

Large time asymptotics of solutions for the modified KdV equation with a fifth order dispersive term

Nakao Hayashi and Pavel I. Naumkin

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Abstract. We study the large time asymptotics of small amplitude solutions to the Cauchy problem for the modified KdV equation with a fifth order dispersive term.

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§1. Introduction

We study the large time asymptotics of small amplitude solutions to the Cauchy problem for the modified Korteweg-de Vries equation with a fifth order dispersive term

$$(1.1) \quad \begin{cases} \partial_t v - \frac{a}{3} \partial_x^3 v + \frac{b}{5} \partial_x^5 v = \partial_x v^3, & t > 0, \ x \in \mathbf{R}, \\ v(0, x) = v_0(x), & x \in \mathbf{R}, \end{cases}$$

where $a, b > 0$ or $a, b < 0$. The case of $a, b > 0$ is called the positive dispersion, whereas the case of $a, b < 0$ is called the negative dispersion in the study of the solitary waves (see [14]). In this paper we assume that $a, b > 0$ since the case of $a, b < 0$ is considered in the same way. In the case of $a, b > 0$ it is expected that the main term of the large time asymptotics of solutions to (1.1) is located in the positive half-line. We also assume that the initial data are real-valued functions. The problem of large time asymptotics of solutions to (1.1) is open for the complex-valued initial data.

Korteweg-de Vries (KdV) equation

$$(1.2) \quad \partial_t v - \frac{a}{3} \partial_x^3 v = \partial_x v^2$$

was introduced in [17] to describe a model for unidirectional propagation of nonlinear dispersive long waves. In [13], it was shown that the system of equations for magneto-acoustic waves of small finite amplitude propagating in a cold collision-free plasma can be reduced to KdV equation. It was also found that the coefficient a depends on masses of ion and electron and has a possibility to be zero. Therefore the higher order dispersive terms should be taken into account. Indeed in [12] it was shown that, when $a = 0$, then the system of equations can be reduced to a simple non linear dispersive equation of the form

$$\partial_t v + \frac{b}{5} \partial_x^5 v = \partial_x v^2.$$

Thus, when a is close to 0, we obtain a generalized KdV equation

$$(1.3) \quad \partial_t v - \frac{a}{3} \partial_x^3 v + \frac{b}{5} \partial_x^5 v = \partial_x v^2.$$

In [14], the steady travelling wave solutions of (1.3) were investigated numerically to reveal how the solitary wave solutions of (1.2) are modified by the fifth order dispersive term. In [12], it was also pointed out that the Alfvén waves are described by the modified KdV equation

$$(1.4) \quad \partial_t v - \frac{a}{3} \partial_x^3 v = \partial_x v^3.$$

Thus we arrive to equation (1.1). Concerning the solitary wave solutions of higher order dispersive equations, there are a lot of works (see, e.g., [11], [15]). However the large time asymptotic behavior of small solutions was not studied well. On the other hand, the Cauchy problem (1.3) was considered in papers [2], [3], [20] and problem (1.1) was studied in [19]. In [2] and [19], the local existence in $\mathbf{H}^r$ with $r > 1/4$ and global existence in $\mathbf{H}^2$ were obtained for (1.3) and (1.1), respectively by using the Strichartz estimates. In [3], it was shown via the bilinear estimates that problem (1.3) has local solutions for $\mathbf{H}^r$ data with $r > -1$. This result combined with the $\mathbf{L}^2$ -conservation law yields global existence of solutions to (1.3) in $\mathbf{L}^2$. In [20] the order of Sobolev space r was reduced to $r \geq -7/5$ by improving the bilinear estimates. And also the global existence in $\mathbf{H}^r$ with $r > -1/2$ was obtained by using the high-low frequency method from paper [1] developed for (1.2). As far as we know the large time asymptotics of solutions to (1.3) is still an open problem.

For the modified KdV equation (1.4) we studied the large time asymptotics of small solutions and showed that the small amplitude solutions are stable in the neighborhood of the self-similar solutions (see [9]).

In the case of the fourth order nonlinear Schrödinger equation

$$(1.5) \quad \begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - \frac{1}{4}\partial_x^4 u = \lambda |u|^2 u, & t > 0, \ x \in \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

we developed the factorization technique used in [7] for the nonlinear Klein-Gordon equation to obtain the large time asymptotics of small solutions. Since the symbol of equation (1.5) $K(\xi) = -\frac{i}{2}\xi^2 - \frac{i}{4}\xi^4$ is inhomogeneous, it seems difficult to apply the operator

$$\mathcal{J}_1 = e^{tK(-i\partial_x)} x e^{-tK(-i\partial_x)} = x + it(\partial_x - \partial_x^3)$$

to (1.5) directly, so we defined the new operator

$$\mathcal{A}_1 = \frac{\overline{M_1}}{\sqrt{K''(\xi)}} \partial_\xi \frac{M_1}{\sqrt{K''(\xi)}}, \quad M_1 = e^{-t(\xi K'(\xi) - K(\xi))}.$$

In the case of the nonlinear Schrödinger equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u$, the symbol is $-\frac{i}{2}\xi^2$, and we have

$$\begin{aligned} \mathcal{A}_2 &= e^{-\frac{it}{2}\xi^2} \partial_\xi e^{\frac{it}{2}\xi^2} = \partial_\xi + it\xi \\ &= -i\mathcal{F}(x + it\partial_x)\mathcal{F}^{-1} = -i\mathcal{F}\mathcal{J}_2\mathcal{F}^{-1}. \end{aligned}$$

Hence the operator $\mathcal{A}_1$ is a generalization of $\mathcal{A}_2$, and $\mathcal{J}_2 = e^{\frac{it}{2}\partial_x^2} x e^{-\frac{it}{2}\partial_x^2} = x + it\partial_x$ was widely used in the study of the large time behavior of solutions to the nonlinear Schrödinger equations (see [10]). Here we denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transform of the function ϕ

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x) dx,$$

then the inverse Fourier transformation $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi) d\xi.$$

Multiplying both sides of (1.5) by $e^{-itK(\xi)}$ and using the factorization technique in [8], we obtained for the gauge invariant nonlinearity

$$\sqrt{K''(\xi)} \left| \frac{\hat{u}(\xi)}{\sqrt{K''(\xi)}} \right|^2 \frac{\hat{u}(\xi)}{\sqrt{K''(\xi)}}.$$

Then we can see that the operator $\mathcal{A}_1$ works well for the gauge invariant nonlinearity in (1.5). In the case of equations (1.1) or (1.4) we encounter another difficulty. By the factorization technique the nonlinearity can be decomposed as a combination of the following four terms $\hat{u}^3$, $|\hat{u}|^2 \hat{u}$, $\overline{\hat{u}}^3$, $|\hat{u}|^2 \overline{\hat{u}}$ with some additional oscillating factors, except the term $|\hat{u}|^2 \hat{u}$. (Note that for the case of the complex-valued solution, we do not have the same representation for the nonlinearity). Due to these oscillation factors the operator $\mathcal{J}$ does not

work on the terms $\widehat{u}^3$, $\overline{\widehat{u}}^3$ and $|\widehat{u}|^2 \overline{\widehat{u}}$. To avoid this difficulty in [9] we applied for (1.4) the dilation operator $x\partial_x + 3t\partial_t$. However the dilation operators only work well for dispersive equations with homogeneous symbols such as (1.4). So we can not overcome this difficulty by introducing the modified dilation operator due to the inhomogeneous symbol in equation (1.3).

In our previous paper [9] we also used the property that the difference between the total mass of the solution and the self-similar solution is equal to zero. However we do not know the existence of the self-similar solution for (1.1). So we assume in this paper that the total mass of the initial data is zero, i.e. $\int v_0(x) dx = 0$. Then by the equation we get $\int v(t, x) dx = 0$ for all $t > 0$. Under this condition, taking $u(t, x) = \int_{-\infty}^x v(t, x) dx$ we can rewrite (1.1) in the potential form

$$(1.6) \quad \begin{cases} \partial_t u - \frac{a}{3} \partial_x^3 u + \frac{b}{5} \partial_x^5 u = (u_x)^3, & t > 0, x \in \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}. \end{cases}$$

This is our equation which we study in this paper.

To state our results precisely we introduce *Notation and Function Spaces*. We denote the Lebesgue space by $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = (\int |\phi(x)|^p dx)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x \in \mathbf{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{k,s} = \left\{ \varphi \in \mathbf{S}'; \|\varphi\|_{\mathbf{H}_p^{k,s}} = \left\| \langle x \rangle^s \langle i\partial_x \rangle^k \phi \right\|_{\mathbf{L}^p} < \infty \right\},$$

$k, s \in \mathbf{R}$, $1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1+x^2}$, $\langle i\partial_x \rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{k,s} = \mathbf{H}_2^{k,s}$, $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I}; \mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C . We define the free evolution group $\mathcal{U}(t) = e^{-it\Lambda(-i\partial_x)} = \mathcal{F}^{-1} E \mathcal{F}$, where the multiplication factor $E(t, \xi) = e^{-it\Lambda(\xi)}$, and $\Lambda(\xi) = \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5$ is the dispersion relation for equation (1.6).

We are now in a position to state our main result. Define the Heaviside function $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$.

Theorem 1.1. *Assume that the initial data $u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^3 \cap \mathbf{H}^{1,1}} \leq \varepsilon$. Then there exists a unique global solution $e^{it\Lambda(-i\partial_x)} u \in \mathbf{C}([0, \infty); \mathbf{H}^3 \cap \mathbf{H}^{1,1})$ of the Cauchy problem (1.6) satisfying the time decay estimates*

$$\|\partial_x u(t)\|_{\mathbf{L}^\infty} + \|\partial_x^2 u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon t^{-\frac{1}{2}}.$$

Moreover there exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ such that the

asymptotics

$$\begin{aligned}
 \partial_x u(t) &= 2Ret^{-\frac{1}{2}} \left| \Lambda'' \left(\frac{x}{t} \right) \right|^{-\frac{1}{2}} e^{it \left(\frac{x}{t} \Lambda' \left(\frac{x}{t} \right) - \Lambda \left(\frac{x}{t} \right) \right)} \\
 \times \theta \left(\frac{x}{t} \right) &= \frac{ix}{t} W_+ \left(\frac{x}{t} \right) \exp \left(3i \left| \Lambda'' \left(\frac{x}{t} \right) \right|^{-1} \left(\frac{x}{t} \right)^3 \left| W_+ \left(\frac{x}{t} \right) \right|^2 \log t - \frac{\pi}{4} i \right) \\
 (1.7) \quad &+ O \left(\varepsilon t^{-\frac{1}{2}-\delta} \right)
 \end{aligned}$$

is valid for $t \rightarrow \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\delta \in (0, \frac{1}{6})$ is a small constant.

We introduce the factorization formula for equation (1.6). We have

$$\begin{aligned}
 \mathcal{U}(t) \mathcal{F}^{-1} \phi &= \mathcal{F}^{-1} E \phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{it \left(\frac{x}{t} \xi - \Lambda(\xi) \right)} \phi(\xi) d\xi \\
 &= \mathcal{D}_t \sqrt{\frac{it}{2\pi}} \int_{\mathbf{R}} e^{it(x\xi - \Lambda(\xi))} \phi(\xi) d\xi,
 \end{aligned}$$

where $\mathcal{D}_t \phi = (it)^{-\frac{1}{2}} \phi(xt^{-1})$. Consider the stationary point defined by the equation $\Lambda'(\xi) = x$. Since $\Lambda'(\xi) = a\xi^2 + b\xi^4$ is monotonous in the domain $\xi \in \mathbf{R}_+$, there exists a stationary point $\xi = \eta(x)$, where $\eta(x) = \frac{1}{\sqrt{2b}} \sqrt{\sqrt{4bx + a^2} - a}$, such that $\Lambda'(\eta(x)) = x$ for $x > 0$. This fact means that the main term of the large time asymptotics of the solutions lies in the positive half-line. We extend $\eta(x)$ for all $x \in \mathbf{R}$ by

$$\eta(x) = \frac{x}{\sqrt{2b}|x|} \sqrt{\sqrt{4b|x| + a^2} - a}.$$

Define the operator $(\mathcal{B}\phi)(x) = |\Lambda''(\eta(x))|^{-\frac{1}{2}} \phi(\eta(x))$. Since $x = \frac{\eta}{|\eta|} \Lambda'(\eta)$, we get

$$\begin{aligned}
 \mathcal{U}(t) \mathcal{F}^{-1} \phi &= \mathcal{D}_t \sqrt{\frac{it}{2\pi}} \int_{\mathbf{R}} e^{-it(\Lambda(\xi) - x\xi)} \phi(\xi) d\xi \\
 &= \mathcal{D}_t \sqrt{\frac{it}{2\pi}} \int_{\mathbf{R}} e^{-it \left(\Lambda(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi \right)} \phi(\xi) d\xi \\
 &= \mathcal{D}_t \mathcal{B} \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_{\mathbf{R}} e^{-it \left(\Lambda(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi \right)} \phi(\xi) d\xi,
 \end{aligned}$$

Since u is a real-valued function, then if we take $u = \mathcal{U}(t) \mathcal{F}^{-1} \hat{\varphi}$ we have

$\widehat{\varphi}(-\xi) = \overline{\widehat{\varphi}(\xi)}$. Hence

$$\begin{aligned}
 \mathcal{U}(t) \mathcal{F}^{-1} \widehat{\varphi} &= \mathcal{D}_t \mathcal{B} \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-it(\Lambda(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi)} \widehat{\varphi}(\xi) d\xi \\
 &\quad + \overline{\mathcal{D}_t \mathcal{B} \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-it(\Lambda(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi)} \widehat{\varphi}(\xi) d\xi} \\
 (1.8) \quad &= 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \widehat{\varphi}(\xi) d\xi = 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} \widehat{\varphi}
 \end{aligned}$$

for $\eta \in \mathbf{R}$, where the multiplication factor $M(t, \eta) = e^{it(\eta \Lambda'(\eta) - \Lambda(\eta))\theta(\eta)} = e^{itH(\eta)}$, $H(\eta) = (\eta \Lambda'(\eta) - \Lambda(\eta))\theta(\eta)$, the phase function

$$S(\xi, \eta) = \Lambda(\xi) - \Lambda(\eta)\theta(\eta) - \frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \eta\theta(\eta))$$

and the operator

$$\mathcal{V}\phi = \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) d\xi.$$

Also we need the representation for the inverse evolution group $\mathcal{FU}(-t)$ for all $\xi \geq 0$

$$\begin{aligned}
 \mathcal{FU}(-t)\phi &= \overline{E}\mathcal{F}\phi = \sqrt{\frac{1}{2\pi}} \int_{\mathbf{R}} e^{it\Lambda(\xi) - ix\xi} \phi(x) dx \\
 &= \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{it(\Lambda(\xi) - x\xi)} \mathcal{D}_t^{-1}\phi(x) dx \\
 &= \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{it(\Lambda(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi)} (\mathcal{B}^{-1} \mathcal{D}_t^{-1}\phi)(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta \\
 (1.9) \quad &= \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{itS(\xi, \eta)} \overline{M} (\mathcal{B}^{-1} \mathcal{D}_t^{-1}\phi)(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta = \mathcal{Q} \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1}\phi,
 \end{aligned}$$

where

$$\mathcal{D}_t^{-1}\phi = (it)^{\frac{1}{2}} \phi(xt), \quad (\mathcal{B}^{-1}\phi)(\eta) = |\Lambda''(\eta)|^{\frac{1}{2}} \phi\left(\frac{\eta}{|\eta|} \Lambda'(\eta)\right),$$

and the operator

$$\mathcal{Q}\phi = \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{itS(\xi, \eta)} \phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta$$

for $\xi \geq 0$, is considered as the inverse operator to $\mathcal{V}$.

Since $\mathcal{FU}(-t)\mathcal{L} = \partial_t \mathcal{FU}(-t)$ with $\mathcal{L} = \partial_t - \frac{a}{3}\partial_x^3 + \frac{b}{5}\partial_x^5$, applying the operator $\mathcal{FU}(-t)$ to equation (1.6) we get for the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t)u(t)$

$$\partial_t \widehat{\varphi} = \partial_t \mathcal{FU}(-t)u = \mathcal{FU}(-t)\mathcal{L}u = \mathcal{FU}(-t)u_x^3.$$

Then by (1.8) and (1.9) we find the following factorization property

$$\begin{aligned} \mathcal{FU}(-t)u_x^3 &= \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}u_x^3 \\ &= \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(\mathcal{D}_t\mathcal{BMV}(i\xi)\widehat{\varphi} + \overline{\mathcal{D}_t\mathcal{BMV}(i\xi)\widehat{\varphi}}\right)^3 \\ &= \mathcal{Q}\overline{M}\mathcal{B}^{-1}\frac{1}{(it)}\left(\mathcal{BMV}(i\xi)\widehat{\varphi} + \frac{(it)^{\frac{1}{2}}}{(-it)^{\frac{1}{2}}}\overline{\mathcal{BMV}(i\xi)\widehat{\varphi}}\right)^3 \\ &= (it)^{-1}\mathcal{Q}\overline{M}\mathcal{B}^{-1}\left(\mathcal{BMV}(i\xi)\widehat{\varphi} + i\overline{\mathcal{BMV}(i\xi)\widehat{\varphi}}\right)^3 \\ &= (it)^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}\left(MV(i\xi)\widehat{\varphi} + i\overline{MV(i\xi)\widehat{\varphi}}\right)^3. \end{aligned}$$

Hence

$$\begin{aligned} &\mathcal{FU}(-t)u_x^3 \\ &= (it)^{-1}\mathcal{Q}|\Lambda''|^{-1}M^2(\mathcal{V}i\xi\widehat{\varphi})^3 + 3t^{-1}\mathcal{Q}|\Lambda''|^{-1}|\mathcal{V}i\xi\widehat{\varphi}|^2\mathcal{V}i\xi\widehat{\varphi} \\ &\quad - 3(it)^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}^2|\mathcal{V}i\xi\widehat{\varphi}|^2\overline{\mathcal{V}i\xi\widehat{\varphi}} - t^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}^4\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)^3. \end{aligned}$$

We need to calculate the commutator of $\mathcal{Q}$ and M since $\partial_\xi \mathcal{Q}M$ yields undesirable time growth, when we wish to estimate the derivative $\partial_\xi \widehat{\varphi}$. Note that in the case of the usual nonlinear Schrödinger equation with cubic nonlinearity u^3 , we have

$$\mathcal{FU}(-t)u^3 = (it)^{-1}\mathcal{Q}M^2(\mathcal{V}\widehat{\varphi})^3$$

with

$$\begin{aligned} \mathcal{Q}\phi &= \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{\frac{it}{2}(\xi-\eta)^2} \phi(\eta) d\eta = \mathcal{F}^{-1}e^{-\frac{i}{2t}|x|^2}\mathcal{F}\phi, \\ \mathcal{V}\phi &= \sqrt{\frac{it}{2\pi}} \int_{\mathbf{R}} e^{-\frac{it}{2}(\xi-\eta)^2} \phi(\xi) d\xi = \mathcal{F}e^{\frac{i}{2t}|\xi|^2}\mathcal{F}^{-1}\phi. \end{aligned}$$

We can see that $M = e^{\frac{it}{2}\eta^2}$ yields an additional time growth since $\partial_\xi \mathcal{Q}M^2$ has a time growing summand like $-it\xi \mathcal{Q}M^2$. We encounter a similar difficulty in our case.

We have for $\alpha \neq -1$

$$\begin{aligned}
& it(S(\xi, \eta) + \alpha H(\eta)) \\
&= it\left(\Lambda(\xi) - \Lambda(\eta)\theta(\eta) - \frac{\eta}{|\eta|}\Lambda'(\eta)(\xi - \eta\theta(\eta))\right. \\
&\quad \left.+ \alpha(\eta\Lambda'(\eta)\theta(\eta) - \Lambda(\eta)\theta(\eta))\right) \\
&= it\left(\Lambda(\xi) - (1+\alpha)\Lambda\left(\frac{\xi}{1+\alpha}\right)\right) \\
&\quad + i(1+\alpha)t\left(\Lambda\left(\frac{\xi}{1+\alpha}\right) - \Lambda(\eta)\theta(\eta) - \frac{\eta}{|\eta|}\Lambda'(\eta)\left(\frac{\xi}{1+\alpha} - \eta\theta(\eta)\right)\right) \\
&= it\left(\Lambda(\xi) - (1+\alpha)\Lambda\left(\frac{\xi}{1+\alpha}\right)\right) + i(1+\alpha)tS\left(\frac{\xi}{1+\alpha}, \eta\right).
\end{aligned}$$

Hence

$$\mathcal{Q}(t)M^\alpha\phi = e^{it(\Lambda(\xi) - (1+\alpha)\Lambda(\frac{\xi}{1+\alpha}))}i^{\frac{1}{2}}\mathcal{D}_{1+\alpha}\mathcal{Q}(t(1+\alpha))\phi.$$

This fact implies that our method does not work for the nonlinearity $|u|^2$, namely for the case of equations (1.2) or (1.3) with a quadratic nonlinearity. Thus we obtain from (1.6) the following equation for the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t)u(t)$

$$\begin{aligned}
(1.10) \quad \partial_t \widehat{\varphi} &= -i^{\frac{3}{2}}t^{-1}e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\frac{1}{|\Lambda''|}(\mathcal{V}i\xi\widehat{\varphi})^3 \\
&\quad + 3t^{-1}\mathcal{Q}(t)\frac{1}{|\Lambda''|}|\mathcal{V}i\xi\widehat{\varphi}|^2\mathcal{V}i\xi\widehat{\varphi} \\
&\quad + 3i^{\frac{3}{2}}t^{-1}\mathcal{D}_{-1}\mathcal{Q}(-t)\frac{1}{|\Lambda''|}|\mathcal{V}i\xi\widehat{\varphi}|^2\overline{\mathcal{V}i\xi\widehat{\varphi}} \\
&\quad - i^{\frac{1}{2}}t^{-1}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\frac{1}{|\Lambda''|}\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)^3,
\end{aligned}$$

where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$.

It is well known that the operator $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t)$ is a useful tool for obtaining the $\mathbf{L}^\infty$ -time decay estimates of solutions and has been used widely for the studying the asymptotic behavior of solutions to various nonlinear dispersive equations. We have

$$\begin{aligned}
\mathcal{J} &= \mathcal{U}(t)x\mathcal{U}(-t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}i\partial_\xi e^{it\Lambda(\xi)}\mathcal{F} \\
&= \mathcal{F}^{-1}(i\partial_\xi - t\Lambda'(\xi))\mathcal{F} = x - t\Lambda'(-i\partial_x),
\end{aligned}$$

where $\Lambda'(-i\partial_x) = -a\partial_x^2 + b\partial_x^4$. Note that the commutators are true $[\mathcal{J}, \mathcal{L}] = 0$, $[\mathcal{J}, \partial_x] = -1$, where $\mathcal{L} = \partial_t + i\Lambda(-i\partial_x) = \partial_t - \frac{a}{3}\partial_x^3 + \frac{b}{5}\partial_x^5$. However, it seems that $\mathcal{J}$ does not work well on the nonlinear terms. Then, instead of the operators

$\mathcal{J}$ and the dilation operator used in [5] we apply the modified dilation operator defined by

$$\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a.$$

This is our idea in this paper. Note that $\mathcal{P}$ acts well on the nonlinear terms as the first order differential operator. Also $\mathcal{J}$ and $\mathcal{P}$ are related via the identity $\mathcal{P} = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I}$, where $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$. In order to get the estimate of $\mathcal{J}\partial_x u$, we will show the a-priori estimates of $\mathcal{P}u$, $t\mathcal{L}u$ and $\mathcal{I}u$. Different point compared to the previous works is to consider the estimate of $\mathcal{I}u$ since $\mathcal{I}u$ contains the third order derivatives $t\partial_x^3 u$ with the additional time growth (if $a \neq 0$). Note that the commutators $[\mathcal{P}, \mathcal{L}] = -\mathcal{L}$ and $[\mathcal{P}, \partial_x] = -\frac{1}{5}\partial_x$ are true. Also we have $[\mathcal{I}, \mathcal{L}] = 0$.

Our main task is to estimate each term of the right-hand side of equation (1.10) in $\mathbf{L}^\infty$ and $\mathbf{H}^1$ norms. We organize the rest of our paper as follows. In Section 2, we state the uniform estimates for the decomposition operator $\mathcal{V}$ (Lemma 2.2). The uniform estimates of $\mathcal{Q}(t)$ and $\mathcal{Q}(-t)$ are also obtained in Lemma 2.3. We prove $\mathbf{L}^2$ -estimates of the derivatives of $\mathcal{Q}$ (Lemma 3.1) and $\mathcal{V}$ (Lemma 3.5) in Section 3. Lemma 3.2 and Lemma 3.3 are prepared for proving Lemma 3.5. In Section 4 we divide the nonlinear term into the main term and the remainder terms (Lemma 4.1). In Section 5 we obtain the estimates of $\|\mathcal{F}\mathcal{U}(-t)u\|_{\mathbf{L}^\infty}$ and $\|\mathcal{J}u(t)\|_{\mathbf{H}^1}$. Section 6 is devoted to the proof of Theorem 1.1.

§2. Estimates in the uniform norm

2.1. Estimates for two kernels

Define two kernels, which describe the main term of the large time asymptotics of the operators $\mathcal{V}$ and $\mathcal{Q}$, respectively,

$$A(t, \eta) = \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \chi(\xi\eta^{-1}) d\xi$$

and

$$A^*(t, \xi) = \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi\eta^{-1}) d\eta,$$

for $\xi, \eta > 0$, where $S(\xi, \eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta)$, $\Lambda(\xi) = \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5$, and the cut off function $\chi(z) \in \mathbf{C}^2(\mathbf{R})$ is such that $\chi(z) = 0$ for $z \leq \frac{1}{3}$, $\chi(z) = 1$ for $\frac{2}{3} \leq z \leq \frac{3}{2}$, and $\chi(z) = 0$ for $z \geq 3$.

Lemma 2.1. *The estimate*

$$\|A(t)\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|A^*(t)\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq C$$

is true for all $t > 0$. Moreover the asymptotics are valid

$$A(t, \eta) = 1 + O\left((t\eta^3)^{-1}\right)$$

for $t^{\frac{1}{3}}\eta \rightarrow \infty$ and

$$A^*(t, \xi) = 1 + O\left((t\xi^3)^{-1}\right)$$

for $t^{\frac{1}{3}}\xi \rightarrow \infty$.

