/3}^{(x+y)^2+RQ/3}\Big|\sum_{X\leqslant n\leqslant Y}(\Lambda(n)\chi(n)-\delta_\chi)\Big|^2\,\mathrm{d}v\bigg)^{1/2},\end{aligned}$$

where $X := U - \frac{1}{4}$ or $[U] + \frac{1}{4}$ according to $U = \max\{(v - RQ/3)^{1/2},\, x - y\}$ is an integer or not, and $Y := [\min\{(v + RQ/3)^{1/2},\, x + y\}] + \frac{1}{4}$.

If $R = 1$, we have

$$\begin{aligned}\Big|\sum_{X\leqslant n\leqslant Y}(\Lambda(n)\chi(n)-\delta_\chi)\Big| = \Big|\sum_{X<n\leqslant Y}(\Lambda(n)-1)\Big| &\leqslant 2(Y-X)\mathcal{L}\\&\ll \big\{(v+Q/3)^{1/2}-(v-Q/3)^{1/2}\big\}\mathcal{L}\\&\ll Qv^{-1/2}\mathcal{L} \ll N^{-1/2}Q\mathcal{L},\end{aligned}$$

which implies, in view of $Q < xy$,

$$\begin{aligned}d^{-(k-2)/2+\varepsilon}\|\widetilde{W}_{\chi_0^*}\|_2 &\ll d^{-(k-2)/2+\varepsilon}Q^{-1}\big((N^{-1/2}Q\mathcal{L})^2(xy+Q)\big)^{1/2}\\&\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\mathcal{L}.\end{aligned} \tag{2.29}$$

For $R \geqslant 2$ and $r \sim R$, we have $\delta_\chi = 0$. Thus we can apply (2.10) of Lemma 2.2 to write

$$\begin{aligned}\|\widetilde{W}_\chi\|_2 &\ll \Big(\frac{y}{x^3}\Big)^{1/2}\int_{|\tau|\leqslant T_0}|F(\kappa+i\tau,\chi)|\,\mathrm{d}\tau\\&\quad+\frac{(xy)^{1/2}}{RQ}\int_{T_0<|\tau|\leqslant T}\frac{|F(\kappa+i\tau,\chi)|}{|\tau|}\,\mathrm{d}\tau+\frac{(xy)^{1/2}}{RQ},\end{aligned} \tag{2.30}$$

since $T_0^{-1} = \log(Y/X) \asymp RQv^{-1} \asymp RQx^{-2}$ and $(x+y)^2 + RQ/3 - (x-y)^2 + RQ/3 \asymp xy$.

As before, the contribution of the first term on the right-hand side of (2.30) to (2.28) is

$$\begin{aligned} &\ll d^{-(k-2)/2+\varepsilon}(x^{-3}y)^{1/2} \sum_{\ell | d,\, \ell \leqslant 2R} (R/\ell)^{-(k-2)/2+\varepsilon}\big(\ell^{-1}R^2T_0x^{11/20} + x\big) \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\big(P^{(5-k)/4+\varepsilon}Q^{-1}N^{31/40} + 1\big)\mathcal{L}^c \qquad (2.31) \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\mathcal{L}^c \end{aligned}$$

in view of our choice of (P,Q); the contribution of the second term on the right-hand side of (2.30) to (2.28) is

$$\begin{aligned} &\ll d^{-(k-2)/2+\varepsilon}(xy)^{1/2}(RQ)^{-1}\mathcal{L}^c \sum_{\ell | d, \ell \leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon} \max_{T_0 \leqslant T' \leqslant T} T'^{-1}M(\ell, R, T', x) \\ &\ll d^{-(k-2)/2+\varepsilon}(xy)^{1/2}(RQ)^{-1}\mathcal{L}^c \sum_{\ell | d, \ell \leqslant R} (R/\ell)^{-(k-2)/2+\varepsilon}\big(\ell^{-1}R^2x^{11/20} + T_0^{-1}x\big)\mathcal{L}^c \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\big(P^{(5-k)/4+\varepsilon}Q^{-1}N^{31/40} + 1\big)\mathcal{L}^c \qquad (2.32) \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\mathcal{L}^c; \end{aligned}$$

