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Decomposition of diamond model for square contingency tables with ordered categories

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Abstract. For square contingency tables with the same row and column ordinal classifications, this paper shows that the diamond model holds if and only if the weighted covariance for the difference between the row and column classifications and the sum of them equals zero and the uniform association diamond model holds. An example is given.

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§1. Introduction

Consider an $R \times R$ square contingency table with the same row and column ordinal classifications. Let X and Y denote the row and column variables, respectively, and let p_{ij} denote the probability that an observation will fall in (i, j)th cell of the table ($i = 1, \ldots, R; j = 1, \ldots, R$). The independence model is defined by

$$p_{ij} = \mu\alpha_i\beta_j \quad \text{for } i = 1, \ldots, R; j = 1, \ldots, R.$$

Goodman (1979) refereed to this model as the null association model. The uniform association model is defined by

$$p_{ij} = \mu\alpha_i\beta_j\theta^{ij} \quad \text{for } i = 1, \ldots, R; j = 1, \ldots, R;$$

see Goodman (1979, 1981) and Agresti (1984, p.78). A special case of this model with $\theta = 1$ is the independence model. If the independence model holds, then the covariance between X and Y equals zero; but the converse does not hold. Tomizawa, Miyamoto and Sakurai (2008) gave the theorem that the independence model holds if and only if the covariance between X and Y equals zero and the uniform association model holds.

The diamond (DD) model (Goodman, 1985) is defined by

$$p_{ij} = \mu\delta_{i-j}\gamma_{i+j} \quad \text{for } i = 1, \ldots, R; j = 1, \ldots, R.$$

As described by Goodman (1985), the DD model states that there is null association between the difference-diagonal classification (i.e., the difference between the row and column classification) and the sum-diagonal classification (i.e., the sum of the row and column classification). Consider the $(2R-1) \times (2R-1)$ table of the diamond shape formed by rotating to the original $R \times R$ table forty-five degrees so that the $2R-1$ difference-diagonals in the original table form the entries in the rows of the diamond, and corresponding $2R-1$ sum-diagonals in the original table form the entries in the columns of the diamond. Let S^* denote a set of cells of the diamond shape in the $(2R-1) \times (2R-1)$ table. Thus,

$$S^* = \{(s,t) | s = i - j, t = i + j; i = 1, \ldots, R; j = 1, \ldots, R\}.$$

Let p^*_{st} denote the corresponding probability for row value s and column value t for $(s,t) \in S^*$, in the $(2R-1) \times (2R-1)$ table, i.e.,

$$p^*_{st} = p_{\frac{s+t}{2}, \frac{t-s}{2}} \quad \text{for } (s,t) \in S^*.$$

Let $\theta^*_{(k<l;s<t)}$ denote the odds ratio for row values k and l and column values s and t in the $(2R-1) \times (2R-1)$ table of the diamond shape. Thus,

$$\theta^*_{(k<l;s<t)} = \frac{p^*_{ks}p^*_{lt}}{p^*_{kt}p^*_{ls}} \quad \text{for } (k,s), (k,t), (l,s), (l,t) \in S^*.$$

Then, the DD model is also expressed as

$$\theta^*_{(k<l;s<t)} = 1 \quad \text{for } (k,s), (k,t), (l,s), (l,t) \in S^*.$$

For the original $R \times R$ table, the uniform association diamond (UADD) model is defined by

$$p_{ij} = \mu\delta_{i-j}\gamma_{i+j}\phi^{(i-j)(i+j)} \quad \text{for } i = 1, \ldots, R; j = 1, \ldots, R;$$

see Tomizawa (1994). A special case of this model with $\phi = 1$ is the DD model. Using the odds-ratios defined for the $(2R-1) \times (2R-1)$ table of the diamond shape, the UADD model is also expressed as

$$\theta^*_{(s<s+2;t<t+2)} = \phi^4 \quad \text{for } (s,t), (s,t+2), (s+2,t), (s+2,t+2) \in S^*.$$

Thus, the UADD model is uniform association model in $(2R-1)\times(2R-1)$ table of the diamond shape. If the DD model holds, then the UADD model holds; but the converse does not hold. Therefore, for the $(2R-1)\times(2R-1)$ table of the diamond shape, we are interested in what covariance structure between the difference-diagonal classification and sum-diagonal classification is necessary for obtaining the DD model, in addition to the UADD model.

The purpose of this paper is to define a covariance structure between the difference-diagonal classification and sum-diagonal classification, and shows that the DD model holds if and only if the covariance structure equals zero and the UADD model holds.