Proof. We use the identity

$$(2.1) \quad e^{-itS(\xi, \eta)} = H_1 \partial_\xi \left((\xi - \eta) e^{-itS(\xi, \eta)} \right)$$

with $H_1 = (1 - it(\xi - \eta) S_\xi(\xi, \eta))^{-1}$ in the domain $\xi, \eta > 0$, and integrate by parts to get

$$A(t, \eta) = -\sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} (\xi - \eta) \partial_\xi (H_1 \chi(\xi \eta^{-1})) d\xi.$$

Hence in view of the estimate

$$(2.2) \quad |(\xi - \eta) \partial_\xi (H_1 \chi(\xi \eta^{-1}))| \leq C \left(1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2\right)^{-1}$$

in the domain $\frac{\eta}{3} \leq \xi \leq 3\eta$, we obtain

$$\begin{aligned} |A(t, \eta)| &\leq Ct^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_{\frac{\eta}{3}}^{3\eta} \frac{d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\ &\leq Ct^{\frac{1}{2}} \eta^{\frac{3}{2}} \langle \eta \rangle \int_{\frac{1}{3}}^3 \frac{dy}{1 + t\Lambda(\eta) (y - 1)^2} \leq C |t\Lambda(\eta)|^{\frac{1}{2}} \langle t\Lambda(\eta) \rangle^{-\frac{1}{2}} \leq C. \end{aligned}$$

In the same manner we use the identity

$$(2.3) \quad e^{itS(\xi, \eta)} = H_2 \partial_\eta \left((\eta - \xi) e^{itS(\xi, \eta)} \right)$$

with $H_2 = (1 + it(\eta - \xi) S_\eta(\xi, \eta))^{-1}$ in the domain $\xi, \eta > 0$, and integrate by parts

$$\begin{aligned} A^*(t, \xi) &= \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi \eta^{-1}) d\eta \\ &= -\sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} (\xi - \eta) \partial_\eta \left(H_2 |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi \eta^{-1}) \right) d\eta. \end{aligned}$$

Then using the estimate

$$(2.4) \quad \left| (\xi - \eta) \partial_\eta \left(H_2 |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi \eta^{-1}) \right) \right| \leq C \xi^{\frac{1}{2}} \langle \xi \rangle \left(1 + t \xi \langle \xi \rangle^2 (\xi - \eta)^2 \right)^{-1}$$

in the domain $\frac{\xi}{3} \leq \eta \leq 3\xi$, we get as above

$$\begin{aligned} |A^*(t, \xi)| &\leq C t^{\frac{1}{2}} \xi^{\frac{1}{2}} \langle \xi \rangle \int_{\frac{\xi}{3}}^{3\xi} \frac{d\eta}{1 + t \xi \langle \xi \rangle^2 (\xi - \eta)^2} \\ &\leq C t^{\frac{1}{2}} \xi^{\frac{3}{2}} \langle \xi \rangle \int_{\frac{1}{3}}^3 \frac{dy}{1 + t \Lambda(\xi) (y - 1)^2} \leq C |t \Lambda(\xi)|^{\frac{1}{2}} \langle t \Lambda(\xi) \rangle^{-\frac{1}{2}} \leq C. \end{aligned}$$

To compute the asymptotics of the functions $A(t, \eta)$ and $A^*(t, \xi)$ for large $\eta t^{\frac{1}{3}}$ and $\xi t^{\frac{1}{3}}$ we apply the stationary phase method (see [4], p. 163). We have the asymptotics

$$(2.5) \quad \int_{\mathbf{R}} e^{irG(x)} f(x) dx = e^{irG(x_0)} f(x_0) \sqrt{\frac{2\pi}{r |G''(x_0)|}} e^{i\frac{\pi}{4} \text{sgn} G''(x_0)} + O(r^{-\frac{3}{2}})$$

for $r \rightarrow +\infty$, where the stationary point x_0 is defined by $G'(x_0) = 0$. We change $\xi = x\eta$, then we get

$$\begin{aligned} A(t, \eta) &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \chi(\xi \eta^{-1}) d\xi \\ &= \eta \sqrt{\frac{it}{2\pi}} |2\eta(a + 2b\eta^2)|^{\frac{1}{2}} \int_0^\infty e^{-it\eta^3(\frac{a}{3}x^3 + \frac{b}{5}x^5\eta^2 - \frac{a}{3} - \frac{b}{5}\eta^2 - (a + b\eta^2)(x-1))} \chi(x) dx \\ &= \sqrt{\frac{ir}{2\pi}} |2(a + 2b\eta^2)|^{\frac{1}{2}} \int_0^\infty e^{irG(x, \eta)} \chi(x) dx, \end{aligned}$$

where $r = t\eta^3$. By virtue of formula (2.5) with $r = t\eta^3$,

$$f(x) = \chi(x), \quad x_0 = 1$$

and

$$G(x, \eta) = - \left(\frac{a}{3} x^3 + \frac{b}{5} x^5 \eta^2 - \frac{a}{3} - \frac{b}{5} \eta^2 - (a + b\eta^2)(x - 1) \right),$$

we get $A(t, \eta) = 1 + O((t\eta^3)^{-1})$ for $t^{\frac{1}{3}}\eta \rightarrow +\infty$. Also changing $\eta = x\xi$ we obtain

$$\begin{aligned} A^*(t, \xi) &= \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} |\Lambda''(\eta)|^{\frac{1}{2}} \chi(x^{-1}) d\eta \\ &= \xi^{\frac{3}{2}} \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{it\xi^3(\frac{a}{3} + \frac{b}{5}\xi^2 - \frac{a}{3}x^3 - \frac{b}{5}x^5\xi^2 - (ax^2 + bx^4\xi^2)(1-x))} \chi(x^{-1}) \\ &\quad \times |2ax + 4bx^3\xi^2|^{\frac{1}{2}} dx \\ &= \sqrt{\frac{r}{2\pi i}} \int_0^\infty e^{irG(x, \eta)} \chi(x^{-1}) (2x(a + 2bx^2\xi^2))^{\frac{1}{2}} dx, \end{aligned}$$

where $r = t\xi^3$. Then by virtue of formula (2.5) with $r = t\xi^3$,

$$f(x) = \chi(x^{-1}) (2x(a + 2bx^2\xi^2))^{\frac{1}{2}}, x_0 = 1$$

and

$$G(x, \xi) = \frac{a}{3} + \frac{b}{5}\xi^2 - \frac{a}{3}x^3 - \frac{b}{5}x^5\xi^2 - (ax^2 + bx^4\xi^2)(1-x),$$

we get $A^*(t, \xi) = 1 + O((t\xi^3)^{-1})$ as $t^{\frac{1}{3}}\xi \rightarrow +\infty$. Lemma 2.1 is proved. $\square$

2.2. Estimates for the operator $\mathcal{V}$

We estimate the operator

$$\mathcal{V}\xi^j\phi = \sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) \xi^j d\xi$$

in the uniform norm. Denote the norm

$$\|\phi\|_{\mathbf{Z}_1} = \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Note that

$$\begin{aligned} |\mathcal{V}\phi| &\leq Ct^{\frac{1}{2}} |\Lambda''|^{\frac{1}{2}} \left\| \langle \xi \rangle^{-2} \right\|_{\mathbf{L}^1(\mathbf{R}_+)} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\leq Ct^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} \left\| \langle \xi \rangle^2 \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}. \end{aligned}$$

Next lemma says that the main term of $(\mathcal{V}\phi)(\eta)$ is $A(t, \eta)\phi(\eta)$.

Lemma 2.2. *Let $j = 0, 1, 2, 3$. Then the estimates are valid for all $t \geq 1$*

$$\left\| |\eta|^\alpha \langle \eta \rangle^\beta (\mathcal{V}\xi^j\phi - A(t, \eta)\eta^j\phi(\eta)) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq C \max\left(t^{-\frac{1}{4}}, t^{-\frac{j+\alpha}{3}}\right) \|\phi\|_{\mathbf{Z}_1}$$

with $\alpha \geq -\frac{1}{2}$, $\beta \leq \frac{7}{4} - j - \alpha$, and

$$\left\| |\eta|^\alpha \langle \eta \rangle^\beta \mathcal{V}\xi^j\phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \leq C \max\left(t^{-\frac{1}{2}} \log \langle t \rangle, t^{-\frac{j+\alpha}{3}}\right) \|\phi\|_{\mathbf{Z}_1}$$

with $\alpha \geq -\frac{1}{2}$, $\beta \leq \frac{5}{2} - j - \alpha$. When $j = 2, 3$, the norm $\|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ can be replaced by $\|\xi \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$.

Proof. By the definition of $\mathcal{V}$ and $A(t, \eta)$, we can write

$$\begin{aligned}
& \mathcal{V} \xi^j \phi - A(t, \eta) \eta^j \phi(\eta) \\
&= \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi, \eta)} (\phi(\xi) \xi^j - \phi(\eta) \eta^j) \chi(\xi \eta^{-1}) d\xi \\
&\quad + \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^{\frac{2\eta}{3}} e^{-itS(\xi, \eta)} (1 - \chi(\xi \eta^{-1})) \phi(\xi) \xi^j d\xi \\
&\quad + \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_{\frac{3\eta}{2}}^{\infty} e^{-itS(\xi, \eta)} (1 - \chi(\xi \eta^{-1})) \phi(\xi) \xi^j d\xi \\
&= \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} (I_1 + I_2 + I_3)
\end{aligned}$$

for $\eta > 0$. We integrate by parts via the identity (2.1)

$$\begin{aligned}
I_1 &= - \int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi, \eta)} (\xi - \eta) \chi(\xi \eta^{-1}) H_1 \phi(\xi) j \xi^{j-1} d\xi \\
&\quad - \int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi, \eta)} (\phi(\xi) \xi^j - \phi(\eta) \eta^j) (\xi - \eta) \partial_\xi (\chi(\xi \eta^{-1}) H_1) d\xi \\
&\quad - \int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi, \eta)} (\xi - \eta) \chi(\xi \eta^{-1}) H_1 \xi^j \phi_\xi(\xi) d\xi.
\end{aligned}$$

Using the estimates

$$(2.6) \quad |H_1| = |1 - it(\xi - \eta) S_\xi(\xi, \eta)|^{-1} \leq C \left(1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2\right)^{-1}$$

and (2.2) in the domain $\frac{\eta}{3} < \xi < 3\eta$, we obtain for $j = 0, 1, 2, 3$

$$\begin{aligned}
|I_1| &\leq C \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} j \eta^{j-1} \int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta| d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\
&\quad + C \eta^j \int_{\frac{\eta}{3}}^{3\eta} \frac{|\phi(\xi) - \phi(\eta)| d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\
&\quad + C \eta^j \int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta| |\phi_\xi(\xi)| d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2}.
\end{aligned}$$

Since

$$\begin{aligned}
& |\phi(\xi) - \phi(\eta)| \leq C \int_\xi^\eta \langle z \rangle^{-1} \langle z \rangle |\partial_z \phi(z)| dz \\
&\leq C \left(\int_\xi^\eta \langle z \rangle^{-2} dz \right)^{\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \langle \eta \rangle^{-1} |\xi - \eta|^{\frac{1}{2}} \|\phi\|_{\mathbf{Z}_1}
\end{aligned}$$

in the domain $\frac{\eta}{3} < \xi < 3\eta$, we get

$$\begin{aligned}
|I_1| &\leq C \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} j \eta^{j-1} \int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta| d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\
&\quad + C \|\phi\|_{\mathbf{Z}_1} \eta^j \langle \eta \rangle^{-1} \int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta|^{\frac{1}{2}} d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\
&\quad + C \|\phi\|_{\mathbf{Z}_1} \eta^j \langle \eta \rangle^{-1} \left(\int_{\frac{\eta}{3}}^{3\eta} \frac{(\xi - \eta)^2 d\xi}{\left(1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2\right)^2} \right)^{\frac{1}{2}}
\end{aligned}$$

for $j = 0, 1, 2, 3$. Note that

$$\begin{aligned}
&\int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta| d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \\
&\leq C \eta^2 \int_{\frac{1}{3}}^3 \frac{|y - 1| dy}{1 + t\Lambda(\eta) (y - 1)^2} \\
&\leq C \langle t\Lambda(\eta) \rangle^{-1} \eta^2 \int_{-\frac{2}{3}t\sqrt{\Lambda(\eta)}}^{2t\sqrt{\Lambda(\eta)}} \frac{|y| dy}{1 + y^2} \leq C \eta^2 \langle t\Lambda(\eta) \rangle^{-1} \log \langle t\Lambda(\eta) \rangle, \\
&\int_{\frac{\eta}{3}}^{3\eta} \frac{|\xi - \eta|^{\frac{1}{2}} d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \leq C \eta^{\frac{3}{2}} \int_{\frac{1}{3}}^3 \frac{|y - 1|^{\frac{1}{2}} dy}{1 + t\Lambda(\eta) (y - 1)^2} \leq C \eta^{\frac{3}{2}} \langle t\Lambda(\eta) \rangle^{-\frac{3}{4}}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{\frac{\eta}{3}}^{3\eta} \frac{(\xi - \eta)^2 d\xi}{\left(1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2\right)^2} \right)^{\frac{1}{2}} \\
&\leq C \eta^{\frac{3}{2}} \left(\int_{\frac{1}{3}}^3 \frac{(y - 1)^2 dy}{\left(1 + t\Lambda(\eta) (y - 1)^2\right)^2} \right)^{\frac{1}{2}} \leq C \eta^{\frac{3}{2}} \langle t\Lambda(\eta) \rangle^{-\frac{3}{4}}.
\end{aligned}$$

Hence we get for $j = 0, 1, 2, 3$

$$\begin{aligned}
&t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta |I_1| \\
&\leq C t^{\frac{1}{2}} \|\phi\|_{\mathbf{Z}_1} \left(j \eta^{j+\frac{3}{2}+\alpha} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \log \langle t\Lambda(\eta) \rangle \right. \\
&\quad \left. + \eta^{j+2+\alpha} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-\frac{3}{4}} \right) \\
(2.7) \quad &\leq C \left(t^{-\frac{1}{4}} + t^{-\frac{j+\alpha}{3}} \right) \|\phi\|_{\mathbf{Z}_1}
\end{aligned}$$

with $j + \alpha + \beta \leq \frac{7}{4}$, $j + 2 + \alpha \geq 0$, since

$$\begin{aligned} & \eta^{j+\frac{3}{2}+\alpha} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \log \langle t\Lambda(\eta) \rangle \\ & \leq C \eta^{j+\frac{3}{2}+\alpha} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \log \left\langle t^{\frac{1}{3}} \eta \right\rangle \\ & \leq C \begin{cases} t^{-\frac{j+\alpha}{3}-\frac{1}{2}}, & 3 > j + \frac{3}{2} + \alpha \geq 0 \\ t^{-1} \log \langle t \rangle, & j + \frac{3}{2} + \alpha \geq 3 \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \eta^{j+2+\alpha} \langle \eta \rangle^{\beta} \langle t\Lambda(\eta) \rangle^{-\frac{3}{4}} \leq C \eta^{j+2+\alpha} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-\frac{9}{4}} \\ & \leq C \begin{cases} t^{-\frac{j+\alpha+2}{3}}, & \frac{9}{4} \geq j + 2 + \alpha \geq 0 \\ t^{-\frac{3}{4}}, & j + 2 + \alpha \geq \frac{9}{4} \end{cases} \end{aligned}$$

for $\eta < 1$, also

$$\begin{aligned} & \eta^{j+\frac{3}{2}+\alpha} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \log \langle t\Lambda(\eta) \rangle \\ & \leq C \eta^{j+\frac{5}{2}+\alpha+\beta} \left\langle t^{\frac{1}{5}} \eta \right\rangle^{-5} \log \left\langle t^{\frac{1}{5}} \eta \right\rangle \leq C t^{-1} \log \langle t \rangle \end{aligned}$$

and

$$\eta^{j+2+\alpha} \langle \eta \rangle^{\beta} \langle t\Lambda(\eta) \rangle^{-\frac{3}{4}} \leq C \eta^{j+2+\alpha+\beta} \left\langle t^{\frac{1}{5}} \eta \right\rangle^{-\frac{15}{4}} \leq C t^{-\frac{3}{4}}$$

for $\eta > 1$ and $j + \alpha + \beta \leq \frac{7}{4}$.

Next we consider

$$I_2 = \int_0^{\frac{2\eta}{3}} e^{-itS(\xi, \eta)} (1 - \chi(\xi \eta^{-1})) \phi(\xi) \xi^j d\xi$$

for $\eta > 0$. We integrate by parts via the identity

$$(2.8) \quad e^{-itS(\xi, \eta)} = H_3 \partial_\xi \left(\xi e^{-itS(\xi, \eta)} \right)$$

with $H_3 = (1 - it\xi(\Lambda'(\xi) - \Lambda'(\eta)))^{-1}$ to get

$$\begin{aligned} I_2 &= - \int_0^{\frac{2\eta}{3}} e^{-itS(\xi, \eta)} \phi(\xi) \xi \partial_\xi (H_3 (1 - \chi(\xi \eta^{-1})) \xi^j) d\xi \\ &\quad - \int_0^{\frac{2\eta}{3}} e^{-itS(\xi, \eta)} H_3 (1 - \chi(\xi \eta^{-1})) \xi^{j+1} \phi_\xi(\xi) d\xi. \end{aligned}$$

Using the estimates $|H_3| \leq C(1 + t\xi\Lambda'(\eta))^{-1}$ and

$$|\xi \partial_\xi (H_3 (1 - \chi(\xi \eta^{-1})) \xi^j)| \leq C \xi^j (1 + t\xi\Lambda'(\eta))^{-1}$$

in the domain $0 < \xi < \frac{2\eta}{3}$, we obtain

$$|I_2| \leq C \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_0^{\frac{2\eta}{3}} \frac{\xi^j d\xi}{1 + t\xi\Lambda'(\eta)} + C \int_0^{\frac{2\eta}{3}} \frac{\xi^{j+\frac{1}{2}} \left| \xi^{\frac{1}{2}} \phi_\xi \right| d\xi}{1 + t\xi\Lambda'(\eta)}.$$

We have

$$\int_0^{\frac{2\eta}{3}} \frac{\xi^j d\xi}{1 + t\xi\Lambda'(\eta)} \leq C\eta^{1+j} \int_0^{\frac{2}{3}} \frac{y^j dy}{1 + t\Lambda(\eta)y} \leq C\eta^{1+j} \langle t\Lambda(\eta) \rangle^{-1}$$

and

$$\int_0^{\frac{2\eta}{3}} \frac{\xi^{2j+1} d\xi}{(1 + t\xi\Lambda'(\eta))^2} \leq C\eta^{2j+2} \int_0^{\frac{2}{3}} \frac{y^{2j+1} dy}{(1 + t\Lambda(\eta)y)^2} \leq C\eta^{2j+2} \langle t\Lambda(\eta) \rangle^{-2}.$$

Hence we get

$$\begin{aligned} & t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta |I_2| \\ & \leq C t^{\frac{1}{2}} \|\phi\|_{\mathbf{Z}_1} \eta^{\alpha+j+\frac{3}{2}} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \\ (2.9) \quad & \leq C \left(t^{-\frac{1}{2}} + t^{-\frac{\alpha+j}{3}} \right) \|\phi\|_{\mathbf{Z}_1} \end{aligned}$$

with $\alpha + j + \frac{3}{2} \geq 0$, $j + \alpha + \beta \leq \frac{5}{2}$, since

$$\begin{aligned} & \eta^{\alpha+j+\frac{3}{2}} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \leq C\eta^{\alpha+j+\frac{3}{2}} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \\ & \leq C \begin{cases} t^{-\frac{\alpha+j}{3}-\frac{1}{2}}, & 3 \geq \alpha + j + \frac{3}{2} \geq 0 \\ t^{-1}, & \alpha + j + \frac{3}{2} \geq 3 \end{cases} \end{aligned}$$

for $\eta < 1$, and

$$\begin{aligned} & \eta^{\alpha+j+\frac{3}{2}} \langle \eta \rangle^{\beta+1} \langle t\Lambda(\eta) \rangle^{-1} \\ & \leq C\eta^{j+\frac{5}{2}+\alpha+\beta} \left\langle t^{\frac{1}{5}} \eta \right\rangle^{-5} \leq Ct^{-1}, \end{aligned}$$

for $\eta > 1$ and $j + \alpha + \beta \leq \frac{5}{2}$.

Finally we estimate

$$I_3 = \int_{\frac{3\eta}{2}}^{\infty} e^{-itS(\xi,\eta)} (1 - \chi(\xi\eta^{-1})) \phi(\xi) \xi^j d\xi$$

for $\eta > 0$. We integrate by parts via the identity (2.8)

$$\begin{aligned} I_3 &= - \int_{\frac{3\eta}{2}}^{\infty} e^{-itS(\xi,\eta)} \phi(\xi) \xi \partial_\xi (H_3(1 - \chi(\xi\eta^{-1})) \xi^j) d\xi \\ &\quad - \int_{\frac{3\eta}{2}}^{\infty} e^{-itS(\xi,\eta)} H_3(1 - \chi(\xi\eta^{-1})) \xi^{j+1} \phi_\xi(\xi) d\xi. \end{aligned}$$

Using the estimates $|H_3| \leq C(1 + t\Lambda(\xi))^{-1}$ and

$$|\xi \partial_\xi (H_3(1 - \chi(\xi\eta^{-1}))\xi^j)| \leq C\xi^j(1 + t\Lambda(\xi))^{-1}$$

in the domain $\xi > \frac{3\eta}{2}$, we obtain

$$|I_3| \leq C \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1 + t\Lambda(\xi)} + C \int_{\frac{3\eta}{2}}^\infty \frac{\xi^{j+\frac{1}{2}} \left| \xi^{\frac{1}{2}} \phi_\xi \right| d\xi}{1 + t\Lambda(\xi)}.$$

Since

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1 + t\Lambda(\xi)} \leq Ct^{-1} \int_{\frac{3\eta}{2}}^\infty \xi^{j-5} d\xi \leq Ct^{-1} \eta^{j-4}$$

for $\eta > 1$, $t \geq 1$, and for $0 < \eta \leq 1$, we find

$$\begin{aligned} \int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1 + t\Lambda(\xi)} &\leq \int_{\frac{3\eta}{2}}^{\frac{3}{2}} \frac{\xi^j d\xi}{1 + t\xi^3} + Ct^{-1} \int_{\frac{3}{2}}^\infty \xi^{j-4} d\xi \\ &\leq Ct^{-\frac{j+1}{3}} \int_{\frac{3\eta}{2}t^{\frac{1}{3}}}^{\frac{3}{2}t^{\frac{1}{3}}} \frac{\xi^j d\xi}{1 + \xi^3} + Ct^{-1} \leq Ct^{-1} \log \langle t \rangle + Ct^{-\frac{j+1}{3}} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{j-2} \end{aligned}$$

for $j = 0, 1, 2$ and for $j = 3$

$$\begin{aligned} \int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1 + t\Lambda(\xi)} &\leq \int_{\frac{3\eta}{2}}^{\frac{3}{2}} \frac{\xi^j d\xi}{1 + t\xi^3} + Ct^{-1} \int_{\frac{3}{2}}^\infty \xi^{j-5} d\xi \\ &\leq Ct^{-\frac{j+1}{3}} \int_{\frac{3\eta}{2}t^{\frac{1}{3}}}^{\frac{3}{2}t^{\frac{1}{3}}} \frac{\xi^j d\xi}{1 + \xi^3} + Ct^{-1} \leq Ct^{-1}. \end{aligned}$$

Similarly

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1 + t\Lambda(\xi))^2} \leq Ct^{-2} \int_{\frac{3\eta}{2}}^\infty \xi^{2j-9} d\xi \leq Ct^{-2} \eta^{2j-8}$$

for $\eta > 1$. For $0 < \eta \leq 1$

$$\begin{aligned} \int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1 + t\Lambda(\xi))^2} &\leq \int_{\frac{3\eta}{2}}^{\frac{3}{2}} \frac{\xi^{2j+1} d\xi}{(1 + t\xi^3)^2} + Ct^{-2} \int_{\frac{3}{2}}^\infty \xi^{2j-9} d\xi \\ &\leq Ct^{-\frac{2j+2}{3}} \int_{\frac{3\eta}{2}t^{\frac{1}{3}}}^{\frac{3}{2}t^{\frac{1}{3}}} \frac{\xi^{2j+1} d\xi}{(1 + \xi^3)^2} + Ct^{-2} \leq Ct^{-2} \log \langle t \rangle + Ct^{-\frac{2j+2}{3}} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{2j-4} \end{aligned}$$

for $j = 0, 1, 2$, and for $j = 3$

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1 + t\Lambda(\xi))^2} \leq Ct^{-2} \int_{\frac{3\eta}{2}}^{\frac{3}{2}} d\xi + Ct^{-2} \int_{\frac{3}{2}}^\infty \xi^{2j-9} d\xi \leq Ct^{-2}.$$

Hence we get

$$\begin{aligned}
& t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta |I_3| \\
& \leq C \|\phi\|_{\mathbf{Z}_1} \langle \eta \rangle^{j-3+\beta} |\eta|^{\frac{1}{2}+\alpha} \left(t^{-\frac{1}{2}} \log \langle t \rangle + t^{\frac{1}{2}-\frac{j+1}{3}} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{j-2} \right) \\
& \leq C \left(t^{-\frac{1}{2}} \log \langle t \rangle + t^{-\frac{j+\alpha}{3}} \right) \|\phi\|_{\mathbf{Z}_1}
\end{aligned}$$

for $j = 0, 1, 2$, and for $j = 3$

$$t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta |I_3| \leq C t^{-\frac{1}{2}} \|\phi\|_{\mathbf{Z}_1} \langle \eta \rangle^\beta |\eta|^{\frac{1}{2}+\alpha} \leq C t^{-\frac{1}{2}} \|\phi\|_{\mathbf{Z}_1}$$

with $j + \beta + \alpha \leq \frac{5}{2}$, $\alpha \geq -\frac{1}{2}$.