the contribution of the last term on the right-hand side of (2.30) to (2.28) is

$$\begin{aligned} &\ll d^{-(k-2)/2+\varepsilon}Q^{-1}(xy)^{1/2} \sum_{\ell | d,\, \ell \leqslant 2R} \sum_{r \sim R,\, \ell | r} (r/\ell)^{-(k-2)/2+\varepsilon} \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}R^{(5-k)/4+\varepsilon}Q^{-1}x \qquad (2.33) \\ &\ll d^{-(k-2)/2+\varepsilon}N^{-1/4}y^{1/2}\mathcal{L}^c, \end{aligned}$$

since $R^{(5-k)/4+\varepsilon}x \leqslant P^{(5-k)/4+\varepsilon}N^{1/2} \leqslant Q$.

Now the estimate (2.28) follows from (2.29), (2.31), (2.32) and (2.33). This proves (2.18).

The estimate (2.19) can be proved in the same way as Lemma 2.3 of [13] and we omit the details. This completes the proof of Proposition 2.1. ■

3. Asymptotic formula for $\mathcal{R}_k(N;\mathfrak{M})$

The aim of this section is to treat the integral $\mathcal{R}_k(N;\mathfrak{M})$.

Proposition 3.1. *Let $k=3,4$. Then for sufficiently large $N\in\mathcal{A}_k$ we have*

$$\mathcal{R}_k(N;\mathfrak{M})=\int_{\mathfrak{M}} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha \sim C_k\mathfrak{S}_k(N)N^{-1/2}y^{k-1}, \tag{3.1}$$

where C_k are some positive constants, $\phi(q)$ is the Euler function and

$$\mathfrak{S}_k(N):=\sum_{q=1}^{\infty}\frac{1}{\phi(q)^k}\sum_{\substack{a=1\\(a,q)=1}}^{q}\Bigg(\sum_{\substack{h=1\\(h,q)=1}}^{q}e^{2\pi iah^2/q}\Bigg)^k e^{-2\pi iaN/q}.$$

Proof. Since $q\leqslant P<x-y$, we have $(p,q)=1$ for all $p\in(x-y,x+y]$. By using the orthogonality relation, we can write

$$\begin{aligned}S(a/q+\beta)&=\sum_{1\leqslant h\leqslant q}e^{2\pi iah^2/q}\sum_{\substack{x-y\leqslant p\leqslant x+y\\ p\equiv h(\bmod q),\,(p,q)=1}}(\log p)e\big(\beta p^2\big)\\ &=\frac{1}{\phi(q)}\sum_{\chi(\bmod q)}\sum_{1\leqslant h\leqslant q}\overline{\chi}(h)e^{2\pi iah^2/q}\sum_{x-y\leqslant p\leqslant x+y}\chi(p)(\log p)e\big(\beta p^2\big).\end{aligned}$$

Introducing notation

$$C(\chi,a):=\sum_{1\leqslant h\leqslant q}\overline{\chi}(h)e^{2\pi iah^2/q}\qquad\text{and}\qquad C(q,a):=C(\chi_0,a), \tag{3.2}$$

where χ_0 is the principal character modulo q, the preceding relation can be written as

$$S(a/q+\beta)=\frac{C(q,a)}{\phi(q)}S^0(\beta)+\frac{1}{\phi(q)}\sum_{\chi(\bmod q)}C(\chi,a)W_\chi(\beta), \tag{3.3}$$

where $S^0(\beta)$ and $W_\chi(\beta)$ are defined as in (2.14) and (2.15), respectively. In view of our choice of P and Q, we have $2P<Q$. Thus the intervals $I(a,q)$ are mutually disjoint and we can write, by using (3.3),