Finally we estimate

$$\mathcal{V}_\xi^j \phi = \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) \xi^j d\xi$$

for $\eta < 0$. We integrate by parts via the identity

$$(2.10) \quad e^{-itS(\xi, \eta)} = H_4 \partial_\xi \left(\xi e^{-itS(\xi, \eta)} \right)$$

with $H_4 = (1 - it\xi (\Lambda'(\xi) + \Lambda'(\eta)))^{-1}$ to get

$$\begin{aligned}
\mathcal{V}_\xi^j \phi &= -\sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) \xi \partial_\xi (H_4 \xi^j) d\xi \\
&\quad - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \xi^{j+1} H_4 \phi_\xi(\xi) d\xi.
\end{aligned}$$

Using the estimates $|H_4| \leq C (1 + t\xi (\Lambda'(\xi) + \Lambda'(\eta)))^{-1}$ and

$$|\xi \partial_\xi (H_4 \xi^j)| \leq C \xi^j (1 + t\xi (\Lambda'(\xi) + \Lambda'(\eta)))^{-1}$$

we obtain

$$\begin{aligned}
|\mathcal{V}_\xi^j \phi| &\leq C \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty \frac{\xi^j d\xi}{1 + t\xi (\Lambda'(\xi) + \Lambda'(\eta))} \\
&\quad + C t^{\frac{1}{2}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty \frac{\xi^{j+\frac{1}{2}} |\xi^{\frac{1}{2}} \phi_\xi| d\xi}{1 + t\xi (\Lambda'(\xi) + \Lambda'(\eta))}.
\end{aligned}$$

As above we get

$$\begin{aligned}
& \int_0^\infty \frac{\xi^j d\xi}{1+t\xi(\Lambda'(\xi)+\Lambda'(\eta))} + \left(\int_0^\infty \frac{\xi^{2j+1} d\xi}{(1+t\xi(\Lambda'(\xi)+\Lambda'(\eta)))^2} \right)^{\frac{1}{2}} \\
& \leq \int_0^{|\eta|} \frac{\xi^j d\xi}{1+t\xi\Lambda'(\eta)} + \int_{|\eta|}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} \\
& \quad + \left(\int_0^{|\eta|} \frac{\xi^{2j+1} d\xi}{(1+t\xi\Lambda'(\eta))^2} \right)^{\frac{1}{2}} + \left(\int_{|\eta|}^\infty \frac{\xi^{2j+1} d\xi}{(1+t\Lambda(\xi))^2} \right)^{\frac{1}{2}} \\
& \leq C\eta^{1+j} \langle t\Lambda(\eta) \rangle^{-1} + C \langle \eta \rangle^{j-4} \begin{cases} t^{-1} \log \langle t \rangle + t^{-\frac{j+1}{3}} \langle t^{\frac{1}{3}} \eta \rangle^{j-2}, & j = 0, 1, 2, \\ t^{-1}, & j = 3. \end{cases}
\end{aligned}$$

Then we find

$$\left\| |\eta|^\alpha \langle \eta \rangle^\beta \mathcal{V} \xi^j \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \leq C \left(t^{-\frac{1}{2}} \log \langle t \rangle + t^{-\frac{\alpha+1}{3}} \right) \|\phi\|_{\mathbf{Z}_1}$$

if $\alpha \geq -\frac{1}{2}$, $\alpha + \beta + j \leq \frac{5}{2}$. Lemma 2.2 is proved. $\square$

2.3. Estimates for the operator $\mathcal{Q}$

In this subsection we consider the estimate of the operator

$$\mathcal{Q}\phi = \mathcal{Q}(t)\phi = \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{itS(\xi, \eta)} \phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta$$

in the uniform norm. Note that

$$\|\mathcal{Q}\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq Ct^{\frac{1}{2}} \int_{\mathbf{R}} |\phi(\eta)| |\Lambda''(\eta)|^{\frac{1}{2}} d\eta \leq Ct^{\frac{1}{2}} \left\| |\Lambda''(\eta)|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^1(\mathbf{R})}.$$

In particular we find

$$\begin{aligned}
& \|\mathcal{Q}(1-\theta)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|\mathcal{D}_{-1}\mathcal{Q}(-t)(1-\theta)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
(2.11) \quad & \leq Ct^{\frac{1}{2}} \left\| |\Lambda''(\eta)|^{\frac{1}{2}} \phi \right\|_{\mathbf{L}^1(\mathbf{R}_-)}.
\end{aligned}$$

The next lemma says that the main term of the asymptotics of $\mathcal{Q}\phi$ in the positive half-line is $A^*(t)\phi$.

Lemma 2.3. *Let $\alpha > -2$, $\alpha_1 > -\frac{3}{2}$, $\beta \leq \frac{3}{4} - \alpha$, $\beta_1 < \frac{5}{2} - \alpha_1$. Then the estimate*

$$\begin{aligned}
& \|\mathcal{Q}\theta\phi - \theta A^*(t)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|\mathcal{Q}(-t)\theta\phi\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \\
& \leq C \max \left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
& \quad + C \max \left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{4}} \right) \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}
\end{aligned}$$

is valid for all $t \geq 1$.

Proof. For the case of $\xi \geq 0$, we write

$$\begin{aligned}
& \mathcal{Q}\theta\phi - \theta A^*(t)\phi \\
&= \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} (\phi(\eta) - \phi(\xi)) |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi\eta^{-1}) d\eta \\
&\quad + \sqrt{\frac{t}{2\pi i}} \int_0^{\frac{2}{3}\xi} e^{itS(\xi, \eta)} \phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) d\eta \\
&\quad + \sqrt{\frac{t}{2\pi i}} \int_{\frac{3}{2}\xi}^\infty e^{itS(\xi, \eta)} \phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) d\eta \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

In the first integral I_1 by using the identity (2.3) we get

$$\begin{aligned}
I_1 &= -\sqrt{\frac{t}{2\pi i}} \int_{\frac{\xi}{3}}^{3\xi} e^{itS(\xi, \eta)} (\phi(\eta) - \phi(\xi)) (\eta - \xi) \\
&\quad \times \partial_\eta \left(H_2 |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi\eta^{-1}) \right) d\eta \\
&\quad - \sqrt{\frac{t}{2\pi i}} \int_{\frac{\xi}{3}}^{3\xi} e^{itS(\xi, \eta)} (\eta - \xi) |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi\eta^{-1}) H_2 \phi(\eta) d\eta.
\end{aligned}$$

Then using (2.4) and the estimates

$$|\phi(\eta) - \phi(\xi)| \leq C\xi^\alpha \langle \xi \rangle^\beta |\eta - \xi|^{\frac{1}{2}} \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

and

$$|H_2| \leq C \left(1 + t\xi \langle \xi \rangle^2 (\xi - \eta)^2 \right)^{-1}$$

in the domain $\frac{\xi}{3} \leq \eta \leq 3\xi$, we find

$$\begin{aligned}
|I_1| &\leq Ct^{\frac{1}{2}} \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \xi^{\frac{1}{2}+\alpha} \langle \xi \rangle^{1+\beta} \\
&\quad \times \left(\int_{\frac{\xi}{3}}^{3\xi} \frac{|\eta - \xi|^{\frac{1}{2}} d\eta}{1 + t\xi \langle \xi \rangle^2 (\xi - \eta)^2} + \left(\int_{\frac{\xi}{3}}^{3\xi} \frac{(\eta - \xi)^2 d\eta}{\left(1 + t\xi \langle \xi \rangle^2 (\xi - \eta)^2 \right)^2} \right)^{\frac{1}{2}} \right).
\end{aligned}$$

Changing $\eta = \xi y$ we get

$$\begin{aligned}
\int_{\frac{\xi}{3}}^{3\xi} \frac{|\eta - \xi|^{\frac{1}{2}} d\eta}{1 + t\xi \langle \xi \rangle^2 (\xi - \eta)^2} &\leq C\xi^{\frac{3}{2}} \int_{\frac{1}{3}}^3 \frac{|y - 1|^{\frac{1}{2}} dy}{1 + t\Lambda(\xi)(1 - y)^2} \\
&\leq C\xi^{\frac{3}{2}} \langle t\Lambda(\xi) \rangle^{-\frac{3}{4}}
\end{aligned}$$

and

$$\begin{aligned} \int_{\frac{\xi}{3}}^{3\xi} \frac{(\eta - \xi)^2 d\eta}{\left(1 + t\xi \langle \xi \rangle^2 (\xi - \eta)^2\right)^2} &\leq C\xi^3 \int_{\frac{1}{3}}^3 \frac{(y - 1)^2 dy}{\left(1 + t\Lambda(\xi)(1 - y)^2\right)^2} \\ &\leq C\xi^3 \langle t\Lambda(\xi) \rangle^{-\frac{3}{2}}. \end{aligned}$$

Then using

$$t^{\frac{1}{2}} \xi^{2+\alpha} \langle \xi \rangle^{1+\beta} \langle t\Lambda(\xi) \rangle^{-\frac{3}{4}} \leq Ct^{\frac{1}{2}} \xi^{2+\alpha} \left\langle t^{\frac{1}{3}} \xi \right\rangle^{-\frac{9}{4}} \leq C \left(t^{-\frac{2\alpha+1}{6}} + t^{-\frac{1}{4}} \right)$$

for $0 < \xi < 1$, if $\alpha \geq -2$ and

$$t^{\frac{1}{2}} \xi^{2+\alpha} \langle \xi \rangle^{1+\beta} \langle t\Lambda(\xi) \rangle^{-\frac{3}{4}} \leq Ct^{\frac{1}{2}} \xi^{3+\alpha+\beta} \left\langle t^{\frac{1}{5}} \xi \right\rangle^{-\frac{15}{4}} \leq Ct^{-\frac{1}{4}}$$

for $\xi \geq 1$, if $\beta \leq \frac{3}{4} - \alpha$, we obtain

$$\begin{aligned} |I_1| &\leq C \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} t^{\frac{1}{2}} \xi^{2+\alpha} \langle \xi \rangle^{1+\beta} \langle t\Lambda(\xi) \rangle^{-\frac{3}{4}} \\ &\leq C \max \left(t^{-\frac{2\alpha+1}{6}}, t^{-\frac{1}{4}} \right) \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

In the second integral I_2 using the identity $e^{itS(\xi, \eta)} = H_5 \partial_\eta (\eta e^{itS(\xi, \eta)})$ with $H_5 = (1 + it\eta S_\eta(\xi, \eta))^{-1}$ we integrate by parts

$$\begin{aligned} I_2 &= -\sqrt{\frac{t}{2\pi i}} \int_0^{\frac{2}{3}\xi} e^{itS(\xi, \eta)} \phi(\eta) \eta \partial_\eta \left(H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \right) d\eta \\ &\quad - \sqrt{\frac{t}{2\pi i}} \int_0^{\frac{2}{3}\xi} e^{itS(\xi, \eta)} \eta H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \phi_\eta(\eta) d\eta. \end{aligned}$$

Since $S_\eta(\xi, \eta) = -\Lambda''(\eta)(\xi - \eta)$ for $\eta > 0$, we have the estimates

$$(2.12) \quad \left| \eta H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \right| \leq C\eta^{\frac{3}{2}} \langle \eta \rangle \left(1 + t(\xi - \eta)\eta^2 \langle \eta \rangle^2 \right)^{-1},$$

and

$$(2.13) \quad \left| \eta \partial_\eta \left(H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \right) \right| \leq C\eta^{\frac{1}{2}} \langle \eta \rangle \left(1 + t(\xi - \eta)\eta^2 \langle \eta \rangle^2 \right)^{-1}.$$

Hence in the domain $0 < \eta < \frac{2}{3}\xi$ we get

$$\begin{aligned} |I_2| &\leq Ct^{\frac{1}{2}} \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_0^{\frac{2}{3}\xi} \frac{\eta^{\frac{1}{2}+\alpha_1} \langle \eta \rangle^{1+\beta_1} d\eta}{1 + t\xi\eta^2 \langle \eta \rangle^2} \\ &\quad + Ct^{\frac{1}{2}} \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_0^{\frac{2}{3}\xi} \frac{\eta^{3+2\alpha} \langle \eta \rangle^{2+2\beta} d\eta}{\left(1 + t\xi\eta^2 \langle \eta \rangle^2\right)^2} \right)^{\frac{1}{2}}. \end{aligned}$$

We get

$$\begin{aligned}
& \int_0^{\frac{2}{3}\xi} \frac{\eta^{\frac{1}{2}+\alpha_1} \langle \eta \rangle^{1+\beta_1} d\eta}{1+t\xi\eta^2 \langle \eta \rangle^2} \leq C \int_0^\infty \frac{\eta^{\frac{1}{2}+\alpha_1} \langle \eta \rangle^{1+\beta_1} d\eta}{1+t\Lambda(\eta)} \\
& \leq C \int_0^1 \frac{\eta^{\frac{1}{2}+\alpha_1} d\eta}{1+t\eta^3} + Ct^{-1} \int_1^\infty \eta^{-\frac{7}{2}+\alpha+\beta} d\eta \\
& \leq Ct^{-\frac{1}{2}-\frac{\alpha_1}{3}} \int_0^{t^{\frac{1}{3}}} \frac{\eta^{\frac{1}{2}+\alpha_1} d\eta}{1+\eta^3} + Ct^{-1} \leq C \left(t^{-\frac{1}{2}-\frac{\alpha_1}{3}} + t^{-1} \log \langle t \rangle \right),
\end{aligned}$$

if $\alpha_1 > -\frac{3}{2}$, $\alpha_1 + \beta_1 < \frac{5}{2}$. Similarly

$$\begin{aligned}
& \int_0^{\frac{2}{3}\xi} \frac{\eta^{3+2\alpha} \langle \eta \rangle^{2+2\beta} d\eta}{\left(1+t\xi\eta^2 \langle \eta \rangle^2\right)^2} \leq C \int_0^\infty \frac{\eta^{3+2\alpha} \langle \eta \rangle^{2+2\beta} d\eta}{(1+t\Lambda(\eta))^2} \\
& \leq C \int_0^1 \frac{\eta^{3+2\alpha} d\eta}{1+t^2\eta^6} + Ct^{-2} \int_1^\infty \eta^{-5+2\alpha+2\beta} d\eta \\
& \leq Ct^{-\frac{4}{3}-\frac{2\alpha}{3}} \int_0^{t^{\frac{1}{3}}} \frac{\eta^{3+2\alpha} d\eta}{1+\eta^6} + Ct^{-2} \leq C \left(t^{-\frac{4}{3}-\frac{2\alpha}{3}} + t^{-2} \log \langle t \rangle \right),
\end{aligned}$$

where $\alpha > -2$, $\alpha + \beta < 2$. Therefore

$$\begin{aligned}
|I_2| & \leq C \max \left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
& \quad + C \max \left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

In the third integral I_3 using the identity $e^{itS(\xi,\eta)} = H_5 \partial_\eta (\eta e^{itS(\xi,\eta)})$ with $H_5 = (1 + it\eta S_\eta(\xi, \eta))^{-1}$ we integrate by parts

$$\begin{aligned}
I_3 & = -\sqrt{\frac{t}{2\pi i}} \int_{\frac{3}{2}\xi}^\infty e^{itS(\xi,\eta)} \phi(\eta) \eta \partial_\eta \left(H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \right) d\eta \\
& \quad - \sqrt{\frac{t}{2\pi i}} \int_{\frac{3}{2}\xi}^\infty e^{itS(\xi,\eta)} \eta H_5 |\Lambda''(\eta)|^{\frac{1}{2}} (1 - \chi(\xi\eta^{-1})) \phi_\eta(\eta) d\eta.
\end{aligned}$$

Since $\frac{3}{2}\xi < \eta$ implies $\frac{1}{3}\eta < (\eta - \xi)$, then by (2.12) and (2.13) we get

$$\begin{aligned}
|I_3| & \leq Ct^{\frac{1}{2}} \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_{\frac{3}{2}\xi}^\infty \frac{\eta^{\frac{1}{2}+\alpha_1} \langle \eta \rangle^{1+\beta_1} d\eta}{1+t\Lambda(\eta)} \\
& \quad + Ct^{\frac{1}{2}} \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_{\frac{3}{2}\xi}^\infty \frac{\eta^{3+2\alpha} \langle \eta \rangle^{2+2\beta} d\eta}{(1+t\Lambda(\eta))^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

In the same way as in the proof of the estimate for I_2 we obtain

$$\begin{aligned} |I_3| \leq & C \max \left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & + C \max \left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} . \end{aligned}$$

Next consider $\mathcal{Q}(-t) \theta \phi$ for $\xi < 0$. Using the identity

$$e^{-itS(\xi, \eta)} = H_6 \partial_\eta \left(\eta e^{-itS(\xi, \eta)} \right)$$

with $H_6 = (1 - it\eta S_\eta(\xi, \eta))^{-1}$, we integrate by parts

$$\begin{aligned} \mathcal{Q}(-t) \theta \phi &= -\sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\eta) \eta \partial_\eta \left(H_6 |\Lambda''(\eta)|^{\frac{1}{2}} \right) d\eta \\ &\quad - \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \eta H_6 |\Lambda''(\eta)|^{\frac{1}{2}} \phi_\eta(\eta) d\eta . \end{aligned}$$

Since $S_\eta(\xi, \eta) = -\Lambda''(\eta)(\xi - \eta)$, $\xi < 0$, $\eta > 0$, then we have the estimates

$$\left| \eta H_6 |\Lambda''(\eta)|^{\frac{1}{2}} \right| \leq C \eta^{\frac{3}{2}} \langle \eta \rangle (1 + t\Lambda(\eta))^{-1} ,$$

and

$$\left| \eta \partial_\eta \left(H_6 |\Lambda''(\eta)|^{\frac{1}{2}} \right) \right| \leq C \eta^{\frac{1}{2}} \langle \eta \rangle (1 + t\Lambda(\eta))^{-1} .$$

Hence we get

$$\begin{aligned} |\mathcal{Q}(-t) \theta \phi| \leq & C t^{\frac{1}{2}} \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R})} \int_0^\infty \frac{|\eta|^{\frac{1}{2}+\alpha_1} \langle \eta \rangle^{1+\beta_1} d\eta}{1 + t\Lambda(\eta)} \\ & + C t^{\frac{1}{2}} \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \left(\int_0^\infty \frac{|\eta|^{3+2\alpha} \langle \eta \rangle^{2+2\beta} d\eta}{(1 + t\Lambda(\eta))^2} \right)^{\frac{1}{2}} . \end{aligned}$$

Then in the same way as in the proof of the estimate for I_2 we obtain

$$\begin{aligned} |\mathcal{Q}(-t) \theta \phi| \leq & C \max \left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha_1} \langle \eta \rangle^{-\beta_1} \phi \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & + C \max \left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}} \log \langle t \rangle \right) \left\| \eta^{-\alpha} \langle \eta \rangle^{-\beta} \partial_\eta \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \end{aligned}$$

for all $\xi \leq 0$. Lemma 2.3 is proved. $\square$

§3. Estimates in the L^2 - norm

3.1. Estimate for derivatives of $\mathcal{Q}$

First we note that

$$\begin{aligned}
 \|\mathcal{Q}\phi\|_{L^2(\mathbf{R}_+)} &= Ct^{\frac{1}{2}} \left\| \int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)\xi} M\phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta \right\|_{L^2(\mathbf{R}_+)} \\
 &= Ct^{\frac{1}{2}} \left\| \int_{\mathbf{R}} e^{-itx\xi} M\phi(\eta(x)) |\Lambda''(\eta(x))|^{-\frac{1}{2}} dx \right\|_{L^2(\mathbf{R}_+)} \\
 (3.1) \quad &\leq C \int_{\mathbf{R}} |\phi(\eta(x))|^2 |\Lambda''(\eta(x))|^{-1} dx = C \|\phi\|_{L^2(\mathbf{R})}.
 \end{aligned}$$

Also we have the relation

$$i\xi\mathcal{Q}\phi = \mathcal{Q}\mathcal{A}\phi,$$

where

$$\mathcal{A} = \frac{\overline{M}}{t|\Lambda''(\eta)|^{\frac{1}{2}}} \partial_{\eta} \frac{M}{|\Lambda''(\eta)|^{\frac{1}{2}}} = \mathcal{A}_0 + i\eta\theta(\eta)$$

with $\mathcal{A}_0 = \frac{1}{t|\Lambda''(\eta)|^{\frac{1}{2}}} \partial_{\eta} \frac{1}{|\Lambda''(\eta)|^{\frac{1}{2}}}$. In the next lemma we obtain the estimates through the operator $\mathcal{A}_0$.

Lemma 3.1. *The estimates*

$$\left\| \langle \xi \rangle^{-4} t \partial_t \mathcal{Q}(t) \phi \right\|_{L^2(\mathbf{R}_+)} \leq C \sum_{l=0}^4 \left\| t \mathcal{A}_0 \eta^l \phi \right\|_{L^2(\mathbf{R}_+)} + Ct \left\| \Lambda' \phi \right\|_{L^2(\mathbf{R}_-)}.$$

and

$$\begin{aligned}
 \left\| \langle \xi \rangle^{-4} t \partial_t \mathcal{Q}(t) \phi \right\|_{L^2(\mathbf{R}_+)} &\leq \left\| t \mathcal{A}_0^2 \phi \right\|_{L^2(\mathbf{R}_+)} + \left\| \eta t \mathcal{A}_0 \phi \right\|_{L^2(\mathbf{R}_+)} \\
 &\quad + \sum_{l=1}^4 \left\| t \mathcal{A}_0 \eta^l \phi \right\|_{L^2(\mathbf{R}_+)} + Ct \left\| \Lambda' \phi \right\|_{L^2(\mathbf{R}_-)}
 \end{aligned}$$

are true.

Proof. By a direct calculation we find

$$\begin{aligned}
 t \partial_t \mathcal{Q} \phi &= (it\Lambda(\xi) \mathcal{Q} \phi - it\mathcal{Q}\Lambda\theta\phi) \\
 &\quad + (it\mathcal{Q}\eta\Lambda'\theta\phi - it\xi\mathcal{Q}\Lambda'\theta\phi) + it\xi\mathcal{Q}(1-\theta)\Lambda'\phi.
 \end{aligned}$$

Applying the commutator

$$i\xi\mathcal{Q}\phi - \mathcal{Q}i\eta\theta\phi = \mathcal{Q}(\mathcal{A} - i\eta\theta)\phi = \mathcal{Q}\mathcal{A}_0\phi$$

we obtain

$$it\mathcal{Q}\eta\Lambda'\theta\phi - it\xi\mathcal{Q}\Lambda'\theta\phi = -\mathcal{Q}t\mathcal{A}_0\Lambda'\theta\phi.$$

By the relation $\mathcal{A} = \mathcal{A}_0 + i\eta\theta(\eta)$ we get

$$(i\xi)^n \mathcal{Q}\phi - \mathcal{Q}(i\eta)^n \theta\phi = \sum_{l=0}^{n-1} (i\xi)^l \mathcal{Q}\mathcal{A}_0 (i\eta)^{n-1-l} \theta\phi.$$

Therefore

$$\begin{aligned} it\Lambda(\xi) \mathcal{Q}\phi - it\mathcal{Q}\Lambda\theta\phi &= it\frac{a}{3} \sum_{l=0}^2 (i\xi)^l \mathcal{Q}\mathcal{A}_0 (i\eta)^{2-l} \theta\phi \\ &+ it\frac{b}{5} \sum_{l=0}^4 (i\xi)^l \mathcal{Q}\mathcal{A}_0 (i\eta)^{4-l} \theta\phi. \end{aligned}$$

Then by inequality (3.1) we obtain the first estimate of the lemma. The second estimate is proved in the same way. Lemma 3.1 is proved. $\square$

3.2. Auxiliary estimates in $\mathbf{L}^2$

In the next two lemmas we estimate some integrals to prove the estimate of the derivative of $\mathcal{V}\phi$. First, we consider the integral

$$\mathcal{I}_1\phi = |\Lambda''(\eta)|^{\frac{1}{2}} t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \phi(\xi) \psi(\xi, \eta) d\xi.$$

Lemma 3.2. *Let $\beta > \alpha > 0$, $\delta = 0$ or $\beta > \alpha = 0$, $\delta > 0$. Suppose that*

$$\left| |\eta|^{-\alpha} \langle \eta \rangle^\beta \langle \xi \rangle^\delta (\eta \partial_\eta)^k \psi(\xi, \eta) \right| \leq C$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then the estimate

$$\|\mathcal{I}_1\phi\|_{\mathbf{L}^2(\mathbf{R})} \leq C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 & \text{for } \alpha > 0 \\ \log \langle t \rangle & \text{for } \alpha = 0 \end{cases}$$

is true for all $t \geq 1$.

Proof. We have

$$\begin{aligned}
\|\mathcal{I}_1 \phi\|_{\mathbf{L}^2(\mathbf{R})}^2 &= Ct \int_{\mathbf{R}} d\eta |\Lambda''(\eta)| \int_0^\infty e^{itS(\xi, \eta)} \overline{\phi(\xi)} \psi(\xi, \eta) d\xi \\
&\times \int_0^\infty e^{-itS(\zeta, \eta)} \phi(\zeta) \psi(\zeta, \eta) d\zeta \\
&= Ct \int_0^\infty e^{it\Lambda(\xi)} \overline{\phi(\xi)} d\xi \int_0^\infty e^{-it\Lambda(\zeta)} \phi(\zeta) d\zeta \\
&\times \int_{\mathbf{R}} d\eta |\Lambda''(\eta)| \overline{\psi(\xi, \eta)} \psi(\zeta, \eta) e^{-it\frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \zeta)} \\
&= C \int_0^\infty d\xi e^{it\Lambda(\xi)} \overline{\phi(\xi)} \int_0^\infty d\zeta e^{-it\Lambda(\zeta)} \phi(\zeta) K(t, \xi, \zeta),
\end{aligned}$$

where

$$K(t, \xi, \zeta) = t \int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \zeta)} |\Lambda''(\eta)| \overline{\psi(\xi, \eta)} \psi(\zeta, \eta) d\eta.$$

We integrate two times by parts via the identity

$$e^{-it\frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \zeta)} = H_6 \partial_\eta \left(\eta e^{-it\frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \zeta)} \right)$$

with $H_6 = (1 - it\eta |\Lambda''(\eta)| (\xi - \zeta))^{-1}$ to get

$$K(t, \xi, \zeta) = t \int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|} \Lambda'(\eta)(\xi - \zeta)} \eta \partial_\eta \left(H_6 \eta \partial_\eta \left(H_6 |\Lambda''(\eta)| \overline{\psi(\xi, \eta)} \psi(\zeta, \eta) \right) \right) d\eta.$$

Then using the estimate

$$\left| \eta \partial_\eta \left(H_6 \eta \partial_\eta \left(H_6 |\Lambda''(\eta)| \overline{\psi(\xi, \eta)} \psi(\zeta, \eta) \right) \right) \right| \leq \frac{C |\eta|^{1+2\alpha} \langle \eta \rangle^{2-2\beta} \langle \xi \rangle^{-\delta} \langle \zeta \rangle^{-\delta}}{(1 + t |\Lambda'(\eta)| |\xi - \zeta|)^2},$$

we get

$$\begin{aligned}
|K(t, \xi, \zeta)| &\leq Ct \int_{\mathbf{R}} \left| \eta \partial_\eta \left(H_6 \eta \partial_\eta \left(H_6 |\Lambda''(\eta)| \overline{\psi(\xi, \eta)} \psi(\zeta, \eta) \right) \right) \right| d\eta \\
&\leq Ct \langle \xi - \zeta \rangle^{-\delta} \left(\int_0^1 \frac{\eta^{1+2\alpha} d\eta}{(1 + t\eta^2 |\xi - \zeta|)^2} + \int_1^\infty \frac{\eta^{3+2\alpha-2\beta} d\eta}{(1 + t\eta^4 |\xi - \zeta|)^2} \right) \\
&\leq Ct \langle \xi - \zeta \rangle^{-\delta} |(\xi - \zeta) t|^{-1-\alpha} \int_0^{(t|\xi-\zeta|)^{\frac{1}{2}}} \frac{y^{1+2\alpha} dy}{(1 + y^2)^2} \\
&\quad + Ct |(\xi - \zeta) t|^{\frac{\beta-\alpha}{2}-1} \int_{(t|\xi-\zeta|)^{\frac{1}{4}}}^\infty \frac{y^{3+2\alpha-2\beta} dy}{(1 + y^4)^2} \\
&\leq Ct \langle \xi - \zeta \rangle^{-\delta} \langle (\xi - \zeta) t \rangle^{-1-\alpha} \\
&\quad + Ct \left(|(\xi - \zeta) t|^{\frac{\beta-\alpha}{2}-1} + 1 \right) \langle (\xi - \zeta) t \rangle^{-\frac{\beta-\alpha}{2}-1}
\end{aligned}$$

if $\beta > \alpha \geq 0$. Then by the Young inequality for convolutions we obtain

$$\begin{aligned}
& \|\mathcal{I}_1 \phi\|_{\mathbf{L}^2(\mathbf{R})}^2 \leq C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty |K(t, \xi, \zeta)| |\phi(\zeta)| d\zeta \right\|_{\mathbf{L}^2} \\
& \leq C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty t \langle \xi - \zeta \rangle^{-\delta} \langle (\xi - \zeta)t \rangle^{-1-\alpha} |\phi(\zeta)| d\zeta \right\|_{\mathbf{L}^2} \\
& + C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty t^{\frac{\beta-\alpha}{2}} |\xi - \zeta|^{\frac{\beta-\alpha}{2}-1} \left(|(\xi - \zeta)t|^{\frac{\beta-\alpha}{2}-1} + 1 \right) |\phi(\zeta)| d\zeta \right\|_{\mathbf{L}^2} \\
& \leq C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}^2 \left(\left\| t \langle \xi t \rangle^{-1-\alpha} \langle \xi \rangle^{-\delta} \right\|_{\mathbf{L}^1} + \left\| t \left(|\xi t|^{\frac{\beta-\alpha}{2}-1} + 1 \right) \langle \xi t \rangle^{-\frac{\beta-\alpha}{2}-1} \right\|_{\mathbf{L}^1} \right) \\
& \leq C \|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)}^2 \begin{cases} 1 & \text{for } \alpha > 0 \\ \log \langle t \rangle & \text{for } \alpha = 0 \end{cases},
\end{aligned}$$

since in the case of $\alpha > 0$, $\delta = 0$ we have

$$\left\| t \langle \xi t \rangle^{-1-\alpha} \langle \xi \rangle^{-\delta} \right\|_{\mathbf{L}^1} \leq C \int_0^t \frac{d\xi}{(1+\xi)^{1+\alpha}} + C \int_1^\infty \xi^{-1-\alpha} d\xi \leq C,$$

and in the case of $\alpha = 0$, $\delta > 0$ we find

$$\left\| t \langle \xi t \rangle^{-1-\alpha} \langle \xi \rangle^{-\delta} \right\|_{\mathbf{L}^1} \leq C \int_0^t \frac{d\xi}{1+\xi} + C \int_1^\infty \xi^{-1-\delta} d\xi \leq C \log \langle t \rangle.$$

Lemma 3.2 is proved. $\square$

In the next lemma we prove another estimate of the integral

$$\mathcal{I}_2 \phi = t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) \psi(\xi, \eta) d\xi$$

in $\mathbf{L}^2$.

Lemma 3.3. *Let $k = 0, 1, 2$, $k - 1 \leq \alpha < k + 1 - \beta$. Suppose that*

$$|\psi(\xi, \eta)| + |(\xi - \eta\theta(\eta)) \partial_\xi \psi(\xi, \eta)| \leq |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta$$

for $\xi > 0$, $\eta \in \mathbf{R}$. Suppose that $\phi(0) = 0$. Then the estimate

$$\|\mathcal{I}_2 \phi\|_{\mathbf{L}^2(\mathbf{R})} \leq C \max \left(1, t^{-\frac{1}{6} - \frac{\alpha-k}{3}} \right) \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

is true for all $t \geq 1$.

Proof. Consider $\eta > 0$. Using identity (2.1) we integrate by parts

$$\begin{aligned}
\mathcal{I}_2 \phi &= -t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) (\xi - \eta) \partial_\xi (H_1 \psi(\xi, \eta)) d\xi \\
&\quad - t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} (\xi - \eta) H_1 \psi(\xi, \eta) \phi_\xi(\xi) d\xi.
\end{aligned}$$

By the estimates $|H_1| \leq C \left(1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)\right)^{-1}$, and

$$|(\xi - \eta) \partial_\xi (H_1 \psi(\xi, \eta))| \leq \frac{C |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)}$$

we obtain

$$\begin{aligned} |\mathcal{I}_2 \phi| &\leq C t^{\frac{1}{2}} \int_0^\infty \frac{|\phi(\xi) - \phi(0)| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)} \\ &+ C t^{\frac{1}{2}} \int_0^\infty \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta |\partial_\xi \phi| d\xi}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)}. \end{aligned}$$

Since $|\phi(\xi) - \phi(0)| \leq C \xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ we get

$$\begin{aligned} |\mathcal{I}_2 \phi| &\leq C t^{\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \int_0^\infty \frac{|\xi + |\eta||^{-k} \xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)} \\ (3.2) \quad &+ C t^{\frac{1}{2}} \int_0^\infty \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta |\partial_\xi \phi| d\xi}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)}. \end{aligned}$$

We estimate the first summand in (3.2) as

$$\begin{aligned} &\int_0^\infty \frac{|\xi + |\eta||^{-k} \xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t(\xi - \eta)^2(\eta + \xi)(1 + \eta^2 + \xi^2)} \\ (3.3) \quad &\leq C \int_0^{2\eta} \frac{|\eta|^{\alpha-k+\frac{1}{2}} \langle \eta \rangle^\beta d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} + C |\eta|^\alpha \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)}. \end{aligned}$$

Then we find

$$\begin{aligned} &\int_0^{2\eta} \frac{|\eta|^{\alpha-k+\frac{1}{2}} \langle \eta \rangle^\beta d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \leq C \int_0^2 \frac{|\eta|^{\alpha-k+\frac{3}{2}} \langle \eta \rangle^\beta dy}{1 + t\Lambda(\eta)(y-1)^2} \\ &\leq C |\eta|^{\alpha-k+\frac{3}{2}} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} &t^{\frac{1}{2}} \left\| |\eta|^{\alpha-k+\frac{3}{2}} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t^{\frac{1}{2}} \left(\int_0^1 \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \eta^{2\alpha-2k+3} d\eta \right)^{\frac{1}{2}} \\ (3.4) \quad &+ C \left(\int_1^\infty \eta^{2\alpha+2\beta-2k-2} d\eta \right)^{\frac{1}{2}} \leq C \max \left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}} \right) \end{aligned}$$

if $\alpha > k - 2$, $\alpha + \beta - k < 1$. Also the second term of the right-hand side of (3.3) is estimated as

$$|\eta|^\alpha \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \leq C t^{-1} \eta^{\alpha+\beta} \int_\eta^\infty \xi^{-k-5} d\xi \leq C t^{-1} \langle \eta \rangle^{\alpha+\beta-k-4}$$

for $\eta > 1$, $t \geq 1$, and

$$\begin{aligned}
& |\eta|^\alpha \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)} \leq C \int_\eta^1 \frac{\xi^{\alpha-k+\frac{1}{2}} d\xi}{1+t\xi^3} + Ct^{-1} \int_1^\infty \xi^{-k-5} d\xi \\
& \leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \int_{\eta t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha-k+\frac{1}{2}} d\xi}{1+\xi^3} + Ct^{-1} \\
& \leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \max \left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{\alpha-k-\frac{3}{2}} \right) + Ct^{-1} \log t
\end{aligned}$$

for $0 < \eta \leq 1$. Hence

$$\begin{aligned}
& \left\| |\eta|^\alpha \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \left(\int_0^1 \max \left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{2\alpha-2k-3} \right) d\eta \right)^{\frac{1}{2}} + Ct^{-1} \log t \\
(3.5) \quad & + Ct^{-1} \left(\int_1^\infty \langle \eta \rangle^{2\alpha+2\beta-2k-8} d\eta \right)^{\frac{1}{2}} \leq C \max \left(t^{-1} \log t, t^{-\frac{2}{3}-\frac{\alpha-k}{3}} \right)
\end{aligned}$$

for $\alpha + \beta - k < \frac{7}{2}$. Thus by (3.3)-(3.5)

$$(3.6) \quad t^{\frac{1}{2}} \left\| \int_0^\infty \frac{|\xi + |\eta||^{-k} \xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \max \left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}} \right)$$

if $\alpha > k-2$, $\alpha + \beta - k < 1$.

Consider the second summand in (3.2) for $0 < \eta < 1$

$$\begin{aligned}
& \int_0^\infty \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta |\partial_\xi \phi| d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \\
& \leq C \int_0^{2\eta} \frac{|\xi - \eta|^{\alpha+1-k} |\partial_\xi \phi| d\xi}{1+t|\eta-\xi|^3} \\
(3.7) \quad & + C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_{2\eta}^\infty \frac{\xi^{2\alpha+2-2k} d\xi}{(1+t\Lambda(\xi))^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

if $\alpha + 1 - k \geq 0$. By the Young inequality

$$\begin{aligned}
& \left\| \int_0^{2\eta} \frac{|\xi - \eta|^{\alpha+1-k} |\partial_\xi \phi| d\xi}{1+t|\eta-\xi|^3} \right\|_{\mathbf{L}^2} \\
& \leq C \|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| |\eta|^{\alpha+1-k} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \right\|_{\mathbf{L}^1(\mathbf{R})} \\
(3.8) \quad & \leq C \max \left(t^{-1} \log t, t^{-\frac{\alpha-k+2}{3}} \right) \|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

Let us consider the second term of the right hand side of (3.7). We have

$$\begin{aligned}
\int_{2\eta}^{\infty} \frac{\xi^{2\alpha+2-2k} d\xi}{(1+t\Lambda(\xi))^2} &\leq \int_{2\eta}^2 \frac{\xi^{2\alpha+2-2k} d\xi}{(1+t\xi^3)^2} + Ct^{-2} \int_2^{\infty} \xi^{2\alpha-2k-8} d\xi \\
&\leq Ct^{-\frac{2\alpha-2k+3}{3}} \int_{2\eta t^{\frac{1}{3}}}^{2t^{\frac{1}{3}}} \frac{\xi^{2\alpha+2-2k} d\xi}{(1+\xi^3)^2} + Ct^{-2} \\
&\leq Ct^{-1-\frac{2\alpha-2k}{3}} \max\left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{2\alpha-2k-3}\right) + Ct^{-2}
\end{aligned}$$

if $\alpha - k < \frac{7}{2}$ and then

$$\begin{aligned}
&\left\| \left(\int_{2\eta}^{\infty} \frac{\xi^{2\alpha+2-2k} d\xi}{(1+t\Lambda(\xi))^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(0,1)} \\
&\leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \left(\int_0^1 \max\left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{2\alpha-2k-3}\right) d\eta \right)^{\frac{1}{2}} + Ct^{-1} \\
(3.9) \quad &\leq C \max\left(t^{-\frac{1}{2}}, t^{-\frac{2}{3}-\frac{\alpha-k}{3}}\right).
\end{aligned}$$

By (3.7)-(3.9)

$$\begin{aligned}
&t^{\frac{1}{2}} \left\| \int_0^{\infty} \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta} |\partial_{\xi} \phi| d\xi}{1+t(\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)} \right\|_{\mathbf{L}^2(0,1)} \\
(3.10) \quad &\leq C \|\phi_{\xi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \max\left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right).
\end{aligned}$$

For $\eta > 1$ we have

$$\begin{aligned}
&\int_0^{\infty} \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta} |\partial_{\xi} \phi| d\xi}{1+t(\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)} \\
&\leq C\eta^{\alpha+\beta-k} \int_0^{2\eta} \frac{|\xi - \eta| |\partial_{\xi} \phi| d\xi}{1+t\eta^3 (\xi - \eta)^2} + Ct^{-1} \eta^{\alpha+\beta} \int_{\eta}^{\infty} \xi^{-k-4} |\partial_{\xi} \phi| d\xi \\
&\leq C \|\phi_{\xi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \eta^{\alpha+\beta-k} \left(\int_0^{2\eta} \frac{(\xi - \eta)^2 d\xi}{(1+t\eta^3 (\xi - \eta)^2)^2} \right)^{\frac{1}{2}} \\
&\quad + Ct^{-1} \|\phi_{\xi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \eta^{\alpha+\beta} \left(\int_{\eta}^{\infty} \xi^{-2k-8} d\xi \right)^{\frac{1}{2}} \\
&\leq C \|\phi_{\xi}\|_{\mathbf{L}^2(\mathbf{R}_+)} t^{-\frac{3}{4}} \eta^{\alpha+\beta-k-\frac{9}{4}},
\end{aligned}$$

and then

$$(3.11) \quad \left\| \int_0^\infty \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta |\partial_\xi \phi| d\xi}{1 + t (\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)} \right\|_{\mathbf{L}^2(1, \infty)} \\ \leq C t^{-\frac{3}{4}} \|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \eta^{\alpha + \beta - k - \frac{9}{4}} \right\|_{\mathbf{L}^2(1, \infty)} \leq C t^{-\frac{3}{4}} \|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

if $\alpha + \beta - k < \frac{7}{4}$. Therefore by (3.10) and (3.11)

$$(3.12) \quad t^{\frac{1}{2}} \left\| \int_0^\infty \frac{|\xi - \eta| |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta |\partial_\xi \phi| d\xi}{1 + t (\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq C \max \left(1, t^{-\frac{1}{6} - \frac{\alpha - k}{3}} \right) \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

By (3.6) and (3.12)

$$\|\mathcal{I}_2 \phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \max \left(1, t^{-\frac{1}{6} - \frac{\alpha - k}{3}} \right) \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

To estimate $\mathcal{I}_2 \phi$ for $\eta < 0$, we integrate by parts via the identity (2.10)

$$\mathcal{I}_2 \phi = -t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} (\phi(\xi) - \phi(0)) \xi \partial_\xi (H_4 \psi(\xi, \eta)) d\xi \\ - t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \xi H_4 \psi(\xi, \eta) \phi_\xi(\xi) d\xi.$$

Using the estimates

$$|H_4| \leq C \left(1 + t\xi \left(\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2 \right) \right)^{-1}, |\psi(\xi, \eta)| \leq C |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta$$

and

$$|\xi \partial_\xi (H_4 \psi(\xi, \eta))| \leq C |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta \left(1 + t\xi \left(\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2 \right) \right)^{-1},$$

we obtain

$$\begin{aligned}
|\mathcal{I}_2\phi| &\leq Ct^{\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2} \int_0^\infty \frac{\xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t\xi (\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2)} \\
&\quad + Ct^{\frac{1}{2}} \int_0^\infty \frac{|\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta \phi_\xi(\xi) \xi d\xi}{1 + t\xi (\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2)} \\
&\leq Ct^{\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2} \left(\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k} \langle \eta \rangle^\beta d\xi}{1 + t\xi \eta^2 \langle \eta \rangle^2} + \int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle \xi \rangle^{\beta-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \right) \\
&\quad + Ct^{\frac{1}{2}} \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2} \\
(3.13) \quad &\times \left(\left(\int_0^{|\eta|} \frac{|\eta|^{2\alpha-2k} \langle \eta \rangle^{2\beta} \xi^2 d\xi}{(1 + t\eta^2 \langle \eta \rangle^2 \xi)^2} \right)^{\frac{1}{2}} + \left(\int_{|\eta|}^\infty \frac{\xi^{2\alpha-2k+2} \langle \xi \rangle^{2\beta} d\xi}{(1 + t\Lambda(\xi))^2} \right)^{\frac{1}{2}} \right).
\end{aligned}$$

For $|\eta| < 1$

$$\begin{aligned}
&\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k} \langle \eta \rangle^\beta d\xi}{1 + t\xi \eta^2 \langle \eta \rangle^2} \leq C \int_0^1 \frac{|\eta|^{\alpha+\frac{3}{2}-k} \langle \eta \rangle^\beta dy}{1 + t\Lambda(\eta)y} \\
&\leq C |\eta|^{\alpha+\frac{3}{2}-k} \int_0^1 \frac{dy}{1 + t|\eta|^3 y} \leq C |\eta|^{\alpha+\frac{3}{2}-k} \left\langle t|\eta|^3 \right\rangle^{-\frac{3}{4}}
\end{aligned}$$

and for $|\eta| \geq 1$

$$\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k} \langle \eta \rangle^\beta d\xi}{1 + t\xi \eta^2 \langle \eta \rangle^2} \leq C |\eta|^{\alpha+\beta+\frac{3}{2}-k} \int_0^1 \frac{dy}{1 + t|\eta|^5 y} \leq Ct^{-\frac{3}{4}} |\eta|^{\alpha+\beta-k-\frac{9}{4}}.$$

Therefore

$$\begin{aligned}
&\left\| \int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k} \langle \eta \rangle^\beta d\xi}{1 + t\xi \eta^2 \langle \eta \rangle^2} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C \left(\int_0^1 |\eta|^{2\alpha+3-2k} \left\langle t^{\frac{1}{3}} |\eta| \right\rangle^{-\frac{9}{2}} d|\eta| \right)^{\frac{1}{2}} \\
(3.14) \quad &+ Ct^{-\frac{3}{4}} \left(\int_1^\infty |\eta|^{2\alpha+2\beta-2k-\frac{9}{2}} d|\eta| \right)^{\frac{1}{2}} \leq C \max \left(t^{-\frac{3}{4}} \log t, t^{-\frac{\alpha-k+2}{3}} \right)
\end{aligned}$$

if $\alpha + \beta - k < \frac{5}{2}$. In the same manner

$$\begin{aligned}
& \left\| \left(\int_0^{|\eta|} \frac{|\eta|^{2\alpha-2k} \langle \eta \rangle^{2\beta} \xi^2 d\xi}{(1 + t\eta^2 \langle \eta \rangle^2 \xi)^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
& \leq \left\| \left(\int_0^1 \frac{|\eta|^{2\alpha-2k+3} \langle \eta \rangle^{2\beta} y^2 dy}{(1 + t\Lambda(\eta)y)^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
& \leq C \left(\int_0^1 |\eta|^{2\alpha-2k+3} \left\langle t|\eta|^3 \right\rangle^{-2} d|\eta| \right)^{\frac{1}{2}} \\
& \quad + Ct^{-1} \left(\int_1^\infty \eta^{2\alpha+2\beta-2k-7} d|\eta| \right)^{\frac{1}{2}} \\
(3.15) \quad & \leq C \max \left(t^{-1} \log t, t^{-\frac{\alpha-k+2}{3}} \right).
\end{aligned}$$

We have

$$\begin{aligned}
& \int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle \xi \rangle^{\beta-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \\
& \leq Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \int_{|\eta|t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha+\frac{1}{2}-k} d\xi}{1 + \xi^3} + Ct^{-1} \int_1^\infty \xi^{\alpha+\beta-k-5} d\xi \\
& \leq Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \min \left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{\alpha-k-\frac{3}{2}} \right) + Ct^{-1}
\end{aligned}$$

for $|\eta| < 1$ if $\alpha - k > -\frac{3}{2}$, $\alpha + \beta - k < 4$. And

$$\int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle \xi \rangle^{\beta-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \leq Ct^{-1} \int_{|\eta|}^\infty \xi^{\alpha+\beta-k-5} d\xi \leq Ct^{-1} |\eta|^{\alpha+\beta-k-4}$$

for $|\eta| \geq 1$. Therefore

$$\begin{aligned}
& \left\| \int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle \xi \rangle^{\beta-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \left(\int_0^1 \min \left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{2\alpha-2k-3} \right) d\eta \right)^{\frac{1}{2}} \\
& \quad + Ct^{-1} \left(\int_0^1 d\eta \right)^{\frac{1}{2}} + Ct^{-1} \left(\int_1^\infty |\eta|^{2\alpha+2\beta-2k-8} d\eta \right)^{\frac{1}{2}} \\
(3.16) \quad & \leq C \min \left(t^{-1}, t^{-\frac{\alpha-k+2}{3}} \right)
\end{aligned}$$

if $\alpha + \beta - k < \frac{7}{2}$. Similarly,

$$\begin{aligned}
 & \left\| \left(\int_{|\eta|}^{\infty} \frac{\xi^{2\alpha-2k+2} \langle \xi \rangle^{2\beta} d\xi}{(1+t\Lambda(\xi))^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
 & \leq C t^{-\frac{\alpha-k}{3}-\frac{1}{2}} \left(\int_0^1 \min \left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{2\alpha-2k-3} \right) d\eta \right)^{\frac{1}{2}} \\
 & \quad + C t^{-1} \left(\int_1^{\infty} d\eta \right)^{\frac{1}{2}} + C t^{-1} \left(\int_1^{\infty} \eta^{2\alpha+2\beta-2k-9} d\eta \right)^{\frac{1}{2}} \\
 (3.17) \quad & \leq C \min \left(t^{-1}, t^{-\frac{\alpha-k+2}{3}} \right).
 \end{aligned}$$

Thus we get by (3.13)-(3.17)

$$\|\mathcal{I}_2 \phi\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C \min \left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}} \right) \|\langle \xi \rangle \phi_{\xi}\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Lemma 3.3 is proved. $\square$

Next we estimate

$$\mathcal{I}_3 = t^{\frac{1}{2}} \int_0^{\infty} e^{-itS(\xi,\eta)} \psi(\xi, \eta) d\xi$$

in $\mathbf{L}^{\infty}$.

Lemma 3.4. *Let $k = 0, 1, 2$, $k-1 \leq \alpha < k+4$, $\alpha + \beta < 1+k$. Suppose that*

$$|\psi(\xi, \eta)| + |(\xi - \eta \theta(\eta)) \partial_{\xi} \psi(\xi, \eta)| \leq |\xi + |\eta||^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta}$$

for $\xi > 0$, $\eta \in \mathbf{R}$. Then the estimate

$$\|\mathcal{I}_3\|_{\mathbf{L}^2(\mathbf{R})} \leq C \max \left(1, t^{-\frac{\alpha-k}{3}} \right)$$

is true for all $t \geq 1$.

Proof. Consider $\eta > 0$. Using identity (2.1) we integrate by parts

$$\begin{aligned}
 \mathcal{I}_3 &= -t^{\frac{1}{2}} \int_0^{\infty} e^{-itS(\xi,\eta)} (\xi - \eta) \partial_{\xi} (H_1 \psi(\xi, \eta)) d\xi \\
 &\quad + O \left(t^{\frac{1}{2}} \eta \psi(0, \eta) \langle t\Lambda(\eta) \rangle^{-1} \right).
 \end{aligned}$$

Using the estimate

$$|(\xi - \eta) \partial_{\xi} (H_1 \psi(\xi, \eta))| \leq \frac{C |\xi + |\eta||^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta}}{1 + t(\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)}$$

we obtain

$$(3.18) \quad \begin{aligned} |\mathcal{I}_3| &\leq C t^{\frac{1}{2}} |\eta|^{\alpha+1-k} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-1} \\ &+ C t^{\frac{1}{2}} \int_0^\infty \frac{|\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t(\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)}. \end{aligned}$$

We estimate the first summand as

$$(3.19) \quad \begin{aligned} &\left\| t^{\frac{1}{2}} |\eta|^{\alpha+1-k} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-1} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\leq C t^{\frac{1}{2}} \left\| \eta^{\alpha+1-k} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \right\|_{\mathbf{L}^2(0,1)} \\ &+ C t^{-\frac{1}{2}} \left\| \eta^{\alpha+\beta-k-4} \right\|_{\mathbf{L}^2(1,\infty)} \leq C \max \left(t^{-\frac{\alpha-k}{3}}, t^{-\frac{1}{2}} \right) \end{aligned}$$

if $\alpha \geq k-1$, $\alpha + \beta - k < \frac{7}{2}$. For the second summand we get

$$\begin{aligned} &\int_0^\infty \frac{|\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1 + t(\xi - \eta)^2 (\eta + \xi) (1 + \eta^2 + \xi^2)} \\ &\leq C \int_0^{2\eta} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} + C |\eta|^{-\frac{1}{3}} \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1 + t\Lambda(\xi)}. \end{aligned}$$

Then we find

$$\begin{aligned} &\int_0^{2\eta} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1 + t\eta \langle \eta \rangle^2 (\xi - \eta)^2} \leq C \int_0^2 \frac{\eta^{\alpha+1-k} \langle \eta \rangle^\beta dy}{1 + t\Lambda(\eta) (y-1)^2} \\ &\leq C \eta^{\alpha+1-k} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} &\left\| \eta^{\alpha+1-k} \langle \eta \rangle^\beta \langle t\Lambda(\eta) \rangle^{-\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \eta^{\alpha+1-k} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-\frac{3}{2}} \right\|_{\mathbf{L}^2(0,1)} \\ &+ C t^{-\frac{1}{2}} \left\| \eta^{\alpha+\beta+1-k-\frac{5}{2}} \right\|_{\mathbf{L}^2(1,\infty)} \leq C \max \left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}}, t^{-\frac{1}{2}} \right) \end{aligned}$$

for $\alpha \geq k-1$, $\alpha + \beta - k < 1$. Also the second term is estimated as

$$|\eta|^{-\frac{1}{3}} \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1 + t\Lambda(\xi)} \leq C t^{-1} \eta^\beta \int_\eta^\infty \xi^{\alpha-k-5} d\xi \leq C t^{-1} \eta^{\alpha+\beta-k-4}$$

for $\eta \geq 1$, $t \geq 1$, $\alpha - k < 4$ and

$$\begin{aligned}
& |\eta|^{-\frac{1}{3}} \langle \eta \rangle^\beta \int_\eta^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\Lambda(\xi)} \\
& \leq C |\eta|^{-\frac{1}{3}} \int_\eta^1 \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\xi^3} + Ct^{-1} |\eta|^{-\frac{1}{3}} \int_1^\infty \xi^{\alpha-k-5+\frac{1}{3}} d\xi \\
& \leq Ct^{-\frac{\alpha}{3}-\frac{1}{9}} |\eta|^{-\frac{1}{3}} \int_{\eta t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+\xi^3} + Ct^{-1} |\eta|^{-\frac{1}{3}} \\
& \leq Ct^{-\frac{\alpha+1-k}{3}-\frac{1}{9}} |\eta|^{-\frac{1}{3}} \min\left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{\alpha-k-2+\frac{1}{3}}\right) + C |\eta|^{-\frac{1}{3}} t^{-1} \log t
\end{aligned}$$

for $0 < \eta \leq 1$, $k-1 \leq \alpha < k+4$. Hence for the second summand

$$\begin{aligned}
& t^{\frac{1}{2}} \left\| \int_0^\infty \frac{|\eta|^\alpha |\xi + |\eta||^{-1} \langle \eta \rangle^\beta \langle \xi \rangle^\delta d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
(3.20) \quad & \leq C \max\left(1, t^{-\frac{\alpha-k}{3}}\right).
\end{aligned}$$

We apply (3.19) and (3.20) to (3.18) to get

$$\|\mathcal{I}_3\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \max\left(1, t^{-\frac{\alpha-k}{3}}\right)$$

for all $t \geq 1$, if $k-1 \leq \alpha < k+4$, $\alpha + \beta - k < 1$.