$$\begin{aligned}\int_{\mathfrak{M}} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha&=\sum_{1\leqslant q\leqslant P}\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}e^{-2\pi iaN/q}\int_{-1/(qQ)}^{1/(qQ)}S(a/q+\beta)^k e(-\beta N)\,\mathrm{d}\beta\\ &=\sum_{0\leqslant j\leqslant k}\frac{k!}{(k-j)!j!}I_j,\end{aligned} \tag{3.4}$$

where

$$I_j := \sum_{1\leqslant q\leqslant P} \frac{1}{\phi(q)^k} \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} C(q,a)^{k-j} e^{-2\pi iaN/q} \times$$
$$\times \int_{-1/(qQ)}^{1/(qQ)} S^0(\beta)^{k-j} \bigg(\sum_{\chi\,(\mathrm{mod}\, q)} C(\chi,a) W_\chi(\beta)\bigg)^j e(-\beta N)\,\mathrm{d}\beta.$$

We shall see that I_0 contributes the main term and the others I_j are as error terms.

By the standard major arcs techniques, we have

$$I_0 = C_k \mathfrak{S}_k(N) y^{k-1} N^{-1/2}\{1+o(1)\}. \tag{3.5}$$

It remains to estimate I_j $(1\leqslant j\leqslant k)$. We shall only treat I_k. The others can be treated similarly (even more easily). We can write

$$I_k = \sum_{1\leqslant q\leqslant P} \sum_{\chi_1\,(\mathrm{mod}\, q)} \cdots \sum_{\chi_k\,(\mathrm{mod}\, q)} B_k(N,q;\chi_1,\ldots,\chi_k) J_k(N,q;\chi_1,\ldots,\chi_k),$$

where

$$B_k(N,q;\chi_1,\ldots,\chi_k) := \frac{1}{\phi(q)^k} \sum_{\substack{a=1\\(a,q)=1}}^{q} C(\chi_1,a)\cdots C(\chi_k,a) e^{-2\pi iaN/q},$$
$$J_k(N,q;\chi_1,\ldots,\chi_k) := \int_{-1/(qQ)}^{1/(qQ)} W_{\chi_1}(\beta)\cdots W_{\chi_k}(\beta) e(-\beta N)\,\mathrm{d}\beta.$$

Suppose that $\chi_k^*\,(\mathrm{mod}\, r_k)$ with $r_k \mid q$ is the primitive character inducing χ_k. Then we can write $\chi_k = \chi_0\chi_k^*$. It is easy to see that $W_{\chi_k}(\beta) = W_{\chi_k^*}(\beta)$. By Cauchy's inequality, it follows that

$$|J_k(N,q;\chi_1,\ldots,\chi_k)| \leqslant W_{\chi_1^*}^\sharp \cdots W_{\chi_{k-2}^*}^\sharp \|W_{\chi_{k-1}^*}\|_2 \|W_{\chi_k^*}\|_2, \tag{3.6}$$

where $W_\chi^\sharp$ and $\|W_\chi\|_2$ are defined as in (2.16) with $r := [r_1,\ldots,r_k]$. From (3.6) and the inequality

$$\sum_{q\leqslant z,\, r|q} |B_k(N,q;\chi_1^*\chi_0,\ldots,\chi_k^*\chi_0)| \ll_\varepsilon r^{-(k-2)/2+\varepsilon} (\log z)^c$$

(see [12] for $k=3$ and [1] for $k=5$. The general case can be treated in the same way.), we deduce

$$I_k \ll \mathcal{L}^c \sum_{r_1\leqslant P} \sideset{}{^*}\sum_{\chi_1\,(\mathrm{mod}\, r_1)} W_{\chi_1}^\sharp \cdots \sum_{r_{k-2}\leqslant P} \sideset{}{^*}\sum_{\chi_{k-2}\,(\mathrm{mod}\, r_{k-2})} W_{\chi_{k-2}}^\sharp \times$$
$$\times \sum_{r_{k-1}\leqslant P} \sideset{}{^*}\sum_{\chi_{k-1}\,(\mathrm{mod}\, r_{k-1})} \|W_{\chi_{k-1}}\|_2 \sum_{r_k\leqslant P} [r_1,\ldots,r_k]^{-(k-2)/2+\varepsilon} \sideset{}{^*}\sum_{\chi_k\,(\mathrm{mod}\, r_k)} \|W_{\chi_k}\|_2.$$