To estimate $\mathcal{I}_3$ for $\eta < 0$, we integrate by parts via the identity (2.10)

$$\mathcal{I}_3 = -t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \xi \partial_\xi (H_4 \psi(\xi, \eta)) d\xi.$$

Using the estimate

$$|\xi \partial_\xi (H_4 \psi(\xi, \eta))| \leq \frac{C |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta}{1+t\xi(\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2)},$$

we obtain

$$\begin{aligned}
|\mathcal{I}_3| & \leq Ct^{\frac{1}{2}} \int_0^\infty \frac{|\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta \rangle^\beta d\xi}{1+t\xi(\xi^2 \langle \xi \rangle^2 + \eta^2 \langle \eta \rangle^2)} \\
(3.21) \quad & \leq Ct^{\frac{1}{2}} \int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1+t\xi\eta^2 \langle \eta \rangle^2} + Ct^{\frac{1}{2}} |\eta|^{-\frac{1}{3}} \int_{|\eta|}^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^\beta d\xi}{1+t\Lambda(\xi)}.
\end{aligned}$$

We have

$$\begin{aligned} \int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1+t\xi\eta^2 \langle \eta \rangle^2} &\leq C |\eta|^{\alpha+1-k} \int_0^1 \frac{dy}{1+t|\eta|^3 y} \\ &\leq C t^{-\frac{\alpha+1-k}{3}} \min \left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-2} \right) + C t^{-1} \log t \end{aligned}$$

for $|\eta| < 1$ if $\alpha \geq k-1$, and

$$\begin{aligned} \int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1+t\xi\eta^2 \langle \eta \rangle^2} &\leq C |\eta|^{\alpha+1-k+\beta} \int_0^1 \frac{dy}{1+t|\eta|^5 y} \\ &\leq C t^{-1} \log t \end{aligned}$$

for $|\eta| \geq 1$ if $\alpha + \beta - k < 4$. Therefore

$$(3.22) \quad \left\| \int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^\beta d\xi}{1+t\xi\eta^2 \langle \eta \rangle^2} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C \left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}} + t^{-1} \log t \right).$$

Next we find

$$\begin{aligned} |\eta|^{-\frac{1}{3}} \int_{|\eta|}^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^\beta d\xi}{1+t\Lambda(\xi)} &\leq C t^{-1} |\eta|^{-\frac{1}{3}} \int_{|\eta|}^\infty \xi^{\alpha+\beta-k-5+\frac{1}{3}} d\eta \\ &\leq C t^{-1} |\eta|^{\alpha+\beta-k-4} \end{aligned}$$

for $|\eta| \geq 1$ if $\alpha + \beta - k < 4$ and

$$\begin{aligned} &|\eta|^{-\frac{1}{3}} \int_{|\eta|}^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^\beta d\xi}{1+t\Lambda(\xi)} \\ &\leq C |\eta|^{-\frac{1}{3}} \int_{|\eta|}^1 \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t|\xi|^3} + C |\eta|^{-\frac{1}{3}} \int_1^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^\beta d\xi}{1+t\Lambda(\xi)} \\ &\leq C |\eta|^{-\frac{1}{3}} \left(t^{-\frac{\alpha+1-k}{3}+\frac{1}{9}} \min \left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{\alpha-k-2+\frac{1}{3}} \right) + t^{-1} \log t \right) \end{aligned}$$

for $|\eta| < 1$. Therefore

$$(3.23) \quad \left\| |\eta|^{-\frac{1}{3}} \int_{|\eta|}^\infty \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^\beta d\xi}{1+t\Lambda(\xi)} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C \left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}} + t^{-1} \log t \right).$$

By (3.21)-(3.23)

$$\|\mathcal{I}_3\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq C \max \left(1, t^{-\frac{\alpha-k}{3}} \right)$$

if $\alpha \geq k-1$, $\alpha + \beta - k < 3$. Lemma 3.4 is proved. $\square$

3.3. Estimate for derivative of $\mathcal{V}$

In the next lemma we estimate the derivative $\partial_\eta \mathcal{V}$ through the operator $\mathcal{A}_0$.

Lemma 3.5. *Let $\alpha + \beta + j < 3$, and $\alpha \geq 0$ if $j = 1, 2, 3$, $\alpha \geq 1$ if $j = 0$. Then the estimate*

$$\begin{aligned} & \left\| |\eta|^\alpha \langle \eta \rangle^\beta t \mathcal{A}_0 \mathcal{V} \xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ & \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 & \text{for } \alpha > 0, j = 1, 2, 3 \text{ or } \alpha > 1, j = 0 \\ \log \langle t \rangle & \text{for } \alpha = 0, j = 1, 2, 3 \text{ or } \alpha = 1, j = 0 \end{cases} \\ & \quad + C \max \left(1, t^{\frac{1}{2} - \frac{\alpha+j}{3}} \right) |\phi(0)| \end{aligned}$$

is true for all $t \geq 1$.

When $j = 2, 3$, $\|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ can be replaced by $\|\xi \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$.

Proof. We have

$$t \mathcal{A}_0 \mathcal{V} \xi^j \phi = |\Lambda''(\eta)|^{-\frac{1}{2}} \partial_\eta \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \xi^j \phi(\xi) d\xi.$$

By the identity

$$\begin{aligned} \partial_\eta e^{-itS(\xi, \eta)} &= \frac{S_\eta(\xi, \eta)}{S_\xi(\xi, \eta)} \partial_\xi e^{-itS(\xi, \eta)} \\ &= -\frac{|\Lambda''(\eta)|(\xi - \eta\theta(\eta))}{\Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta)} \partial_\xi e^{-itS(\xi, \eta)} \end{aligned}$$

we integrate by parts

$$\begin{aligned} & |\eta|^\alpha \langle \eta \rangle^\beta t \mathcal{A}_0 \mathcal{V} \xi^j \phi \\ &= |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \partial_\xi \left(\frac{\xi - \eta\theta(\eta)}{\Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta)} \xi^j \phi(\xi) \right) d\xi \\ &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_1(\xi, \eta) \phi_\xi(\xi) d\xi \\ & \quad + \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_2(\xi, \eta) \phi(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned} \psi_1(\xi, \eta) &= |\eta|^\alpha \langle \eta \rangle^\beta \frac{(\xi - \eta\theta(\eta)) \xi^j}{\Lambda'(\xi) - \frac{\eta}{|\eta|}\Lambda'(\eta)} \\ &= \begin{cases} \frac{|\eta|^\alpha \langle \eta \rangle^\beta \xi^j}{(\eta + \xi)(a + b\eta^2 + b\xi^2)} & \text{for } \eta > 0 \\ \frac{|\eta|^\alpha \langle \eta \rangle^\beta \xi^{j+1}}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned}\psi_2(\xi, \eta) &= |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \partial_\xi \frac{(\xi - \eta\theta(\eta)) \xi^j}{\Lambda'(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta)} \\ &= |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \begin{cases} \partial_\xi \frac{\xi^j}{(\eta+\xi)(a+b\eta^2+b\xi^2)} & \text{for } \eta > 0 \\ \partial_\xi \frac{\xi^{j+1}}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases}.\end{aligned}$$

By a direct calculation

$$\begin{aligned} |(\eta \partial_\eta)^k \psi_1(\xi, \eta)| &\leq C \frac{|\eta|^\alpha \langle \eta \rangle^\beta \xi^j}{(|\eta| + \xi) (\langle \xi \rangle^2 + \langle \eta \rangle^2)} \\ &\leq C \begin{cases} |\eta|^\alpha \langle \eta \rangle^{\beta+j-3}, & j = 1, 2, 3 \\ |\eta|^{\alpha-1} \langle \eta \rangle^{\beta+\delta-2} \langle \xi \rangle^{-\delta}, & j = 0 \end{cases} \end{aligned}$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then we apply Lemma 3.2 with β replaced by $-\beta - j + 3$ for $j = 1, 2, 3$, and with α replaced by $\alpha - 1$, and β replaced by $-\beta - \delta + 2$ for $j = 0$, and $\delta > 0$ small, to find

$$\begin{aligned} &\left\| \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_1(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ &\leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 & \text{for } \alpha > 0, j = 1, 2, 3 \text{ or } \alpha > 1, j = 0 \\ \log \langle t \rangle & \text{for } \alpha = 0, j = 1, 2, 3 \text{ or } \alpha = 1, j = 0 \end{cases} \end{aligned}$$

if $-\beta - j + 3 > \alpha \geq 0$ for $j = 1, 2, 3$, and $-\beta + 2 > \alpha - 1 \geq 0$ for $j = 0$. Also we have

$$\begin{aligned} &|\psi_2(\xi, \eta)| + |(\xi - \eta\theta(\eta)) \partial_\xi \psi_2(\xi, \eta)| \\ &\leq C |\eta|^{\alpha+\frac{1}{2}} |\xi|^j |\xi + |\eta||^{-2} \langle \eta \rangle^{\beta+1} (\langle \xi \rangle + \langle \eta \rangle)^{-2}. \end{aligned}$$

Then we apply Lemma 3.3 with α, β replaced by $\alpha + \frac{1}{2}, \beta - 1$, respectively, and $k = 2 - j$, $j = 0, 1, 2$ and $k = 0$, $j = 3$, then

$$\left\| \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_2(\xi, \eta) (\phi(\xi) - \phi(0)) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

In the same manner by Lemma 3.4 we get

$$\left\| \sqrt{\frac{it}{2\pi}} \phi(0) \int_0^\infty e^{-itS(\xi, \eta)} \psi_2(\xi, \eta) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \max\left(1, t^{\frac{1}{2} - \frac{\alpha+j}{3}}\right) |\phi(0)|.$$

Lemma 3.5 is proved. $\square$

In the next lemma we estimate the derivative $\partial_t \mathcal{V}\phi$.

Lemma 3.6. *Let $\alpha > 0$, $\alpha + \beta < -1$. Then the estimate*

$$\left\| |\eta|^\alpha \langle \eta \rangle^\beta t \partial_t \mathcal{V}\phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\phi(0)|$$

is true for all $t \geq 1$.

Proof. We have

$$\begin{aligned} \partial_a \mathcal{V}\phi &= |\eta| |\Lambda''(\eta)|^{-1} \mathcal{V}\phi \\ &\quad - it \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \partial_a S(\xi, \eta) \phi(\xi) d\xi \end{aligned}$$

and

$$\begin{aligned} \partial_b \mathcal{V}\phi &= 2 |\eta|^3 |\Lambda''(\eta)|^{-1} \mathcal{V}\phi \\ &\quad - it \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \partial_b S(\xi, \eta) \phi(\xi) d\xi. \end{aligned}$$

Using the identity

$$e^{-itS(\xi, \eta)} = -\frac{1}{itS_\xi(\xi, \eta)} \partial_\xi e^{-itS(\xi, \eta)},$$

we integrate by parts

$$\begin{aligned} \partial_a \mathcal{V}\phi &= |\eta| |\Lambda''(\eta)|^{-\frac{1}{2}} \mathcal{V}\phi \\ &\quad - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \frac{\partial_a S(\xi, \eta)}{S_\xi(\xi, \eta)} \phi_\xi(\xi) d\xi \\ &\quad - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \left(\partial_\xi \frac{\partial_a S(\xi, \eta)}{S_\xi(\xi, \eta)} \right) \phi(\xi) d\xi. \end{aligned}$$

Thus we get

$$\begin{aligned} |\eta|^\alpha \langle \eta \rangle^\beta \partial_a \mathcal{V}\phi &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_1(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \\ &\quad + \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_2(\xi, \eta) \phi(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned} \psi_1(\xi, \eta) &= -|\eta|^\alpha \langle \eta \rangle^\beta \langle \xi \rangle^{-1} \frac{\partial_a S(\xi, \eta)}{S_\xi(\xi, \eta)} \\ &= -\frac{1}{3} |\eta|^\alpha \langle \eta \rangle^\beta \langle \xi \rangle^{-1} \begin{cases} \frac{(2\eta+\xi)(\xi-\eta)}{(\eta+\xi)(a+b\eta^2+b\xi^2)} & \text{for } \eta > 0 \\ \frac{\xi^2(\xi^2+3\eta^2)}{\Lambda'(\xi)+\Lambda'(\eta)} & \text{for } \eta < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned}\psi_2(\xi, \eta) &= |\eta|^{\alpha+1} \langle \eta \rangle^\beta - |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \left(\partial_\xi \frac{\partial_a S(\xi, \eta)}{S_\xi(\xi, \eta)} \right) \\ &= |\eta|^{\alpha+1} \langle \eta \rangle^\beta - \frac{1}{3} |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \begin{cases} \partial_\xi \frac{(2\eta+\xi)(\xi-\eta)}{(\eta+\xi)(a+b\eta^2+b\xi^2)} & \text{for } \eta > 0 \\ \partial_\xi \frac{\xi(\xi^2+3\eta^2)}{\Lambda'(\xi)+\Lambda'(\eta)} & \text{for } \eta < 0 \end{cases}.\end{aligned}$$

We have

$$\left| (\eta \partial_\eta)^k \psi_1(\xi, \eta) \right| \leq C |\eta|^\alpha \langle \eta \rangle^\beta$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then applying Lemma 3.2 we get for $\alpha + \beta < 0$, $\alpha > 0$

$$\begin{aligned}& \left\| \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_1(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ & \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\end{aligned}$$

Also we have

$$|\psi_2(\xi, \eta)| + |(\xi - \eta\theta(\eta)) \partial_\xi \psi_2(\xi, \eta)| \leq |\eta|^{\alpha+\frac{1}{2}} \langle \eta \rangle^{\beta-1}$$

for $\xi > 0$, $\eta \in \mathbf{R}$. We apply Lemma 3.3 and Lemma 3.4 with α and β replaced by $\alpha + \frac{1}{2}$ and $1 - \beta$, respectively, to obtain

$$\begin{aligned}& \left\| \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_2(\xi, \eta) \phi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ & \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\phi(0)|.\end{aligned}$$

Similarly, using the identity

$$e^{-itS(\xi, \eta)} = -\frac{1}{itS_\xi(\xi, \eta)} \partial_\xi e^{-itS(\xi, \eta)},$$

we integrate by parts

$$\begin{aligned}\partial_b \mathcal{V} \phi &= 2 |\eta|^3 |\Lambda''(\eta)|^{-1} \mathcal{V} \phi \\ & - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \frac{\partial_b S(\xi, \eta)}{S_\xi(\xi, \eta)} \phi_\xi(\xi) d\xi \\ & - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \left(\partial_\xi \frac{\partial_b S(\xi, \eta)}{S_\xi(\xi, \eta)} \right) \phi(\xi) d\xi.\end{aligned}$$

Thus we get

$$\begin{aligned} |\eta|^\alpha \langle \eta \rangle^\beta \partial_b \mathcal{V} \phi &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_3(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \\ &+ \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_4(\xi, \eta) \langle \xi \rangle \phi(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned} \psi_3(\xi, \eta) &= -|\eta|^\alpha \langle \eta \rangle^\beta \langle \xi \rangle^{-1} \frac{\partial_b S(\xi, \eta)}{S_\xi(\xi, \eta)} \\ &= -\frac{1}{5} |\eta|^\alpha \langle \eta \rangle^\beta \langle \xi \rangle^{-1} \begin{cases} \frac{(4\eta^3 + \xi^3 + 2\eta\xi^2 + 3\eta^2\xi)(\xi - \eta)}{(\eta + \xi)(a + b\eta^2 + b\xi^2)} & \text{for } \eta > 0 \\ \frac{\xi(5\eta^4 + \xi^4)}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \psi_4(\xi, \eta) &= 2|\eta|^{\alpha+3} \langle \eta \rangle^\beta \xi \langle \xi \rangle^{-1} - |\eta|^\alpha \langle \eta \rangle^\beta \langle \xi \rangle^{-1} |\Lambda''(\eta)|^{\frac{1}{2}} \left(\partial_\xi \frac{\partial_b S(\xi, \eta)}{S_\xi(\xi, \eta)} \xi \right) \\ &= 2|\eta|^{\alpha+3} \langle \eta \rangle^\beta \xi \langle \xi \rangle^{-1} \\ &\quad - \frac{1}{5} |\eta|^\alpha \langle \eta \rangle^\beta |\Lambda''(\eta)|^{\frac{1}{2}} \langle \xi \rangle^{-1} \begin{cases} \partial_\xi \frac{(4\eta^3 + \xi^3 + 2\eta\xi^2 + 3\eta^2\xi)(\xi - \eta)\xi}{(\eta + \xi)(a + b\eta^2 + b\xi^2)} & \text{for } \eta > 0 \\ \partial_\xi \frac{\xi^2(5\eta^4 + \xi^4)}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases}. \end{aligned}$$

We have

$$|(\eta \partial_\eta)^k \psi_3(\xi, \eta)| \leq C |\eta|^\alpha \langle \eta \rangle^{\beta+1}$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then applying Lemma 3.2 we get for $\beta < -\alpha - 1$, $\alpha > 0$

$$\left\| \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_3(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Also we have

$$|\psi_4(\xi, \eta)| + |(\xi - \eta\theta(\eta)) \partial_\xi \psi_4(\xi, \eta)| \leq |\eta|^{\alpha+\frac{1}{2}} \langle \eta \rangle^{\beta+2}$$

for $\xi > 0$, $\eta \in \mathbf{R}$, $\alpha + \beta < -1$. We apply Lemma 3.3 and Lemma 3.4 to get

$$\begin{aligned} &\left\| \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_4(\xi, \eta) \phi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ &\leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\phi(0)|. \end{aligned}$$

Using the above estimates and the following relation

$$t\partial_t \mathcal{V}\phi = a\partial_a \mathcal{V}\phi + b\partial_b \mathcal{V}\phi - |\eta| |\Lambda''(\eta)|^{-1} (1 + 2\eta^2) \mathcal{V}\phi$$

we find

$$\left\| |\eta|^\alpha \langle \eta \rangle^\beta t\partial_t \mathcal{V}\phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\phi(0)|.$$

Lemma 3.6 is proved. $\square$

In the next lemma we estimate the commutator $[\eta, \mathcal{V}] \xi^j \phi$, for $j = 2, 3$.

Lemma 3.7. *Let $\alpha \in [0, 1)$, $\beta > \alpha$, $j = 2, 3$. Then the estimate*

$$\begin{aligned} & \left\| |\eta|^\alpha \langle \eta \rangle^{-\beta} [\eta, \mathcal{V}] \xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C t^{\frac{1}{6} - \frac{\alpha}{3}} \|\langle \xi \rangle \phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & + C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 & \text{for } \alpha > 0 \\ \log \langle t \rangle & \text{for } \alpha = 0 \end{cases} \end{aligned}$$

is true for all $t \geq 1$.

Proof. We have

$$[\eta, \mathcal{V}] \xi^j \phi = \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} (\eta - \xi) \xi^j \phi(\xi) d\xi.$$

Using the identity

$$e^{-itS(\xi, \eta)} = -\frac{1}{itS_\xi(\xi, \eta)} \partial_\xi e^{-itS(\xi, \eta)},$$

we integrate by parts

$$\begin{aligned} [t\eta, \mathcal{V}] \xi^j \phi &= t \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} (\eta - \xi) \xi^j(\xi) d\xi \\ &= -\sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \frac{(\eta - \xi) \xi^j}{iS_\xi(\xi, \eta)} \phi_\xi(\xi) d\xi \\ &\quad - \sqrt{\frac{it |\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \left(\partial_\xi \frac{(\eta - \xi) \xi^j}{iS_\xi(\xi, \eta)} \right) \phi(\xi) d\xi. \end{aligned}$$

Thus we get

$$\begin{aligned} |\eta|^\alpha \langle \eta \rangle^{-\beta} [\eta, \mathcal{V}] \xi^j \phi &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_5(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \\ &\quad + \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_6(\xi, \eta) \phi(\xi) d\xi, \end{aligned}$$

where

$$\begin{aligned}\psi_5(\xi, \eta) &= -|\eta|^\alpha \langle \eta \rangle^{-\beta} \langle \xi \rangle^{-1} \frac{(\eta - \xi) \xi^j}{iS_\xi(\xi, \eta)} \\ &= -|\eta|^\alpha \langle \eta \rangle^{-\beta} \langle \xi \rangle^{-1} \begin{cases} \frac{-\xi^j}{(\eta + \xi)(a + b\eta^2 + b\xi^2)} & \text{for } \eta > 0 \\ \frac{\xi^{j+1}}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases}\end{aligned}$$

and

$$\begin{aligned}\psi_6(\xi, \eta) &= -|\eta|^\alpha \langle \eta \rangle^{-\beta} |\Lambda''(\eta)|^{\frac{1}{2}} \left(\partial_\xi \frac{(\eta - \xi) \xi^j}{iS_\xi(\xi, \eta)} \right) \\ &= -|\eta|^\alpha \langle \eta \rangle^{-\beta} |\Lambda''(\eta)|^{\frac{1}{2}} \begin{cases} \partial_\xi \frac{-\xi^j}{(\eta + \xi)(a + b\eta^2 + b\xi^2)} & \text{for } \eta > 0 \\ \partial_\xi \frac{\xi^{j+1}}{\Lambda'(\xi) + \Lambda'(\eta)} & \text{for } \eta < 0 \end{cases}.\end{aligned}$$

We have

$$\left| (\eta \partial_\eta)^k \psi_5(\xi, \eta) \right| \leq C |\eta|^\alpha \langle \eta \rangle^{-\beta} \langle \xi \rangle^{-1}$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then applying Lemma 3.2 we get for $\beta > \alpha \geq 0$

$$\begin{aligned}& \left\| \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_5(\xi, \eta) \langle \xi \rangle \phi_\xi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ & \leq C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 & \text{for } \alpha > 0 \\ \log \langle t \rangle & \text{for } \alpha = 0 \end{cases}.\end{aligned}$$

Also we have

$$|\psi_6(\xi, \eta)| + |(\xi - \eta \theta(\eta)) \partial_\xi \psi_6(\xi, \eta)| \leq |\eta|^{\alpha + \frac{1}{2}} \langle \eta \rangle^{-\beta + \frac{1}{2}} \langle \xi \rangle$$

for $\xi > 0$, $\eta \in \mathbf{R}$. We apply Lemma 3.3 with $\alpha \in [0, 1)$, $\alpha - \beta < 0$

$$\begin{aligned}& \left\| \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \psi_6(\xi, \eta) \phi(\xi) d\xi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ & \leq C t^{\frac{1}{6} - \frac{\alpha}{3}} \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + C \|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.\end{aligned}$$

Lemma 3.7 is proved. $\square$

§4. Estimate for the nonlinearity

In the next lemma we study the large time behavior of the nonlinearity $\mathcal{FU}(-t)(u_x)^3$. Note that

$$\|\widehat{\varphi}\|_{\mathbf{Z}} = \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + t^{-\gamma} \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

is bounded in time for small $\gamma > 0$.

Lemma 4.1. *Assume that $\|\widehat{\varphi}\|_{\mathbf{Z}} \leq \varepsilon$. Then the asymptotics*

$$\begin{aligned} \mathcal{FU}(-t) \partial_x^j (u_x)^3 &= -i^{\frac{3}{2}} t^{-1} e^{it\Omega} \mathcal{D}_3 |\Lambda''(\xi)|^{-1} (i\xi)^{j+3} \widehat{\varphi}^3(\xi) \\ &+ 3it^{-1} \xi^3 |\Lambda''(\xi)|^{-1} |\widehat{\varphi}(\xi)|^2 (i\xi)^j \widehat{\varphi}(\xi) + O\left(\varepsilon^3 t^{-\frac{9}{8}}\right) \end{aligned}$$

is true for all $t \geq 1$, $\xi \geq 0$, $j = 0, 1, 2$, where $\widehat{\varphi}(t) = \mathcal{FU}(-t) u(t)$.

Proof. We have $\partial_x (u_x)^3 = 3(u_x)^2 u_{xx}$ and

$$\partial_x^2 (u_x)^3 = 3(u_x)^2 u_{xxx} + 6u_x (u_{xx})^2.$$

So we need to consider the term $\mathcal{FU}(-t) \left(\partial_x^j u\right)^2 \partial_x^k u$ with $j = 1, 2$, $k = 1, 2, 3$ for $3 \leq 2j + k \leq 5$. In view of (1.10) we find for the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t) u(t)$

$$\begin{aligned} &\mathcal{FU}(-t) \left(\partial_x^j u\right)^2 \partial_x^k u \\ &= -i^{\frac{3}{2}} t^{-1} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \frac{1}{|\Lambda''|} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\ &\quad + t^{-1} \mathcal{Q}(t) \frac{1}{|\Lambda''|} \left(2 \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right) \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \right. \\ &\quad \quad \left. + \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right)\right) \\ &\quad + 3i^{\frac{3}{2}} t^{-1} \mathcal{D}_{-1} \mathcal{Q}(-t) \frac{1}{|\Lambda''|} \left(\left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \right. \\ &\quad \left. + 2 \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right) \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right) \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right)\right) \\ &\quad - i^{\frac{1}{2}} t^{-1} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \frac{1}{|\Lambda''|} \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right), \end{aligned}$$

where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$.

By Lemma 2.3 with $\alpha_1 = \frac{1}{2}$, $\alpha = -\frac{1}{8}$, $\beta_1 < 2$, $\beta = \frac{7}{8}$ we have

$$\begin{aligned} &\mathcal{Q}(3t) \frac{\theta}{|\Lambda''|} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\ &= \theta A^*(3t) |\Lambda''|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\ &\quad + Ct^{-\frac{1}{6}} \left\| \eta^{-\frac{1}{2}} \langle \eta \rangle^{-\beta_1} |\Lambda''|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\quad + Ct^{-\frac{1}{8}} \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \partial_\eta |\Lambda''|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

By Lemma 2.2 with $\alpha = -\frac{1}{2}$, $j = 1, 2, 3$

$$(4.1) \quad \left| \mathcal{V}(i\xi)^j \widehat{\varphi} \right| \leq C |\eta|^j |\widehat{\varphi}| + C |\eta|^{\frac{1}{2}} \langle \eta \rangle^{j-\frac{9}{4}} t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}.$$

Hence for $3 \leq 2j + k \leq 5$

$$\begin{aligned} & \left\| \eta^{-\frac{1}{2}} \langle \eta \rangle^{-\beta_1} |\Lambda''|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & \leq C \left(\left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 + t^{-\frac{1}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^3 \right) \leq C\varepsilon^3. \end{aligned}$$

Using the relation

$$\partial_\eta |\Lambda''|^{-1} = |\Lambda''| t \mathcal{A}_0 |\Lambda''|^{-1} + \frac{\Lambda'''}{2 |\Lambda''|^2}$$

and the Leibnitz rule

$$\begin{aligned} \mathcal{A}_0 \left(|\Lambda''|^{-1} \phi_1 \phi_2 \phi_3 \right) &= |\Lambda''|^{-1} \phi_2 \phi_3 \mathcal{A}_0 \phi_1 \\ &\quad + |\Lambda''|^{-1} \phi_1 \phi_3 \mathcal{A}_0 \phi_2 + |\Lambda''|^{-1} \phi_1 \phi_2 \mathcal{A}_0 \phi_3 \end{aligned}$$

we get

$$\begin{aligned} & \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \partial_\eta |\Lambda''|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \left\| \eta^{\frac{1}{8}-2} \langle \eta \rangle^{-\frac{7}{8}-2} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \quad + C \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(t \mathcal{A}_0 \mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \quad + C \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \left(t \mathcal{A}_0 \mathcal{V}(i\xi)^j \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

By (4.1) we have

$$\begin{aligned} & \left\| \eta^{\frac{1}{8}-2} \langle \eta \rangle^{-\frac{7}{8}-2} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \left(\left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 + t^{-\frac{1}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^3 \right) \leq C\varepsilon^3 \end{aligned}$$

and

$$\begin{aligned} & \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(t \mathcal{A}_0 \mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \quad + C \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \left(t \mathcal{A}_0 \mathcal{V}(i\xi)^j \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \|\widehat{\varphi}\|_{\mathbf{Z}}^2 \left(\left\| \eta^{\frac{1}{2}} t \mathcal{A}_0 \mathcal{V}(i\xi)^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\| \eta^{\frac{1}{2}} \langle \eta \rangle^{-1} t \mathcal{A}_0 \mathcal{V}(i\xi)^3 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right). \end{aligned}$$

By Lemma 3.5 with $\alpha = \frac{1}{2}$

$$\begin{aligned} & \left\| \eta^{\frac{1}{2}} t \mathcal{A}_0 \mathcal{V} (i\xi)^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\| \eta^{\frac{1}{2}} \langle \eta \rangle^{-1} t \mathcal{A}_0 \mathcal{V} (i\xi)^3 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \left(\|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + |\phi(0)| \right) \end{aligned}$$

which implies

$$\begin{aligned} & \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \partial_\eta |\Lambda''|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \varepsilon^3 + \|\widehat{\varphi}\|_{\mathbf{Z}}^2 \left(\|\langle \xi \rangle \phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + |\phi(0)| \right) \leq C \varepsilon^3 t^\gamma. \end{aligned}$$

By (2.11)

$$\begin{aligned} & \left\| \mathcal{Q}(3t) (1 - \theta) |\Lambda''|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & \leq C t^{\frac{1}{2}} \left\| |\Lambda''|^{-\frac{1}{2}} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^1(\mathbf{R}_-)}. \end{aligned}$$

Applying estimate of Lemma 2.2 for $\eta < 0$ with $\alpha = 0$, we have

$$|\mathcal{V} \xi^j \widehat{\varphi}| \leq C \max \left(t^{-\frac{1}{2}} \log \langle t \rangle, t^{-\frac{j}{3}} \right) \langle \eta \rangle^{j-\frac{5}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1},$$

which implies

$$\begin{aligned} & \left\| \mathcal{Q}(3t) (1 - \theta) |\Lambda''|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ & \leq C t^{\frac{1}{2}} \left\| |\Lambda''|^{-\frac{1}{2}} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^1(\mathbf{R}_-)} \\ & \leq C t^{-\frac{1}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^3 \left\| |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-1} \right\|_{\mathbf{L}^1(\mathbf{R}_-)} \leq C t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \leq C t^{-\frac{1}{6}} \varepsilon^3. \end{aligned}$$

Thus we find

$$\begin{aligned} & \mathcal{Q}(3t) \frac{1}{|\Lambda''|} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \\ & = \theta A^*(3t) |\Lambda''|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) + O \left(t^{-\frac{1}{8}} \varepsilon^3 \right). \end{aligned}$$

Next by Lemma 2.1 $A(t, \xi) = 1 + O \left(\langle t \xi^3 \rangle^{-1} \right)$ and $A^*(t, \xi) = 1 + O \left(\langle t \xi^3 \rangle^{-1} \right)$ for $t^{\frac{1}{3}} \xi \rightarrow \infty$. And by Lemma 2.2 with $\alpha = 0$

$$\mathcal{V} (i\xi)^j \phi = A(t, \xi) (i\xi)^j \phi(\xi) + O \left(t^{-\frac{1}{4}} \langle \xi \rangle^{j-\frac{7}{4}} \|\phi\|_{\mathbf{Z}_1} \right).$$

Hence

$$\begin{aligned} & \mathcal{Q}(3t) \frac{1}{|\Lambda''|} \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \\ &= |\Lambda''(\xi)|^{-1} (i\xi)^{2j+k} \widehat{\varphi}^3(\xi) + O\left(t^{-\frac{1}{8}} \varepsilon^3\right). \end{aligned}$$

In the same manner we find for the second summand

$$\begin{aligned} & \mathcal{Q}(t) \frac{1}{|\Lambda''|} \left(2 \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}} \right) \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) + \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}} \right) \right) \\ &= \frac{i^k \xi^{2j+k}}{|\Lambda''(\xi)|} \left(2 + (-1)^{j-k} \right) |\widehat{\varphi}(\xi)|^2 \widehat{\varphi}(\xi) + O\left(t^{-\frac{1}{8}} \varepsilon^3\right). \end{aligned}$$

In the case of the nonlinearity $(u_x)^3$ we have $j = k = 1$, for the nonlinearity $\partial_x (u_x)^3 = 3(u_x)^2 u_{xx}$ we have $j = 1, k = 2$, finally, in the case of the nonlinearity $\partial_x^2 (u_x)^3 = 3(u_x)^2 u_{xxx} + 6u_x (u_{xx})^2$ we have $j = 1, k = 3$ or $j = 2, k = 1$.

Next applying the second estimate of Lemma 2.3 as above we get for the third and fourth terms

$$\begin{aligned} & \mathcal{D}_{-1} \mathcal{Q}(-t) \frac{1}{|\Lambda''|} \left(\left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}} \right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi} \right) \right. \\ & \quad \left. + 2 \left(\mathcal{V}(i\xi)^j \widehat{\varphi} \right) \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}} \right) \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}} \right) \right) = O\left(t^{-\frac{1}{8}} \varepsilon^3\right) \end{aligned}$$

and

$$\mathcal{D}_{-3} \mathcal{Q}(-3t) \frac{1}{|\Lambda''|} \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}} \right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}} \right) = O\left(t^{-\frac{1}{8}} \varepsilon^3\right).$$

Lemma 4.1 is proved. $\square$

§5. A priori estimates

Local existence and uniqueness of solutions to the Cauchy problem (1.6) was shown in paper [19]. However we do not have any local result involving the operator $\mathcal{J} = \mathcal{U}(t) x \mathcal{U}(-t)$.

Theorem 5.1. *Assume that the initial data $u_0 \in \mathbf{H}^3 \cap \mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^3 \cap \mathbf{H}^{1,1}} < \infty$. Then there exists a time T such that the Cauchy problem (1.6) has a unique local solution $\mathcal{U}(-t)u \in \mathbf{C}([0, T]; \mathbf{H}^3 \cap \mathbf{H}^{1,1})$.*

Proof. Let us consider the linearized equation of (1.6) such that

$$(5.1) \quad \partial_t u - \frac{a}{3} \partial_x^3 u + \frac{b}{5} \partial_x^5 u = (v_x)^3,$$

$$(5.2) \quad \partial_t u_x - \frac{a}{3} \partial_x^3 u_x + \frac{b}{5} \partial_x^5 u_x = 3(v_x)^2 \partial_x u_x,$$

where $v \in \mathbf{C}([0, T]; \mathbf{H}^3 \cap \mathbf{H}^{1,1})$ and

$$\sup_{t \in [0, T]} \|v(t)\|_{\mathbf{H}^3 \cap \mathbf{H}^{1,1}} \leq 2\rho, \|u_0\|_{\mathbf{H}^3 \cap \mathbf{H}^{1,1}} \leq \rho.$$

By applying the usual energy method to (5.1) and (5.2), we have

$$\begin{aligned} \|u(t)\|_{\mathbf{H}^3} &\leq \|u_0\|_{\mathbf{H}^3} + C \int_0^t \|v(s)\|_{\mathbf{H}^1}^3 ds + C \int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|u(s)\|_{\mathbf{H}^3} ds \\ &\leq \|u_0\|_{\mathbf{H}^3} + \rho^3 CT + \rho^2 CT \sup_{t \in [0, T]} \|u(t)\|_{\mathbf{H}^3}. \end{aligned}$$

Therefore we have $\mathbf{H}^3$ solution. Note that $\|xu(t)\|_{\mathbf{H}^1} \leq C \left\| \langle i\partial_x \rangle^{-1} xu(t) \right\|_{\mathbf{L}^2} + C \|xu_x(t)\|_{\mathbf{L}^2} + C \|u(t)\|_{\mathbf{L}^2}$. Multiplying both sides of (5.1) by $\langle i\partial_x \rangle^{-1} x$, we get

$$\begin{aligned} &\partial_t \langle i\partial_x \rangle^{-1} xu - \frac{a}{3} \partial_x^3 \langle i\partial_x \rangle^{-1} xu + \frac{b}{5} \partial_x^5 \langle i\partial_x \rangle^{-1} xu \\ &= -a \langle i\partial_x \rangle^{-1} \partial_x^2 u + b \langle i\partial_x \rangle^{-1} \partial_x^4 u + \langle i\partial_x \rangle^{-1} \left((v_x)^2 xv_x \right) \end{aligned}$$

from which with the energy method it follows that

$$\begin{aligned} &\left\| \langle i\partial_x \rangle^{-1} xu(t) \right\|_{\mathbf{L}^2} \\ &\leq \left\| \langle i\partial_x \rangle^{-1} xu_0 \right\|_{\mathbf{L}^2} + C \int_0^t \left(\|u(s)\|_{\mathbf{H}^3} + \|v(s)\|_{\mathbf{H}^1}^2 \|xv_x(s)\|_{\mathbf{L}^2} \right) ds \\ &\leq \left\| \langle i\partial_x \rangle^{-1} xu_0 \right\|_{\mathbf{L}^2} + CT \sup_{t \in [0, T]} \|u(t)\|_{\mathbf{H}^3} + \rho^3 CT. \end{aligned}$$

In order to avoid the derivative loss we consider the identity $\mathcal{P} = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I}$, where the modified dilation operator $\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a$, and $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$. Similarly to (5.2) we consider

$$(5.3) \quad \partial_t \mathcal{P}u - \frac{a}{3} \partial_x^3 \mathcal{P}u + \frac{b}{5} \partial_x^5 \mathcal{P}u = 3(v_x)^2 \partial_x \mathcal{P}u$$

and

$$(5.4) \quad \partial_t \mathcal{I}u - \frac{a}{3} \partial_x^3 \mathcal{I}u + \frac{b}{5} \partial_x^5 \mathcal{I}u = 3(v_x)^2 \partial_x \mathcal{I}u + t\partial_x^3 (v_x)^2 \partial_x u - t(v_x)^2 \partial_x^4 u.$$

By applying the usual energy method to (5.3) and (5.4) we have

$$\begin{aligned} \|\mathcal{P}u(t)\|_{\mathbf{L}^2} &\leq \|x\partial_x u_0\|_{\mathbf{L}^2} + C \int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|\mathcal{P}u(s)\|_{\mathbf{L}^2} ds \\ &\leq \|u_0\|_{\mathbf{H}^{1,1}} + \rho^2 CT \sup_{t \in [0, T]} \|\mathcal{P}u(t)\|_{\mathbf{L}^2} \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{I}u(t)\|_{\mathbf{L}^2} &\leq C \int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|\mathcal{I}u(t)\|_{\mathbf{L}^2} ds \\ &\leq \rho^2 CT \sup_{t \in [0, T]} \|\mathcal{I}u(t)\|_{\mathbf{L}^2} + \rho^2 CT \sup_{t \in [0, T]} \|u(t)\|_{\mathbf{H}^3}. \end{aligned}$$

Hence by equation (5.1) we find $\|\mathcal{J}\partial_x u(t)\|_{\mathbf{L}^2} \leq 5\|\mathcal{P}u(t)\|_{\mathbf{L}^2} + 2a\|\mathcal{I}u(t)\|_{\mathbf{L}^2} + 5t\|\mathcal{L}u(t)\|_{\mathbf{L}^2} \leq \|u_0\|_{\mathbf{H}^{1,1}} + \rho^3 CT$. Therefore we have $\mathbf{H}^{1,1}$ solution. $\square$

We can take $T > 1$ if the data are small in $\mathbf{H}^3 \cap \mathbf{H}^{1,1}$ and we may assume that

$$(5.5) \quad \sup_{t \in [0, 1]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} + \|\mathcal{J}u(t)\|_{\mathbf{H}^1} \right) \leq \varepsilon.$$

To get the desired results, we prove the a priori estimates of solutions uniformly in time. Define the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [1, T]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} + t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} \right).$$

Lemma 5.2. *Assume that*

$$\sup_{t \in [1, T]} \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} = \sup_{t \in [1, T]} \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty} \leq \varepsilon$$

holds. Then there exists an ε such that the estimate

$$\sup_{t \in [1, T]} t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} < 100\varepsilon$$

is true for all $T > 1$.

Proof. Arguing by the contradiction, we can assume that there exists a time $T > 0$ such that $\sup_{t \in [1, T]} t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} = 100\varepsilon$. We have the identity

$$\|\mathcal{J}u(t)\|_{\mathbf{L}^2} + \|\partial_x \mathcal{J}u(t)\|_{\mathbf{L}^2} = \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} + \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2}.$$

Since

$$\begin{aligned} \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2}^2 &= \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi| < 1)}^2 + \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2(|\xi| \geq 1)}^2 \\ &\leq C \left\| \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2}^2 + \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2}^2, \end{aligned}$$

we obtain

$$(5.6) \quad \|\mathcal{J}u(t)\|_{\mathbf{H}^1} = \|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} + \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} \leq C \left\| \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} + 2 \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2}.$$

To estimate the norm $\left\| \langle \xi \rangle^{-4} \partial_\xi \widehat{\phi} \right\|_{\mathbf{L}^2}$ we use equation (1.10). We have

$$\begin{aligned} \partial_t \langle \xi \rangle^{-4} \partial_\xi \widehat{\phi} &= \langle \xi \rangle^{-4} \partial_\xi \mathcal{F}\mathcal{U}(-t) u_x^3 \\ &= (it)^{-1} \langle \xi \rangle^{-4} \partial_\xi \mathcal{Q} |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\phi} + i\overline{M\mathcal{V}(i\xi) \widehat{\phi}} \right)^3. \end{aligned}$$

Since $\partial_\xi S(\xi, \eta) = \Lambda'(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi$, $\Lambda'(\xi) = a\xi^2 + b\xi^4$, we find

$$\begin{aligned} \partial_\xi \mathcal{Q}\phi &= it\Lambda'(\xi) \mathcal{Q}\phi - it\mathcal{Q} \frac{\eta}{|\eta|} \Lambda'(\eta) \phi \\ &= ita(\xi^2 \mathcal{Q} - \mathcal{Q}\eta|\eta|) \phi + itb(\xi^4 \mathcal{Q} - \mathcal{Q}\eta^3|\eta|) \phi. \end{aligned}$$

Since $i\xi \mathcal{Q}\phi = \mathcal{Q}\mathcal{A}\phi$, and $\mathcal{A} = \mathcal{A}_0 + i\eta\theta(\eta)$, we get $[i\xi, \mathcal{Q}] = \mathcal{Q}(\mathcal{A} - i\eta) = \mathcal{Q}\mathcal{A}_0 - i\mathcal{Q}\eta(1 - \theta)$. Hence we find

$$\begin{aligned} (\xi^2 \mathcal{Q} - \mathcal{Q}\eta|\eta|) \phi &= -\mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) \phi + \mathcal{Q}(\eta^2\theta\phi - \eta|\eta|) \phi \\ &= -\mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) \phi + \mathcal{Q}\eta^2(1 - \theta) \phi \end{aligned}$$

and

$$\begin{aligned} (\xi^4 \mathcal{Q} - \mathcal{Q}\eta^3|\eta|) \phi &= \xi^2(\xi^2 \mathcal{Q} - \mathcal{Q}\eta|\eta|) \phi + (\xi^2 \mathcal{Q} - \mathcal{Q}\eta^2) \eta|\eta| \phi \\ &= -(1 + \xi^2) \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) \phi + \xi^2 \mathcal{Q}\eta^2(1 - \theta) \phi + \mathcal{Q}(1 - \theta) \eta^4 \phi. \end{aligned}$$

Hence we get

$$\begin{aligned} \partial_\xi \mathcal{F}\mathcal{U}(-t) u_x^3 &= \partial_\xi \mathcal{Q} \overline{M} B^{-1} \mathcal{D}_t^{-1} u_x^3 \\ &= (it)^{-1} \partial_\xi \mathcal{Q} |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\phi} + i\overline{M\mathcal{V}(i\xi) \widehat{\phi}} \right)^3 \\ &= -(a + b + b\xi^2) \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\phi} + i\overline{M\mathcal{V}(i\xi) \widehat{\phi}} \right)^3 + R_0 \end{aligned}$$

with

$$\begin{aligned} R_0 &= (a + b\xi^2) \mathcal{Q}\eta^2(1 - \theta) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\phi} + i\overline{M\mathcal{V}(i\xi) \widehat{\phi}} \right)^3 \\ &\quad + b\mathcal{Q}(1 - \theta) \eta^4 |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\phi} + i\overline{M\mathcal{V}(i\xi) \widehat{\phi}} \right)^3. \end{aligned}$$

By inequality (3.1)

$$\begin{aligned} \left\| \langle \xi \rangle^{-4} R_0 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C \left\| \eta^2 \langle \eta \rangle^2 |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}i\xi \widehat{\phi} + i\overline{M\mathcal{V}i\xi \widehat{\phi}} \right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ &\leq C \left\| |\eta| \langle \eta \rangle^{-1} |\mathcal{V}\xi \widehat{\phi}|^3 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \leq Ct^{-\frac{9}{8}} \|\widehat{\phi}\|_{\mathbf{Z}_1}^3 \leq C\varepsilon^3 t^{-\frac{9}{8}}, \end{aligned}$$

since by the second estimate of Lemma 2.2 with $j = 1$, $\alpha = \frac{1}{8}$ we have $|\mathcal{V}\xi\widehat{\varphi}| \leq Ct^{-\frac{3}{8}} |\eta|^{-\frac{1}{8}} \langle \eta \rangle^{-1} \|\widehat{\varphi}\|_{\mathbf{Z}_1}$ for all $\eta < 0$. Note that

$$\begin{aligned} & \mathcal{A} |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M}\mathcal{V}(i\xi) \widehat{\varphi} \right)^3 \\ &= \frac{\overline{M}}{t |\Lambda''(\eta)|^{\frac{1}{2}}} \partial_\eta \left(\frac{M}{|\Lambda''(\eta)|^{\frac{1}{2}}} \mathcal{V}(i\xi) \widehat{\varphi} + i \frac{\overline{M}}{|\Lambda''(\eta)|^{\frac{1}{2}}} \overline{\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\ &= 3 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi_1}), \end{aligned}$$

where we denote $\psi = \mathcal{V}(i\xi) \widehat{\varphi}$, $\psi_1 = \mathcal{V}(i\xi)^2 \widehat{\varphi}$

$$\begin{aligned} & (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M}\mathcal{V}(i\xi) \widehat{\varphi} \right)^3 \\ &= (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^3 \\ &= 3\mathcal{A}_0\overline{M} |\Lambda''|^{-1} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi_1}) \\ &\quad + i\eta\theta\mathcal{A}_0\overline{M} |\Lambda''|^{-1} (M\psi + i\overline{M}\overline{\psi})^3 \end{aligned}$$

We have the commutator $[\mathcal{A}_0, M^n] = i\eta\theta M^n$, and the Leibnitz rule

$$\begin{aligned} \mathcal{A}_0 \left(|\Lambda''|^{-1} \phi_1 \phi_2 \phi_3 \right) &= |\Lambda''|^{-1} \phi_2 \phi_3 \mathcal{A}_0 \phi_1 \\ &\quad + |\Lambda''|^{-1} \phi_1 \phi_3 \mathcal{A}_0 \phi_2 + |\Lambda''|^{-1} \phi_1 \phi_2 \mathcal{A}_0 \phi_3 \end{aligned}$$

then we get

$$\begin{aligned} & (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M}\mathcal{V}(i\xi) \widehat{\varphi} \right)^3 \\ &= 3\mathcal{A}_0 \left(M^2 |\Lambda''|^{-1} \psi^2 \psi_1 + |\Lambda''|^{-1} (2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi_1}) \right) \\ &\quad - 3\mathcal{A}_0 \left(\overline{M}^2 |\Lambda''|^{-1} (\overline{\psi}^2 \psi_1 + 2\psi\overline{\psi}\overline{\psi_1}) + i\overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^2 \overline{\psi_1} \right) \\ &\quad + i\eta\theta\mathcal{A}_0 \left(M^2 |\Lambda''|^{-1} \psi^3 + 3i\psi^2\overline{\psi} - 3\overline{M}^2 |\Lambda''|^{-1} \psi\overline{\psi}^2 - i\overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^3 \right) \\ &= 6i\eta\theta M^2 |\Lambda''|^{-1} \psi^2 \psi_1 - 2\eta^2\theta M^2 |\Lambda''|^{-1} \psi^3 \\ &\quad + 6i\eta\theta\overline{M}^2 |\Lambda''|^{-1} (\overline{\psi}^2 \psi_1 + 2\psi\overline{\psi}\overline{\psi_1}) - 6\eta^2\theta\overline{M}^2 |\Lambda''|^{-1} \psi\overline{\psi}^2 \\ &\quad - 12\eta\theta\overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^2 \overline{\psi_1} - 4i\eta^2\theta\overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^3 + R_1, \end{aligned}$$

where

$$\begin{aligned} R_1 &= 3M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi^2 \psi_1 + 3\mathcal{A}_0 |\Lambda''|^{-1} (2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi_1}) \\ &\quad - 3\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} (\overline{\psi}^2 \psi_1 + 2\psi\overline{\psi}\overline{\psi_1}) - 3i\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^2 \overline{\psi_1} \\ &\quad + i\eta\theta M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi^3 - 3\eta\theta\mathcal{A}_0 |\Lambda''|^{-1} \psi^2\overline{\psi} \\ &\quad - 3i\eta\theta\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} \psi\overline{\psi}^2 + \eta\theta\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^3. \end{aligned}$$

By inequality (3.1)

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \mathcal{Q}R_1 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \|R_1\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq C \left\| |\eta|^{-1} \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi^2 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \quad + C \left\| |\eta|^{-1} \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}| |\mathcal{V}\xi^2 \widehat{\varphi}| \mathcal{A}_0 \mathcal{V}\xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \quad + C \left\| \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

By Lemma 2.2 with $\alpha = -\frac{1}{2}$, $j = 1, 2$ we have $\left| \mathcal{V}(i\xi)^j \widehat{\varphi} \right| \leq C |\eta|^j |\widehat{\varphi}| + C |\eta|^{\frac{1}{2}} \langle \eta \rangle^{j-\frac{9}{4}} t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}$. Using Lemma 3.5 with $\alpha = 0$ and $\alpha = 1$ we obtain for $j = 1, 2$

$$\left\| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V}\xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t^{\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1} \leq C \varepsilon t^{\frac{1}{6}}$$

and

$$\left\| |\eta| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V}\xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\widehat{\varphi}(0)| \leq C \varepsilon t^\gamma.$$

Therefore

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \mathcal{Q}R_1 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t^{-1} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^2 \left\| |\eta| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V}\xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \quad + C t^{-\frac{4}{3}+2\gamma} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^2 \left\| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V}\xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

Using the relation $\psi_1 = \mathcal{V}(i\xi)^2 \widehat{\varphi} = \mathcal{A}\mathcal{V}(i\xi) \widehat{\varphi} = i\eta\theta\mathcal{V}(i\xi) \widehat{\varphi} + \mathcal{A}_0\mathcal{V}(i\xi) \widehat{\varphi} = i\eta\theta\psi + \mathcal{A}_0\psi$ we get

$$\begin{aligned}
& (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\
& = -8\eta^2\theta M^2 |\Lambda''|^{-1} \psi^3 + 8i\eta^2\theta \overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^3 + R_2,
\end{aligned}$$

where

$$\begin{aligned}
R_2 &= 6i\eta\theta M^2 |\Lambda''|^{-1} \psi^2 \mathcal{A}_0\psi \\
& \quad + 6i\eta\theta \overline{M}^2 |\Lambda''|^{-1} \left(\overline{\psi}^2 (\mathcal{A}_0\psi) + 2\psi \overline{\psi} (\overline{\mathcal{A}_0\psi}) \right) \\
& \quad + 12i\eta\theta \overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^2 \overline{\mathcal{A}_0\psi} + R_1,
\end{aligned}$$