By noticing that $[r_1,\dots,r_k]=[[r_1,\dots,r_{k-1}],r_k]$, we use consecutively (2.18) (2 times), (2.17) ($k-3$ times) and (2.19) (1 time) of Proposition 2.1 to write

$$
\begin{aligned}
I_k &\ll N^{-1/4}y^{1/2}\mathcal{L}^c \sum_{r_1\leqslant P}\ \sideset{}{^*}\sum_{\chi_1\,(\mathrm{mod}\,r_1)} W^{\sharp}_{\chi_1}\cdots \sum_{r_{k-2}\leqslant P}\ \sideset{}{^*}\sum_{\chi_{k-2}\,(\mathrm{mod}\,r_{k-2})} W^{\sharp}_{\chi_{k-2}}\times\\
&\quad\times \sum_{r_{k-1}\leqslant P}[r_1,\dots,r_{k-1}]^{-(k-2)/2+\varepsilon}\ \sideset{}{^*}\sum_{\chi_{k-1}\,(\mathrm{mod}\,r_{k-1})}\|W_{\chi_{k-1}}\|_2\\
&\ll N^{-1/2}y\mathcal{L}^c \sum_{r_1\leqslant P}\ \sideset{}{^*}\sum_{\chi_1\,(\mathrm{mod}\,r_1)} W^{\sharp}_{\chi_1}\cdots \sum_{r_{k-2}\leqslant P}[r_1,\dots,r_{k-2}]^{-(k-2)/2+\varepsilon}\times\\
&\quad\times \sideset{}{^*}\sum_{\chi_{k-2}\,(\mathrm{mod}\,r_{k-2})} W^{\sharp}_{\chi_{k-2}}\\
&\ll N^{-1/2}y^{k-2}\mathcal{L}^c \sum_{r_1\leqslant P} r_1^{-(k-2)/2+\varepsilon}\ \sideset{}{^*}\sum_{\chi_1\,(\mathrm{mod}\,r_1)} W^{\sharp}_{\chi_1}\\
&\ll N^{-1/2}y^{k-1}\mathcal{L}^{-A}
\end{aligned}
\tag{3.7}
$$

for any fixed $A>0$.

Now the required asymptotic formula follows from (3.4), (3.5) and (3.7). ■

4. Proof of Theorem 1.1

In order to bound $S(\alpha)$ on the minor arcs $\mathfrak{m}$, we need two estimates for exponential sums over prime numbers, which are due to Liu, Lü & Zhan [8] and Liu & Zhan [10], respectively.

Lemma 4.1. ([8], Theorem 1.1) *Let $j\in\mathbb{N}$, $2\leqslant v\leqslant u$ and $\alpha=a/q+\beta$ be a real number with with $1\leqslant a\leqslant q$ and $(a,q)=1$. Define*

$$\Xi := |\beta|u^j+(u/v)^2.$$

Then for any fixed $\varepsilon>0$, we have

$$
\begin{aligned}
&\sum_{u\leqslant n\leqslant u+v}\Lambda(n)e(\alpha n^j)\\
&\quad\ll (qu)^{\varepsilon}\big\{v(q\Xi/u)^{1/2}+(qu)^{1/2}\Xi^{1/6}+u^{3/10}v^{1/2}+u^{4/5}\Xi^{-1/6}+u(q\Xi)^{-1/2}\big\},
\end{aligned}
$$

where $\Lambda(n)$ is von Mangoldt's function and the implied constant depends on ε and j only.