As above

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \mathcal{Q}R_2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq C \|\widehat{\varphi}\|_{\mathbf{Z}_1}^2 \left\| |\eta| \langle \eta \rangle^{-2} \mathcal{A}_0 \mathcal{V}\xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

In the same manner we use $\psi^3 = -i\eta^3\theta(\mathcal{V}\widehat{\varphi})^3 + R_3$, with

$$\begin{aligned} R_3 = & -\eta^2\theta(\mathcal{V}\widehat{\varphi})^2(\mathcal{A}_0\mathcal{V}\widehat{\varphi}) + i\eta\theta(\mathcal{V}i\xi\widehat{\varphi})(\mathcal{V}\widehat{\varphi})(\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \\ & + (\mathcal{V}i\xi\widehat{\varphi})^2(\mathcal{A}_0\mathcal{V}\widehat{\varphi}), \end{aligned}$$

which can be estimates as

$$\begin{aligned} & \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \mathcal{Q}\eta^2\theta M^2 |\Lambda''|^{-1} R_3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \eta \langle \eta \rangle^{-2} R_3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq & C \left\| \eta^3 \langle \eta \rangle^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C \left\| \eta^2 \langle \eta \rangle^{-2} (\mathcal{V}i\xi\widehat{\varphi})(\mathcal{V}\widehat{\varphi})(\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C \left\| \eta \langle \eta \rangle^{-2} (\mathcal{V}i\xi\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq & Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{Z}}^2 \|\eta^2 t \mathcal{A}_0 \mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}, \end{aligned}$$

since by Lemma 3.5 with $\alpha = 2$, $j = 0$ we have

$$\|\eta^2 t \mathcal{A}_0 \mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} + C |\widehat{\varphi}(0)| \leq C\varepsilon t^\gamma.$$

Therefore

$$\begin{aligned} & (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\ = & 8i\eta^5\theta M^2 |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 - 8\eta^5\theta \overline{M}^4 |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 \\ & - 8\eta^2\theta M^2 |\Lambda''|^{-1} R_3 + 8i\eta^2\theta \overline{M}^4 |\Lambda''|^{-1} \overline{R_3} + R_2. \end{aligned}$$

Next consider

$$\begin{aligned} & \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\ = & 8i\mathcal{Q}M^2\eta^5\theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 - 8\mathcal{Q}\overline{M}^4\eta^5\theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + R_4, \end{aligned}$$

where the remainder

$$R_4 = -8\mathcal{Q}\eta^2\theta M^2 |\Lambda''|^{-1} R_3 + 8i\mathcal{Q}\eta^2\theta \overline{M}^4 |\Lambda''|^{-1} \overline{R_3} + \mathcal{Q}R_2$$

has the estimate $\left\| \langle \xi \rangle^{-4} (a + b\xi^2) R_4 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}$. We use the identities $\mathcal{Q}(t) M^2 \phi = i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \phi$ and $\mathcal{Q}(t) \overline{M}^4 \phi = i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \phi$, where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$. Then we get

$$\begin{aligned} & \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\ = & 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \eta^5\theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & - 8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \eta^5\theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + R_4. \end{aligned}$$

Define $\chi_1 \in \mathbf{C}^\infty(\mathbf{R})$ such that $\chi_1(x) = 1$ for $x \leq 1$ and $\chi_1(x) = 0$ for $x \geq 2$, $\chi_2(x) = 1 - \chi_1(x)$ and write

$$\begin{aligned} & \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} \left(M\mathcal{V}(i\xi) \widehat{\varphi} + i\overline{M\mathcal{V}(i\xi) \widehat{\varphi}} \right)^3 \\ = & 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_3 \xi^3 \mathcal{Q}(3t) \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & - 8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} \xi^3 \mathcal{Q}(-3t) \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + R_5, \end{aligned}$$

where

$$\begin{aligned} R_5 = & 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \chi_1(t^\nu \eta) \eta^5 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & - 8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \chi_1(t^\nu \eta) \eta^5 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 \\ & + 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_3 [\xi^3, \mathcal{Q}(3t)] \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & - 8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} [\xi^3, \mathcal{Q}(-3t)] \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + R_4. \end{aligned}$$

Note that

$$\begin{aligned} & \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \mathcal{Q}(3t) \chi_1(t^\nu \eta) \eta^5 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq & C\varepsilon^3 t^{\frac{1}{2}} \int_0^{t^{-\nu}} \eta^{\frac{9}{2}} d\eta \leq C\varepsilon^3 t^{\frac{1}{2} - \frac{11}{2}\nu} \leq C\varepsilon^3 t^{\gamma-1} \end{aligned}$$

if we choose $\nu = \frac{3}{11}$. Also we write

$$\begin{aligned} & [\xi^3, \mathcal{Q}(3t)] \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ = & -\xi \mathcal{Q}(3t) \mathcal{A}_0^2 \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & -\xi \mathcal{Q}(3t) \eta \theta \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & -i\xi \mathcal{Q}(3t) \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ (5.7) \quad & -i\mathcal{Q}(3t) \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^4 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3. \end{aligned}$$

Using the identity $\mathcal{A} = \mathcal{A}_0 + i\eta\theta$ and the commutator $[\mathcal{A}_0, \mu] = \frac{1}{t|\Lambda''(\eta)|} \partial_\eta \mu$

by a direct calculation we get

$$\begin{aligned}
& \mathcal{A}_0^2 \chi_2 (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\
= & t^{2\nu-2} \chi_2'' (t^\nu \eta) \frac{\eta^2 \theta}{|\Lambda''|^3} (\mathcal{V}\widehat{\varphi})^3 + t^{\nu-2} \chi_2' (t^\nu \eta) \left(\frac{\eta^2 \theta}{|\Lambda''|} \right)' |\Lambda''|^{-2} (\mathcal{V}\widehat{\varphi})^3 \\
& + t^{\nu-2} \chi_2' (t^\nu \eta) \frac{2\eta\theta}{|\Lambda''|^3} (\mathcal{V}\widehat{\varphi})^3 + t^{-2} \chi_2 (t^\nu \eta) \left(\frac{2\eta\theta}{|\Lambda''|} \right)' |\Lambda''|^{-2} (\mathcal{V}\widehat{\varphi})^3 \\
& + 6t^{\nu-1} \chi_2' (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& + 6t^{-1} \chi_2 (t^\nu \eta) 2\eta\theta |\Lambda''|^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& + 3\chi_2 (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi}) (\mathcal{V}i\xi\widehat{\varphi}) (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& - 6i\chi_2 (t^\nu \eta) \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& - 3it^{-1} \chi_2 (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-2} (\mathcal{V}\widehat{\varphi})^3 - 3\chi_2 (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}i\xi\widehat{\varphi}).
\end{aligned}$$

Then we estimate

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \xi \mathcal{Q}(3t) \mathcal{A}_0^2 \chi_2 (t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq & Ct^{\nu-2} \left(t^\nu \left\| \chi_2'' (t^\nu \eta) \frac{\eta^2 \theta}{|\Lambda''|^3} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\| \chi_2' (t^\nu \eta) \left(\frac{\eta^2 \theta}{|\Lambda''|} \right)' \frac{1}{|\Lambda''|^2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right) \\
& \times \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 \\
+ & Ct^{\nu-2} \left(\left\| \chi_2' (t^\nu \eta) \frac{\eta\theta}{|\Lambda''|^3} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + t^{-2} \left\| \chi_2 (t^\nu \eta) \left(\frac{2\eta\theta}{|\Lambda''|} \right)' \frac{1}{|\Lambda''|^2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right) \\
& \times \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 \\
+ & Ct^{\nu-1} \left\| \chi_2' (t^\nu \eta) \eta^{\frac{1}{2}} \theta |\Lambda''|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^2 \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ & Ct^{-1} \left\| \chi_2 (t^\nu \eta) \eta^{-\frac{1}{2}} \theta |\Lambda''|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^2 \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ & C \left\| \chi_2 (t^\nu \eta) \eta\theta |\Lambda''|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{-\frac{1}{2}} \mathcal{V}i\xi\widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ & C \left\| \eta\theta |\Lambda''|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^2 \left\| \eta^2 \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ & Ct^{-1} \left\| \eta^2 \theta |\Lambda''|^{-2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 \\
+ & C \left\| \eta\theta |\Lambda''|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^2 \|\eta \mathcal{A}_0 \mathcal{V}i\xi\widehat{\varphi}\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

In the same manner we estimate

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-4} (a + b\xi^2) \xi \mathcal{Q}(3t) \eta \theta \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + \left\| \langle \xi \rangle^{-4} (a + b\xi^2) i\xi \mathcal{Q}(3t) \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + \left\| \langle \xi \rangle^{-4} (a + b\xi^2) i\mathcal{Q}(3t) \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^4 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq C t^{\nu-1} \left\| \chi_2'(t^\nu \eta) \eta^3 \theta |\Lambda''|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 \\
& + C t^{-1} \left\| \chi_2(t^\nu \eta) \eta^2 |\Lambda''|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^3 \\
& + C \left\| \chi_2(t^\nu \eta) \eta^{\frac{3}{2}} \theta |\Lambda''|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)}^2 \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& \leq C \varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

Thus we get $\left\| \langle \xi \rangle^{-4} (a + b\xi^2) R_5 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma-1}$. Therefore we obtain the equation $\partial_t \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} = \xi^3 e^{it\Omega} \Phi + O(\varepsilon^3 t^{\gamma-1})$, where

$$\begin{aligned}
\Phi &= \langle \xi \rangle^{-4} (a + b\xi^2) 8i^{\frac{3}{2}} 3^{-3} \mathcal{D}_3 \mathcal{Q}(3t) \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\
&+ \langle \xi \rangle^{-4} (a + b\xi^2) 8i^{\frac{1}{2}} 3^{-3} \mathcal{D}_{-3} \mathcal{Q}(-3t) \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3.
\end{aligned}$$

In the first summand of the right-hand side of the above equation we represent

$$\xi^3 e^{it\Omega} \Phi = \partial_t \left(\frac{\xi^3}{i\Omega} e^{it\Omega} \Phi \right) - \frac{\xi^3}{i\Omega} e^{it\Omega} \partial_t \Phi.$$

Hence

$$\partial_t \left(\langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} - \frac{\xi^3}{i\Omega} e^{it\Omega} \Phi \right) = -\frac{\xi^3}{i\Omega} e^{it\Omega} \partial_t \Phi + O(\varepsilon^3 t^{\gamma-1}).$$

By Lemma 2.2

$$(5.8) \quad \left\| \frac{\xi^3}{i\Omega} e^{it\Omega} \Phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3.$$

Also we find

$$\begin{aligned}
& \left\| \langle \xi \rangle^{-2} \partial_t \Phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \langle \xi \rangle^{-4} (\partial_t \mathcal{Q}(3t)) \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + C t^{\nu-1} \left\| \chi_2'(t^\nu \eta) \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + C \left\| \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\partial_t \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

By the second estimate of Lemma 3.1, as above in (5.7), we have

$$\begin{aligned} & \left\| \langle \xi \rangle^{-4} \partial_t \mathcal{Q}(3t) \phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \mathcal{A}_0^2 \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + \left\| \eta \theta \mathcal{A}_0 \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + \sum_{l=1}^4 \left\| \mathcal{A}_0 \eta^l \theta \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma-1}. \end{aligned}$$

Then by Lemma 3.6 and Lemma 2.2 we find

$$\begin{aligned} & \left\| \chi_2(t^\nu \eta) \eta^2 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\partial_t \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & \leq C \left\| \langle \eta \rangle \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R})}^2 \left\| \eta \langle \eta \rangle^{-3} \partial_t \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \varepsilon^3 t^{\gamma-1}. \end{aligned}$$

Hence we get $\partial_t \left(\langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} - \frac{\xi^3}{i\Omega} e^{it\Omega} \Phi \right) = O(\varepsilon^3 t^{\gamma-1})$. Integrating in time we find

$$(5.9) \quad \left\| \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2} \leq \varepsilon + C \varepsilon^3 t^\gamma.$$

We next consider the estimate of $\|\partial_x \mathcal{I}u(t)\|_{\mathbf{L}^2} = \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2}$. In order to avoid the derivative loss we consider the modified dilation operator $\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a$. Then $\mathcal{P} = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I}$, where $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$. Note that the commutators $[\mathcal{P}, \mathcal{L}] = -\mathcal{L}$ and $[\mathcal{P}, \partial_x] = -\frac{1}{5}\partial_x$ are true. Also we have $[\mathcal{I}, \mathcal{L}] = 0$. Note that

$$\begin{aligned} \mathcal{I}u &= \mathcal{I}\mathcal{U}(t) \mathcal{F}^{-1} \widehat{\varphi} = -\mathcal{F}^{-1} \left(\partial_a + \frac{1}{3}it\xi^3 \right) E\widehat{\varphi} \\ &= -\mathcal{F}^{-1} E \partial_a \widehat{\varphi} = \mathcal{F}^{-1} E \widetilde{\mathcal{I}} \widehat{\varphi} = \mathcal{U}(t) \mathcal{F}^{-1} \widetilde{\mathcal{I}} \widehat{\varphi}, \end{aligned}$$

where $\widetilde{\mathcal{I}} = -\partial_a$. Hence $\|\mathcal{I}u\|_{\mathbf{L}^2} = \|\widetilde{\mathcal{I}} \widehat{\varphi}\|_{\mathbf{L}^2}$. To estimate $\|\mathcal{P}u\|_{\mathbf{L}^2}$ we apply $\mathcal{P}$ to equation (1.6) to obtain

$$\mathcal{L}\mathcal{P}u = (\mathcal{P} + 1) \mathcal{L}u = (\mathcal{P} + 1) u_x^3 = 3u_x^2 \mathcal{P}\partial_x u + u_x^3.$$

Hence

$$\begin{aligned} \frac{d}{dt} \|\mathcal{P}u\|_{\mathbf{L}^2}^2 &= 3(\mathcal{P}u, u_x^2 \mathcal{P}\partial_x u) + (\mathcal{P}u, u_x^3) \\ &\leq C \|u_x u_{xx}\|_{\mathbf{L}^\infty} \|\mathcal{P}u\|_{\mathbf{L}^2}^2 + \|\mathcal{P}u\|_{\mathbf{L}^2} \|u_x\|_{\mathbf{L}^6}^3. \end{aligned}$$

We have

$$\begin{aligned} |u_x| + |u_{xx}| &= |2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} i \xi \widehat{\varphi}| + |2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} (i\xi)^2 \widehat{\varphi}| \\ &\leq C t^{-\frac{1}{2}} |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-1} \left(|\mathcal{V} i \xi \widehat{\varphi}| + |\mathcal{V} (i\xi)^2 \widehat{\varphi}| \right). \end{aligned}$$

Therefore by the second estimate of Lemma 2.2

$$|u_x| + |u_{xx}| \leq Ct^{-\frac{1}{2}-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1} \leq Ct^{-\frac{1}{2}-\frac{1}{6}} \left(\varepsilon + \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \right)$$

for $x < 0$ and by

$$|\mathcal{V}\xi^j \widehat{\varphi}| \leq C |\eta| |\eta^{j-1} \widehat{\varphi}| + C |\eta|^{\frac{1}{2}} \langle \eta \rangle^{-\frac{1}{4}} t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}$$

which follows from the first estimate of Lemma 2.2

$$|u_x| + |u_{xx}| \leq C\varepsilon t^{-\frac{1}{2}} + Ct^{-\frac{1}{2}-\frac{1}{6}} \left(\varepsilon + \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \right)$$

for $x \geq 0$. We can use the estimates $\|u_x u_{xx}\|_{\mathbf{L}^\infty} \leq Ct^{-1} \left(\varepsilon^2 + t^{-\frac{1}{3}} \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2}^2 \right)$ and

$$\|u_x\|_{\mathbf{L}^6}^3 \leq \|u_x\|_{\mathbf{L}^\infty}^2 \|u_x\|_{\mathbf{L}^2} \leq Ct^{-1} \left(\varepsilon^2 + t^{-\frac{1}{3}} \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2}^2 \right) \|u_x\|_{\mathbf{L}^2}.$$

Then we have

$$\begin{aligned} \frac{d}{dt} \|\mathcal{P}u\|_{\mathbf{L}^2} &\leq Ct^{-1} \left(\varepsilon^2 + t^{-\frac{1}{3}} \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2}^2 \right) \|\mathcal{P}u\|_{\mathbf{L}^2} \\ (5.10) \quad &+ Ct^{-1} \left(\varepsilon^2 + t^{-\frac{1}{3}} \|\langle \xi \rangle \widehat{\varphi}_\xi\|_{\mathbf{L}^2}^2 \right) \|u_x\|_{\mathbf{L}^2} \end{aligned}$$

from which it follows that

$$(5.11) \quad \|\mathcal{P}u\|_{\mathbf{L}^2} \leq \varepsilon + C\varepsilon^3 t^\gamma.$$

We have

$$\begin{aligned} \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} &= \|\partial_x \mathcal{I}u\|_{\mathbf{L}^2} \leq 5 \|\mathcal{P}u\|_{\mathbf{L}^2} + t \|\mathcal{L}u\|_{\mathbf{L}^2} + \frac{2a}{5} \|\mathcal{I}u\|_{\mathbf{L}^2} \\ &\leq 5 \|\mathcal{P}u\|_{\mathbf{L}^2} + Ct \|u_x\|_{\mathbf{L}^6}^3 + C \|\mathcal{I}u\|_{\mathbf{L}^2} \\ (5.12) \quad &\leq 5\varepsilon + C\varepsilon^3 t^\gamma + C \|\mathcal{I}u\|_{\mathbf{L}^2}. \end{aligned}$$

To estimate the operator $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$ we use the commutator $[\mathcal{I}, \mathcal{L}] = 0$, then we get

$$\begin{aligned} \mathcal{L}\mathcal{I}u &= \mathcal{I}\mathcal{L}u = \mathcal{I}u_x^3 = -\partial_a u_x^3 + \frac{1}{3}t\partial_x^3 u_x^3 \\ &= -3u_x^2 \partial_a u_x + t\partial_x^2 (u_x^2 u_{xx}) \\ &= 3u_x^2 \mathcal{I}\partial_x u + t(\partial_x^2 (u_x^2 u_{xx}) - u_x^2 \partial_x^4 u). \end{aligned}$$

Consider the estimate of the term $N = t(\partial_x^2(u_x^2 u_{xx}) - u_x^2 \partial_x^4 u)$. Using the factorization formula as above we find with

$$\begin{aligned}
& \mathcal{F}\mathcal{U}(-t)N = -t\xi^2 \mathcal{F}\mathcal{U}(-t)(u_x^2 u_{xx}) - t\mathcal{F}\mathcal{U}(-t)(u_x^2 \partial_x^4 u) \\
&= -\xi^2 \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}(u_x^2 u_{xx}) - \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}(u_x^2 \partial_x^4 u) \\
&= -(it)^{-1}\xi^2 \mathcal{Q}|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) \\
&\quad - (it)^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M\mathcal{V}(i\xi)^4\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)^4\widehat{\varphi}}\right) \\
&= -(it)^{-1}[\xi^2, \mathcal{Q}]|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) + R_6 \\
&= -\mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0)|\Lambda''|^{-1}\overline{M}(M\psi + i\overline{M\psi})^2(M\psi_1 + i\overline{M\psi_1}) + R_6,
\end{aligned}$$

where $\psi = \mathcal{V}(i\xi)\widehat{\varphi}$, $\psi_1 = \mathcal{V}(i\xi)^2\widehat{\varphi}$, $\psi_2 = \mathcal{V}(i\xi)^3\widehat{\varphi}$

$$\begin{aligned}
R_6 &= -(it)^{-1}\mathcal{Q}\eta^2|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) \\
&\quad - (it)^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M\mathcal{V}(i\xi)^4\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)^4\widehat{\varphi}}\right) \\
&= -(it)^{-1}\mathcal{Q}|\Lambda''|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
&\quad \times \left(M[\eta^2, \mathcal{V}](i\xi)^2\widehat{\varphi} + i\overline{M[\eta^2, \mathcal{V}](i\xi)^2\widehat{\varphi}}\right).
\end{aligned}$$

We have by (3.1), (4.1) and Lemma 3.7

$$\begin{aligned}
\|R_6\|_{\mathbf{L}^2} &\leq Ct^{-1}\left\|\left|\Lambda''\right|^{-1}|\mathcal{V}\xi\widehat{\varphi}|^2([\eta^2, \mathcal{V}]\xi^2\widehat{\varphi})\right\|_{\mathbf{L}^2(\mathbf{R})} \\
&\leq C\varepsilon^2t^{-1}\left\|\left|\eta\right|\langle\eta\rangle^{-4}[\eta, \mathcal{V}]\xi^2\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R})} \\
&\quad + C\varepsilon^2t^{-\frac{4}{3}}\left\|\langle\eta\rangle^{-3}[\eta, \mathcal{V}]\xi^3\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R})} \leq C\varepsilon^3t^{\gamma-1}.
\end{aligned}$$

As above we write

$$\begin{aligned}
& (\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\
= & 2\mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi}) (M\psi_1 + i\overline{M}\overline{\psi}_1)^2 \\
& + \mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_2 + i\overline{M}\overline{\psi}_2) \\
& + i\eta\theta \mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1).
\end{aligned}$$

Then by the commutator $[\mathcal{A}_0, M^n] = int\eta\theta M^n$, and the Leibnitz rule

$$\begin{aligned}
& \mathcal{A}_0 \left(|\Lambda''|^{-1} \phi_1 \phi_2 \phi_3 \right) \\
= & |\Lambda''|^{-1} \phi_2 \phi_3 \mathcal{A}_0 \phi_1 + |\Lambda''|^{-1} \phi_1 \phi_3 \mathcal{A}_0 \phi_2 + |\Lambda''|^{-1} \phi_1 \phi_2 \mathcal{A}_0 \phi_3
\end{aligned}$$

we get

$$\begin{aligned}
& (\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\
= & 2\mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi}) (M\psi_1 + i\overline{M}\overline{\psi}_1)^2 \\
& + \mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_2 + i\overline{M}\overline{\psi}_2) \\
& + i\eta\theta \mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1),
\end{aligned}$$

$$\begin{aligned}
& 2\mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi}) (M\psi_1 + i\overline{M}\overline{\psi}_1)^2 \\
= & 4it\eta\theta |\Lambda''|^{-1} M^2 \psi \psi_1^2 + 4it\eta\theta |\Lambda''|^{-1} \overline{M}^2 (\psi \overline{\psi}_1^2 + 2\overline{\psi} \psi_1 \overline{\psi}_1) \\
& - 4t\eta\theta |\Lambda''|^{-1} \overline{M}^4 \overline{\psi} \psi_1^2 \\
& + 2M^2 \mathcal{A}_0 |\Lambda''|^{-1} \psi \psi_1^2 + \mathcal{A}_0 |\Lambda''|^{-1} (i\overline{\psi} \psi_1^2 + 2i\psi \psi_1 \overline{\psi}_1) \\
& - 2\overline{M}^2 \mathcal{A}_0 |\Lambda''|^{-1} (\psi \overline{\psi}_1^2 + 2\overline{\psi} \psi_1 \overline{\psi}_1) - i\overline{M}^4 \mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi} \psi_1^2,
\end{aligned}$$

$$\begin{aligned}
& \mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_2 + i\overline{M}\overline{\psi}_2) \\
= & 2it\eta\theta |\Lambda''|^{-1} M^2 \psi^2 \psi_2 + 2it\eta\theta |\Lambda''|^{-1} \overline{M}^2 (\overline{\psi}^2 \psi_2 + 2\psi \overline{\psi} \overline{\psi}_2) \\
& - 4t\eta\theta |\Lambda''|^{-1} \overline{M}^4 \overline{\psi}^2 \overline{\psi}_2 \\
& + M^2 \mathcal{A}_0 |\Lambda''|^{-1} \psi^2 \psi_2 + \mathcal{A}_0 |\Lambda''|^{-1} (2i\psi \overline{\psi} \psi_2 + i\psi^2 \overline{\psi}_2) \\
& - \overline{M}^2 \mathcal{A}_0 |\Lambda''|^{-1} (\overline{\psi}^2 \psi_2 + 2\psi \overline{\psi} \overline{\psi}_2) - i\overline{M}^4 \mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^2 \overline{\psi}_2,
\end{aligned}$$

$$\begin{aligned}
& i\eta\theta\mathcal{A}_0 |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\
= & -2t\eta^2\theta |\Lambda''|^{-1} M^2\psi^2\psi_1 - 2t\eta^2\theta |\Lambda''|^{-1} \overline{M}^2 (\overline{\psi}^2\psi_1 + 2\psi\overline{\psi}\overline{\psi}_1) \\
& -4it\eta^2\theta |\Lambda''|^{-1} \overline{M}^4\overline{\psi}^2\overline{\psi}_1 \\
& +i\eta\theta M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi^2\psi_1 + i\eta\theta\mathcal{A}_0 |\Lambda''|^{-1} (2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi}_1) \\
& -i\eta\theta\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} (\overline{\psi}^2\psi_1 + 2\psi\overline{\psi}\overline{\psi}_1) + \eta\theta\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^2\overline{\psi}_1.
\end{aligned}$$

Hence

$$\begin{aligned}
& (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\
= & 2it\eta\theta M^2 |\Lambda''|^{-1} (2\psi\psi_1^2 + \psi^2\psi_2 + i\eta\psi^2\psi_1) + 2it\eta\theta\overline{M}^2 \\
& \times |\Lambda''|^{-1} (2\psi\overline{\psi}_1^2 + 4\overline{\psi}\psi_1\overline{\psi}_1 + \overline{\psi}^2\psi_2 + 2\psi\overline{\psi}\overline{\psi}_2 + i\eta\overline{\psi}^2\psi_1 + 2i\eta\psi\overline{\psi}\overline{\psi}_1) \\
& -4t\eta\theta\overline{M}^4 |\Lambda''|^{-1} (\overline{\psi}\psi_1^2 + \overline{\psi}^2\overline{\psi}_2 + i\eta\overline{\psi}^2\overline{\psi}_1) + R_7,
\end{aligned}$$

where

$$\begin{aligned}
R_7 = & 2M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi\psi_1^2 + \mathcal{A}_0 |\Lambda''|^{-1} (i\overline{\psi}\psi_1^2 + 2i\psi\psi_1\overline{\psi}_1) \\
& -2\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} (\psi\overline{\psi}_1^2 + 2\overline{\psi}\psi_1\overline{\psi}_1) - i\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}\psi_1^2 \\
& +M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi^2\psi_2 + \mathcal{A}_0 |\Lambda''|^{-1} (2i\psi\overline{\psi}\psi_2 + i\psi^2\overline{\psi}_2) \\
& -\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} (\overline{\psi}^2\psi_2 + 2\psi\overline{\psi}\overline{\psi}_2) - i\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^2\overline{\psi}_2 \\
& +i\eta\theta M^2\mathcal{A}_0 |\Lambda''|^{-1} \psi^2\psi_1 + i\eta\theta\mathcal{A}_0 |\Lambda''|^{-1} (2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi}_1) \\
& -i\eta\theta\overline{M}^2\mathcal{A}_0 |\Lambda''|^{-1} (\overline{\psi}^2\psi_1 + 2\psi\overline{\psi}\overline{\psi}_1) + \eta\theta\overline{M}^4\mathcal{A}_0 |\Lambda''|^{-1} \overline{\psi}^2\overline{\psi}_1.
\end{aligned}$$

As above

$$\begin{aligned}
& \|R_7\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| |\eta|^{-1} \langle \eta \rangle^{-2} |\mathcal{V}\xi \langle \xi \rangle \widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi^2 \langle \xi \rangle \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + C \left\| \langle \eta \rangle^{-2} |\mathcal{V}\xi \langle \xi \rangle \widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi \langle \xi \rangle \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq & Ct^{-1} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^2 \left\| |\eta| \langle \eta \rangle^{-2} t\mathcal{A}_0 \mathcal{V}\xi^2 \langle \xi \rangle \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
& + Ct^{-\frac{4}{3}+2\gamma} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^2 \left\| \langle \eta \rangle^{-2} t\mathcal{A}_0 \mathcal{V}\xi^2 \langle \xi \rangle \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq & C\varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

Using the relations $\psi_1 = i\eta\theta\psi + \mathcal{A}_0\psi$, $\psi_1^2 = -\eta^2\theta\psi^2 + i\eta\theta\psi\mathcal{A}_0\psi + \psi_1\mathcal{A}_0\psi$ and

$$\psi_2 = \mathcal{V}(i\xi)^3 \widehat{\varphi} = -\eta^2\theta\psi + i\eta\theta\mathcal{A}_0\psi + \mathcal{A}_0\psi_1,$$

we get

$$\begin{aligned} & (\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\ &= -8i\eta^3\theta M^2 |\Lambda''|^{-1} \psi^3 + 4\eta^3\theta \overline{M}^4 |\Lambda''|^{-1} \overline{\psi}^3 + R_8, \end{aligned}$$

where

$$\begin{aligned} R_8 &= 2i\eta\theta M^2 |\Lambda''|^{-1} (2\psi (i\eta\theta\psi\mathcal{A}_0\psi + \psi_1\mathcal{A}_0\psi) \\ &\quad + \psi^2 (i\eta\theta\mathcal{A}_0\psi + \mathcal{A}_0\psi_1) + i\eta\psi^2 (\mathcal{A}_0\psi)) \\ &+ 2i\eta\theta \overline{M}^2 |\Lambda''|^{-1} \left(\begin{aligned} &2\psi (-i\eta\theta\overline{\psi}\mathcal{A}_0\overline{\psi} + \overline{\psi}_1\mathcal{A}_0\overline{\psi}) + 4\overline{\psi} (i\eta\theta\psi\overline{\mathcal{A}_0\psi} + \overline{\psi}_1\mathcal{A}_0\overline{\psi}) \\ &+ \overline{\psi}^2 (i\eta\theta\mathcal{A}_0\overline{\psi} + \mathcal{A}_0\overline{\psi}_1) + 2\psi\overline{\psi} (-i\eta\theta\overline{\mathcal{A}_0\psi} + \overline{\mathcal{A}_0\psi}_1) \\ &+ i\eta\overline{\psi}^2 \mathcal{A}_0\overline{\psi} + 2i\eta\psi\overline{\psi} (\overline{\mathcal{A}_0\psi}) \end{aligned} \right) \\ &- 4\eta\theta \overline{M}^4 |\Lambda''|^{-1} (\overline{\psi} (-i\eta\theta\overline{\psi}\mathcal{A}_0\overline{\psi} + \overline{\psi}_1\mathcal{A}_0\overline{\psi}) \\ &+ \overline{\psi}^2 (-i\eta\theta\overline{\mathcal{A}_0\psi} + \overline{\mathcal{A}_0\psi}_1) + i\eta\overline{\psi}^2 (\overline{\mathcal{A}_0\psi})) \\ &+ R_7. \end{aligned}$$

We have

$$\begin{aligned} \|R_8\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C \left\| \eta \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C \left\| \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}| |\mathcal{V}\xi^2\widehat{\varphi}| \mathcal{A}_0 \mathcal{V}\xi\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+ C \left\| \langle \eta \rangle^{-2} |\mathcal{V}\xi\widehat{\varphi}|^2 \mathcal{A}_0 \mathcal{V}\xi^2\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}. \end{aligned}$$

Then applying $\psi^3 = -i\eta^3\theta (\mathcal{V}\widehat{\varphi})^3 + R_3$, with

$$R_3 = -\eta^2\theta (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) + i\eta\theta (\mathcal{V}i\xi\widehat{\varphi}) (\mathcal{V}\widehat{\varphi}) (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) + (\mathcal{V}i\xi\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi})$$

we obtain

$$\begin{aligned} & (\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\ &= -8\eta^6\theta M^2 |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 + 8i\eta^6\theta \overline{M}^4 |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + R_9, \end{aligned}$$

with $R_9 = -8i\eta^3\theta M^2 |\Lambda''|^{-1} R_3 + 4\eta^3\theta \overline{M}^4 |\Lambda''|^{-1} \overline{R_3} + R_8$. Then as above

$$\begin{aligned} & \mathcal{Q} (\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\overline{\psi})^2 (M\psi_1 + i\overline{M}\overline{\psi}_1) \\ &= -8\mathcal{Q} M^2 \eta^6\theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 + 8i\mathcal{Q} \overline{M}^4 \eta^6\theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + \mathcal{Q} R_9 \\ &= -8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_3 \mathcal{Q} (3t) \eta^6\theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad + 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q} (-3t) \eta^6\theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + \mathcal{Q} R_9, \end{aligned}$$

where $\Omega = \Lambda\left(\frac{\xi}{3}\right) - 3\Lambda\left(\frac{\xi}{3}\right)$ and by Lemma 3.5

$$\begin{aligned} \|\mathcal{Q}R_9\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C \left\| \eta^2 \langle \eta \rangle^{-2} R_3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\leq C \left\| \eta^4 \langle \eta \rangle^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\quad + \left\| \eta^3 \langle \eta \rangle^{-2} (\mathcal{V}\widehat{\varphi}) (\mathcal{V}\xi\widehat{\varphi}) (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\quad + \left\| \eta^2 \langle \eta \rangle^{-2} (\mathcal{V}\xi\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}. \end{aligned}$$

Since $\frac{\xi^2}{1+\xi^2} + \frac{1}{1+\xi^2} = 1$ we write

$$\begin{aligned} &\mathcal{Q}(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\psi)^2 (M\psi_1 + i\overline{M}\psi_1) \\ &= -8t \mathcal{Q} M^2 \eta^6 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 + 8it \mathcal{Q} \overline{M}^4 \eta^6 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + \mathcal{Q}R_9 \\ &= -\frac{8i^{\frac{1}{2}} t \xi^2}{1+\xi^2} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \eta^6 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad + \frac{8i^{\frac{3}{2}} t \xi^2}{1+\xi^2} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \eta^6 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 \\ &\quad - \frac{8i^{\frac{1}{2}} t}{1+\xi^2} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \eta^6 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad + \frac{8i^{\frac{3}{2}} t}{1+\xi^2} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \eta^6 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + \mathcal{Q}R_9. \end{aligned}$$

Next we use $\mathcal{Q}(i\eta)^n \theta = (i\xi)^n \mathcal{Q} - \sum_{l=0}^{n-1} (i\xi)^{n-1-l} \mathcal{Q} \mathcal{A}_0 (i\eta)^l \theta$, then

$$\begin{aligned} &\mathcal{Q}(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0) |\Lambda''|^{-1} \overline{M} (M\psi + i\overline{M}\psi)^2 (M\psi_1 + i\overline{M}\psi_1) \\ &= \xi^3 e^{it\Omega} \Phi_1 + R_{10}, \end{aligned}$$

where

$$\begin{aligned} \Phi_1 &= \frac{8}{1+\xi^2} \left(\frac{i}{3}\right)^3 i^{\frac{3}{2}} \mathcal{D}_3 \mathcal{Q}(3t) \eta^5 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad + \frac{8}{1+\xi^2} i^{\frac{1}{2}} \left(\frac{i}{3}\right)^3 \mathcal{D}_{-3} \mathcal{Q}(-3t) \eta^5 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 \\ &\quad + \frac{8}{1+\xi^2} \left(\frac{i}{3}\right)^3 i^{\frac{3}{2}} \mathcal{D}_3 \mathcal{Q}(3t) \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad + \frac{8}{1+\xi^2} i^{\frac{1}{2}} \left(\frac{i}{3}\right)^3 \mathcal{D}_{-3} \mathcal{Q}(-3t) \eta^3 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 \end{aligned}$$

and

$$\begin{aligned} R_{10} &= -\frac{8}{1+\xi^2} \sum_{l=0}^2 (i\xi)^{2-l} 8i^{\frac{3}{2}} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \mathcal{A}_0 (i\eta)^l \eta^3 \theta |\Lambda''|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ &\quad - \frac{8}{1+\xi^2} \sum_{l=0}^2 (i\xi)^{2-l} 8i^{\frac{1}{2}} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \mathcal{A}_0 (i\eta)^l \eta^3 \theta |\Lambda''|^{-1} (\overline{\mathcal{V}\widehat{\varphi}})^3 + \mathcal{Q}R_9. \end{aligned}$$

We have

$$\begin{aligned} \|R_{10}\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C \left\| \eta^2 (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\quad + Ct^{-1} \left\| \eta \langle \eta \rangle^{-2} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}. \end{aligned}$$

In the same way as above we represent

$$\xi^3 e^{it\Omega} \Phi_1 = \partial_t \left(e^{it\Omega} \frac{\xi^3}{i\Omega} \Phi_1 \right) - e^{it\Omega} \frac{\xi^3}{i\Omega} \partial_t \Phi_1.$$

And using Lemma 3.1 and Lemma 3.6 we find

$$N = \mathcal{U}(t) \mathcal{F}^{-1} \partial_t \left(\frac{\xi^3}{i\Omega} e^{it\Omega} \Phi_1 \right) + O(\varepsilon^3 t^{\gamma-1}) = \mathcal{L}\Psi + O(\varepsilon^3 t^{\gamma-1}),$$

where we denote $\Psi = \mathcal{U}(t) \mathcal{F}^{-1} \left(\frac{\xi^3}{i\Omega} e^{it\Omega} \Phi_1 \right)$. Thus we get

$$\begin{aligned} \mathcal{L}\mathcal{I}u &= 3u_x^2 \mathcal{I} \partial_x u + \mathcal{U}(t) \mathcal{F}^{-1} (t\xi^3 e^{it\Omega} \Phi) + \mathcal{U}(t) \mathcal{F}^{-1} R_{10} \\ &= 3u_x^2 \mathcal{I} \partial_x u + \mathcal{U}(t) \mathcal{F}^{-1} \partial_t \left(e^{it\Omega} \frac{\xi^3}{i\Omega} \Phi_1 \right) \\ &\quad - \mathcal{U}(t) \mathcal{F}^{-1} e^{it\Omega} \frac{\xi^3}{i\Omega} \partial_t \Phi_1 + \mathcal{U}(t) \mathcal{F}^{-1} R_{10} \\ &= 3u_x^2 \mathcal{I} \partial_x u + O(\varepsilon^3 t^{-1}) - \mathcal{L}\Psi. \end{aligned}$$

Hence

$$\mathcal{L}(\mathcal{I}u + \Psi) = 3u_x^2 \partial_x (\mathcal{I}u + \Psi) + O(\varepsilon^3 t^{-1}) - 3u_x^2 \partial_x \Psi.$$

We have $\|\Psi\|_{\mathbf{L}^2} + \|\partial_x \Psi\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{\gamma-1}$, then we get $\|\mathcal{I}u\|_{\mathbf{L}^2} \leq C\varepsilon^3 t^{\gamma-1}$. Therefore by (5.12)

$$(5.13) \quad \|\xi \partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} \leq 5\varepsilon + C\varepsilon^3 t^\gamma.$$

We apply (5.9) and (5.13) to (5.6) to get $t^{-\gamma} \|\mathcal{I}u(t)\|_{\mathbf{H}^1} \leq 2(6\varepsilon + C\varepsilon^3) \leq 20\varepsilon$. This is the desired contradiction. Lemma 5.2 is proved. $\square$

Finally we estimate the norm $\|\mathcal{FU}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}}$.

Lemma 5.3. *Assume that $\sup_{t \in [1, T]} t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} \leq \varepsilon$ holds for small $\gamma > 0$. Then there exists an ε such that the estimate*

$$\sup_{t \in [1, T]} \|\mathcal{FU}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} = \sup_{t \in [1, T]} \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty} < 100\varepsilon$$

is true for all $T > 1$.

Proof. Assume that there exists a time $T > 0$ such that $\sup_{t \in [1, T]} \left\| \langle \xi \rangle^2 \widehat{\varphi} \right\|_{\mathbf{L}^\infty} = 100\varepsilon$. By equation (1.10) for $\widehat{\varphi} = \mathcal{FU}(-t)u(t)$, using Lemma 4.1 we get

$$\begin{aligned} \partial_t (i\xi)^j \widehat{\varphi}(t, \xi) &= \mathcal{FU}(-t) \partial_x^j (u_x)^3 \\ &= -i^{\frac{3}{2}} t^{-1} e^{it\Omega} \mathcal{D}_3 |\Lambda''(\xi)|^{-1} (i\xi)^{j+3} \widehat{\varphi}^3(t, \xi) \\ &\quad + 3it^{-1} \xi^3 |\Lambda''(\xi)|^{-1} |\widehat{\varphi}(t, \xi)|^2 (i\xi)^j \widehat{\varphi}(t, \xi) + O\left(\varepsilon^3 t^{-\frac{9}{8}}\right). \end{aligned}$$

Choosing $\Psi_1(t, \xi) = \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 \int_1^t |\widehat{\varphi}(\tau, \xi)|^2 \frac{d\tau}{\tau}\right)$, we get

$$\begin{aligned} &\partial_t \left((i\xi)^j \widehat{\varphi}(t, \xi) \Psi_1(t, \xi) \right) \\ &= -t^{-1} e^{it\Omega} (i\xi)^j \mathcal{D}_3 |\Lambda''(\xi)|^{-1} \xi^3 \widehat{\varphi}^3(t, \xi) \Psi_1(t, \xi) + O\left(\varepsilon^3 t^{-\frac{7}{6}+3\gamma}\right). \end{aligned}$$

Integrating in time, we obtain

$$\begin{aligned} &\left| (i\xi)^j \widehat{\varphi}(t, \xi) \Psi_1(t, \xi) \right| \\ &\leq \left| (i\xi)^j \widehat{\varphi}(1, \xi) \right| + C \left| \int_1^t e^{i\tau\Omega} (i\xi)^j \mathcal{D}_3 |\Lambda''(\xi)|^{-1} \xi^3 \widehat{\varphi}^3(\tau, \xi) \Psi_1(\tau, \xi) \frac{d\tau}{\tau} \right| \\ &\quad + O(\varepsilon^3). \end{aligned}$$

Integrating by parts we get $\left| (i\xi)^j \widehat{\varphi}(t, \xi) \right| \leq \varepsilon + O(\varepsilon^3) < 2\varepsilon$ for $\xi > 0$. Since the solution u is real, we have $\overline{\widehat{\varphi}(t, \xi)} = \widehat{\varphi}(t, -\xi)$. Therefore $\|\mathcal{FU}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} < 10\varepsilon$. This is the desired contradiction. Lemma 5.3 is proved. $\square$

§6. Proof of Theorem 1.1

By Lemma 5.2 and Lemma 5.3, we see that a priori estimate

$$(6.1) \quad \sup_{t \in [1, T]} t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} + \sup_{t \in [1, T]} \|\mathcal{FU}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} \leq C\varepsilon$$

is true for all $T > 0$. We also get $\sup_{t \in [1, T]} t^{-\gamma} \|u(t)\|_{\mathbf{H}^3} \leq C\varepsilon$ by the energy method and (6.1). Therefore the global existence of solutions of the Cauchy problem (1.6) follows by a standard continuation argument by the local existence Theorem 5.1. Thus we have global in time of solutions to the Cauchy problem (1.6). Time decay of solutions in $\mathbf{L}^\infty$ follows from Lemma 2.2.

Now we turn to the proof of the asymptotic formula (1.7) for the solutions u of the Cauchy problem (1.6). We need to compute the asymptotics of the function $\widehat{\varphi}(t, \xi)$. As in the proof of Lemma 5.2 we get

$$\begin{aligned} & \partial_t (\widehat{\varphi}(t, \xi) \Psi(t, \xi)) \\ &= -t^{-1} e^{it\Omega} \mathcal{D}_3 |\Lambda''(\xi)|^{-1} \xi^3 \widehat{\varphi}^3(t, \xi) \Psi(t, \xi) + O\left(\varepsilon^3 t^{-\frac{7}{6}+\gamma}\right). \end{aligned}$$

Integrating by parts implies for $y(t, \xi) = \widehat{\varphi}(t, \xi) \Psi(t, \xi)$

$$\begin{aligned} & y(t, \xi) - y(s, \xi) \\ &= \int_s^t e^{i\tau\Omega} \mathcal{D}_3 |\Lambda''(\xi)|^{-1} \xi^3 \widehat{\varphi}^3(\tau, \xi) \Psi(\tau, \xi) \frac{d\tau}{\tau} + O\left(\varepsilon^3 \int_s^t \tau^{-\frac{7}{6}+\gamma} d\tau\right). \end{aligned}$$

Hence we obtain

$$\|y(t) - y(s)\|_{\mathbf{L}^\infty} \leq C\varepsilon^3 \int_s^t \tau^{-\frac{7}{6}+\gamma} d\tau \leq C\varepsilon^3 s^{-\frac{1}{6}+\gamma}$$

for any $t > s > 0$. Therefore there exists a unique final state $y_+ \in \mathbf{L}^\infty$ such that

$$(6.2) \quad \|y(t) - y_+\|_{\mathbf{L}^\infty} \leq C\varepsilon^3 t^{-\frac{1}{6}+\gamma}$$

for all $t > 0$. We write

$$(6.3) \quad \int_1^t |\widehat{\varphi}(\tau, \xi)|^2 \frac{d\tau}{\tau} = \int_1^t |y(\tau, \xi)|^2 \frac{d\tau}{\tau} = |y_+|^2 \log t + \Phi_2(t).$$

We study the asymptotics in time of the remainder term $\Phi_2(t)$. We have

$$\Phi_2(t) - \Phi_2(s) = \int_s^t \left(|y(\tau)|^2 - |y(t)|^2 \right) \frac{d\tau}{\tau} + \left(|y(t)|^2 - |y_+|^2 \right) \log \frac{t}{s}.$$

By (6.2) we obtain $\|\Phi_2(t) - \Phi_2(s)\|_{\mathbf{L}^\infty} \leq C\varepsilon^3 s^{-\delta}$ for any $t > s > 0$. Hence there exists a unique real-valued function $\Phi_+ \in \mathbf{L}^\infty$ such that

$$(6.4) \quad \|\Phi_2(t) - \Phi_+\|_{\mathbf{L}^\infty} \leq C\varepsilon^3 t^{-\delta}$$

for all $t > 0$. Representation (6.3) and estimate (6.4) yield

$$\begin{aligned} & \left\| \Psi(t, \xi) - \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |y_+|^2 \log t + 3i |\Lambda''(\xi)|^{-1} \xi^3 \Phi_+\right) \right\|_{\mathbf{L}^\infty} \\ & \leq Ct^{-\frac{1}{6}+\gamma} \end{aligned}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\|\widehat{\varphi}(t, \xi) - y_+ \Psi(t, \xi)\|_{\mathbf{L}^\infty} = \|y(t) - y_+\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}+\gamma}$$

and

$$\begin{aligned} & \left\| y_+ \Psi(t, \xi) - y_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |y_+|^2 \log t + 3i |\Lambda''(\xi)|^{-1} \xi^3 \Phi_+\right) \right\|_{\mathbf{L}^\infty} \\ & \leq Ct^{-\frac{1}{6}+\gamma}. \end{aligned}$$

Therefore we obtain the estimate

$$\left\| \widehat{\varphi}(t, \xi) - W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}+\gamma}$$

with $W_+ = y_+ e^{3i |\Lambda''(\xi)|^{-1} \xi^3 \Phi_+}$. Similarly, we get

$$\left\| \xi \widehat{\varphi}(t, \xi) - \xi W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right\|_{\mathbf{L}^\infty} \leq Ct^{-\frac{1}{6}+\gamma}.$$

Using the factorization of $\mathcal{U}(t)$ we have

$$\begin{aligned} & \left\| \partial_x u(t) - 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} i \xi \left(W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right) \right\|_{\mathbf{L}^\infty} \\ & \leq C \left\| \mathcal{D}_t \mathcal{B} M \left(i \xi \widehat{\varphi}(t) - i \xi W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right) \right\|_{\mathbf{L}^\infty} \\ & \leq Ct^{-\frac{1}{2}-\frac{1}{6}+\gamma}. \end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned} & \left| 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} i \xi \left(W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right) \right| \\ & \leq Ct^{-\frac{1}{2}} |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-1} \left| \mathcal{V} i \xi \left(W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right) \right| \\ & \leq Ct^{-\frac{1}{2}-\frac{1}{6}} \left(\|W_+\|_{\mathbf{L}^\infty} + \left\| \xi \partial_\xi W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right\|_{\mathbf{L}^2} \right) \\ & \leq C \varepsilon t^{-\frac{1}{2}-\frac{1}{6}+\gamma} \end{aligned}$$

for $x < 0$ and

$$\begin{aligned} & \left| 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M \mathcal{V} i \xi \left(W_+ \exp\left(3i |\Lambda''(\xi)|^{-1} \xi^3 |W_+|^2 \log t\right) \right) \right. \\ & \quad \left. - 2\operatorname{Re} \mathcal{D}_t \mathcal{B} M i \eta \left(W_+ \exp\left(3i |\Lambda''(\eta)|^{-1} \eta^3 |W_+|^2 \log t\right) \right) \right| \\ & \leq C \varepsilon t^{-\frac{1}{2}-\frac{1}{6}+\gamma} \end{aligned}$$

for $x \geq 0$. This completes the proof of asymptotics (1.7). Theorem 1.1 is proved.

§7. Appendix

We collect definitions of operators used in this paper.

$$\begin{aligned}
\mathcal{L} &= \partial_t - \frac{a}{3}\partial_x^3 + \frac{b}{5}\partial_x^5, \\
(\mathcal{D}_t\phi)(x) &= (it)^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right), \mathcal{D}_t^{-1}\phi = (it)^{\frac{1}{2}}\phi(xt), \\
\mathcal{U}(t) &= \mathcal{F}^{-1}E\mathcal{F}, \quad E = E(t, \xi) = e^{-it\Lambda(\xi)}, \\
\mathcal{F}\phi &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x) dx, \quad \mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(\xi) d\xi, \\
\Lambda(\xi) &= \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5, \\
M &= M(t, \eta) = e^{it(\eta\Lambda'(\eta) - \Lambda(\eta))\theta(\eta)}, \\
\theta(x) &= 1 \text{ for } x \geq 0; \quad \theta(x) = 0 \text{ for } x < 0, \\
\mathcal{V}\phi &= \sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}} \int_0^\infty e^{-itS(\xi, \eta)} \phi(\xi) d\xi, \\
\mathcal{Q}\phi &= \sqrt{\frac{t}{2\pi i}} \int_{\mathbf{R}} e^{itS(\xi, \eta)} \phi(\eta) |\Lambda''(\eta)|^{\frac{1}{2}} d\eta, \\
A(t, \eta) &= \sqrt{\frac{it}{2\pi}} |\Lambda''(\eta)|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi, \eta)} \chi(\xi\eta^{-1}) d\xi, \\
A^*(t, \xi) &= \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi, \eta)} |\Lambda''(\eta)|^{\frac{1}{2}} \chi(\xi\eta^{-1}) d\eta, \\
\chi(z) &\in \mathbf{C}^2(\mathbf{R}) : \chi(z) = 0 \text{ for } z \leq \frac{1}{3}; \\
\chi(z) &= 1 \text{ for } \frac{2}{3} \leq z \leq \frac{3}{2}; \quad \chi(z) = 0 \text{ for } z \geq 3, \\
S(\xi, \eta) &= \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta), \\
(\mathcal{B}\phi)(x) &= |\Lambda''(\eta(x))|^{-\frac{1}{2}} \phi(\eta(x)), \\
(\mathcal{B}^{-1}\phi)(\eta) &= |\Lambda''(\eta)|^{\frac{1}{2}} \phi\left(\frac{\eta}{|\eta|} \Lambda'(\eta)\right), \\
\eta(x) &= \frac{x}{\sqrt{2b}|x|} \sqrt{\sqrt{4b|x| + a^2} - a}, \\
\mathcal{A} &= \frac{\overline{M}}{t|\Lambda''(\eta)|^{\frac{1}{2}}} \partial_\eta \frac{M}{|\Lambda''(\eta)|^{\frac{1}{2}}} = \mathcal{A}_0 + i\eta\theta(\eta),
\end{aligned}$$

$$\mathcal{A}_0 = \frac{1}{t |\Lambda''(\eta)|^{\frac{1}{2}}} \partial_\eta \frac{1}{|\Lambda''(\eta)|^{\frac{1}{2}}},$$

$$\mathcal{J} = x - t\Lambda'(-i\partial_x),$$

$$\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I},$$

$$\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3,$$

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in [1, T]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}_\infty^{0,2}} + t^{-\gamma} \|\mathcal{J}u(t)\|_{\mathbf{H}^1} \right).$$

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Nakao Hayashi
 Department of Mathematics
 Graduate School of Science, Osaka University, Toyonaka, Osaka, Japan.
E-mail: nhayashi@math.sci.osaka-u.ac.jp

Pavel I. Naumkin
 Centro de Ciencias Matemáticas

UNAM Campus Morelia, AP 61-3 (Xangari)

Morelia CP 58089, Michoacán, Mexico.

E-mail: `pavelni@matmor.unam.mx`