Lemma 4.2. ([10], Theorem 2) *Let $1\leqslant a\leqslant q\leqslant uv$ with $(a,q)=1$ and $u,v\geqslant 1$ and let $\alpha\in\mathbb{R}$ such that $|\alpha-a/q|<1/q^2$. Then for any $\varepsilon>0$ we have*

$$\sum_{u\leqslant n\leqslant u+v}\Lambda(n)e(\alpha n^2)\ll_{\varepsilon} v^{1+\varepsilon}\big(q^{-1/4}+u^{1/8}v^{-1/4}+u^{1/3}v^{-1/2}+(qu)^{1/4}v^{-3/4}\big), \tag{4.1}$$

where the implied constant depends on ε only.

The next proposition gives the required estimate for $S(\alpha)$ on the minor arcs $\mathfrak{m}$.

Proposition 4.1. *With the previous notation, we have*

$$\max_{\alpha\in\mathfrak{m}} |S(\alpha)| \ll_\varepsilon N^{-2\varepsilon} y \quad (k=3,4). \tag{4.2}$$

The implied constant depends on ε only.

Proof. Let

$$Q' := N^{-1/2-10\varepsilon} y^3. \tag{4.3}$$

By Dirichlet's lemma, each $\alpha\in\mathfrak{m}$ can be written as

$$\alpha = a/q+\beta \quad \text{with} \quad 1\leqslant a\leqslant q\leqslant Q', \qquad (a,q)=1 \quad \text{and} \quad |\beta|\leqslant 1/(qQ').$$

We discuss two possibilities according to the size of q:

(i) If $P\leqslant q\leqslant Q'$, we can use Lemma 4.2 with $(u,v)=(x-y,2y)$ to write

$$|S(\alpha)| \ll_\varepsilon N^{-2\varepsilon} y. \tag{4.4}$$

(ii) If $q\leqslant P$, we must have $1/(qQ) < |\alpha - a/q| \leqslant 1/(qQ')$. Since $P^{-1}Q^{-1}\geqslant y^{-2}$, by Lemma 4.1 with $j=2$ and $(u,v)=(x-y,2y)$ we have

$$NQ^{-1} \ll q\Xi \asymp q|\beta|N \ll NQ'^{-1}.$$

Thus we have, for $k=3,4$,

$$\begin{aligned} |S(\alpha)| &\ll_\varepsilon N^{\varepsilon/10}\big\{N^{-1/4}y(q\Xi)^{1/2} + N^{1/4}q^{1/3}(q\Xi)^{1/6} \\ &\qquad + N^{3/20}y^{1/2} + N^{2/5}\Xi^{-1/6} + N^{1/2}(q\Xi)^{-1/2}\big\} \\ &\ll_\varepsilon N^{\varepsilon/10}\big\{N^{1/4}Q'^{-1/2}y + N^{5/12}P^{1/3}Q'^{-1/6} \\ &\qquad + N^{3/20}y^{1/2} + N^{2/5}(N^{-1}PQ)^{1/6} + Q^{1/2}\big\} \\ &\ll_\varepsilon N^{\varepsilon/10}\big\{N^{1/2+10\varepsilon}y^{-1/2} + N^{3/20}y^{1/2} + N^{7/30}y^{1/3} + N^{-3\varepsilon}y\big\} \\ &\ll_\varepsilon N^{-2\varepsilon}y, \end{aligned}$$

provided $y\geqslant N^{1/2-3/20+8\varepsilon}$. ■

We also need a preliminary lemma, which can be regarded as a generalization of Hua's lemma ([17], Lemma 2.5) in the case of short intervals.

Lemma 4.3. *Let $X\geqslant Y\geqslant 2$ and*

$$S_2^*(\alpha) := \sum_{X-Y\leqslant n\leqslant X+Y} e(\alpha n^2).$$

Then for any $\varepsilon>0$, we have

$$\int_0^1 |S_2^*(\alpha)|^4 \,\mathrm{d}\alpha \ll_\varepsilon X^\varepsilon Y^2.$$

Proof. We first write

$$\begin{aligned}\int_0^1 |S_2^*(\alpha)|^4 \,\mathrm{d}\alpha &= \sum_{\substack{n_1^2+n_4^2=n_2^2+n_3^2\\ X-Y\leqslant n_i\leqslant X+Y}} 1 = \sum_{\substack{n_1^2-n_2^2=n_3^2-n_4^2\\ X-Y\leqslant n_i\leqslant X+Y}} 1\\ &= \sum_{\substack{(n_1-n_2)(n_1+n_2)=(n_3-n_4)(n_3+n_4)\\ X-Y\leqslant n_i\leqslant X+Y}} 1\\ &\ll Y^2 + \sum_{X-Y\leqslant n_1\neq n_2\leqslant X+Y} \tau(|(n_1-n_2)(n_1+n_2)|)\\ &\ll X^\varepsilon Y^2,\end{aligned}$$

where $\tau(d)$ is the divisor function. This completes the proof. ■

Now we are ready to complete the proof of Theorem 1.1. Let $k=3$ or 4 and denote by $\mathcal{E}_k^*(z)$ the set of integers $N\in\mathcal{A}_k\cap[z/2,z]$ such that

$$N\neq p_1^2+\cdots+p_k^2 \quad \text{with} \quad \big|p_j-(N/k)^{1/2}\big|\leqslant N^{1/2-9/80+\varepsilon} \quad (1\leqslant j\leqslant k).$$

Introduce the generating function

$$Z(\alpha) := \sum_{N\in\mathcal{E}_k^*(z)} e(-\alpha N).$$

Clearly we have

$$\int_0^1 S(\alpha)^k Z(\alpha)\,\mathrm{d}\alpha = 0.$$

By using Proposition 3.1 with $k=3,4$, it follows that

$$\begin{aligned}\bigg|\int_{\mathfrak{m}} S(\alpha)^k Z(\alpha)\,\mathrm{d}\alpha\bigg| &= \bigg|\int_{\mathfrak{M}} S(\alpha)^k Z(\alpha)\,\mathrm{d}\alpha\bigg|\\ &= \sum_{N\in\mathcal{E}_k^*(z)} \int_{\mathfrak{M}} S(\alpha)^k e(-\alpha N)\,\mathrm{d}\alpha\\ &\gg |\mathcal{E}_k^*(z)| z^{-1/2} y^{k-1}.\end{aligned}$$

From this and (4.2), we deduce that

$$\begin{aligned}|\mathcal{E}_k^*(z)| &\ll z^{1/2} y^{-k+1} \int_{\mathfrak{m}} \big|S(\alpha)^k Z(\alpha)\big|\,\mathrm{d}\alpha\\ &\ll z^{1/2-2(k-2)\varepsilon} y^{-1} \int_0^1 \big|S(\alpha)^2 Z(\alpha)\big|\,\mathrm{d}\alpha\\ &\ll z^{1/2-2(k-2)\varepsilon} y^{-1} \bigg(\int_0^1 |Z(\alpha)|^2\,\mathrm{d}\alpha\bigg)^{1/2} \bigg(\int_0^1 |S(\alpha)|^4\,\mathrm{d}\alpha\bigg)^{1/2}.\end{aligned}$$

Clearly

$$\int_0^1 |Z(\alpha)|^2 \,\mathrm{d}\alpha = |\mathcal{E}_k^*(z)|$$

and Lemma 4.3 implies

$$\int_0^1 |S(\alpha)|^4 \,\mathrm{d}\alpha \ll \log^4 z \int_0^1 |S_2^*(\alpha)|^4 \,\mathrm{d}\alpha \ll z^\varepsilon y^2.$$

Thus

$$|\mathcal{E}_k^*(z)| \ll z^{1/2-(2k-5)\varepsilon} |\mathcal{E}_k^*(z)|^{1/2},$$

which is equivalent to

$$\mathcal{E}_k^*(z) \ll z^{1-(4k-10)\varepsilon}.$$

This completes the proof of Theorem 1.1. ∎

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