§2. Decomposition

Let the random variables U and V denote $U = X - Y$ and $V = X + Y$. For the $(2R-1)\times(2R-1)$ table of the diamond shape, we express p^*_{st} as $\mu\delta_s\gamma_t\psi_{st}$ for $(s,t)\in S^*$. We note that for the original $R\times R$ table, if we express p_{ij} as $\lambda\alpha_i\beta_j\omega_{ij}$ $(i = 1,\dots,R; j = 1,\dots,R)$, then we see

$$\mu = \lambda, \quad \delta_s = \alpha_{\frac{s+t}{2}}, \quad \gamma_t = \beta_{\frac{t-s}{2}}, \quad \psi_{st} = \omega_{\frac{s+t}{2},\frac{t-s}{2}},$$

namely,

$$p^*_{st} = \lambda\alpha_{\frac{s+t}{2}}\beta_{\frac{t-s}{2}}\omega_{\frac{s+t}{2},\frac{t-s}{2}}.$$

We express $\mathrm{P}(U = s, V = t \mid |U| = k)$ as $p^*_{st(k)}$ for $(s,t)\in S^*$ and $k = 0,1,\dots R-1$. Then we have

$$p^*_{st(k)} = \frac{\delta_s\gamma_t\psi_{st}}{\sum\sum_{(u,v)\in S^*_k}\delta_u\gamma_v\psi_{uv}} = \mu_k\delta_s\gamma_t\psi_{st},$$

where

$$S^*_k = \{(s,t)|s = i-j, t = i+j; |s| = k; i = 1,\dots,R; j = 1,\dots,R\},$$

$$\mu_k = \frac{1}{\sum\sum_{(u,v)\in S^*_k}\delta_u\gamma_v\psi_{uv}}.$$

We define the weighted covariance between U and V as

$$\mathrm{Cov}(U,V \mid |U|) = \sum_{k=1}^{R-2} w_k \mathrm{Cov}(U,V \mid |U| = k),$$

where $w_k > 0$ and $\sum_{k=1}^{R-2} w_k = 1$. For instance,

$$w_k = \frac{\sum\sum_{(u,v)\in S^*_k} p^*_{uv}}{\sum\sum_{(u,v)\in S^*-S^*_0-S^*_{R-1}} p^*_{uv}} \quad (k = 1,\dots,R-2),$$

or $\{w_k = 1/(R-2)\}$ is considered. Since the DD model is expressed as $p^*_{st} = \mu\delta_s\gamma_t$ for $(s,t) \in S^*$, under the DD model, we see

$$
\begin{aligned}
&\mathrm{Cov}(U, V \mid |U| = k) \\
&\quad = \mathrm{E}(UV \mid |U| = k) - \mathrm{E}(U \mid |U| = k)\mathrm{E}(V \mid |U| = k) \\
&\quad = \sum\sum_{(s,t)\in S^*_k} st\mu_k\delta_s\gamma_t - \Big(\sum\sum_{(s,t)\in S^*_k} s\mu_k\delta_s\gamma_t\Big)\Big(\sum\sum_{(s,t)\in S^*_k} t\mu_k\delta_s\gamma_t\Big) \\
&\quad = \mu_k\Big(\sum_s s\delta_s\Big)\Big(\sum_t t\gamma_t\Big) - \mu_k^2\Big(\sum_s s\delta_s\Big)\Big(\sum_t \gamma_t\Big)\Big(\sum_s \delta_s\Big)\Big(\sum_t t\gamma_t\Big) \\
&\quad = \mu_k\Big(\sum_s s\delta_s\Big)\Big(\sum_t t\gamma_t\Big) - \mu_k\Big(\sum_s s\delta_s\Big)\Big(\sum_t t\gamma_t\Big) \\
&\quad = 0,
\end{aligned}
$$

for $k = 1, \ldots, R-2$. Therefore, if the DD model holds, then the weighted covariance $\mathrm{Cov}(U, V \mid |U|)$ equals zero. We obtain the following lemma.

Lemma 2.1. *For $k = 1, \ldots, R-2$, $\mathrm{Cov}(U, V \mid |U| = k)$ is equivalent to*

$$
2k \sum\sum_{s<t} (t-s) p^*_{ks(k)} p^*_{-k,t(k)} (\theta^*_{(-k<k;s<t)} - 1).
$$

Proof. We have

$$
\begin{aligned}
&\mathrm{Cov}(U,V \mid |U|=k)\\
&= \Big(\sum\sum_{(j,t)\in S_k^*} jt\mu_k\delta_j\gamma_t\psi_{jt}\Big) - \Big(\sum\sum_{(i,s)\in S_k^*} i\mu_k\delta_i\gamma_s\psi_{is}\Big)\Big(\sum\sum_{(j,t)\in S_k^*} t\mu_k\delta_j\gamma_t\psi_{jt}\Big)\\
&= \Big(\sum\sum_{(i,s)\in S_k^*} \mu_k\delta_i\gamma_s\psi_{is}\Big)\Big(\sum\sum_{(j,t)\in S_k^*} jt\mu_k\delta_j\gamma_t\psi_{jt}\Big)\\
&\qquad\qquad - \Big(\sum\sum_{(i,s)\in S_k^*} i\mu_k\delta_i\gamma_s\psi_{is}\Big)\Big(\sum\sum_{(j,t)\in S_k^*} t\mu_k\delta_j\gamma_t\psi_{jt}\Big)\\
&= \sum\sum\sum\sum_{(i,s),(j,t)\in S_k^*} jt\mu_k^2\delta_i\delta_j\gamma_s\gamma_t\psi_{is}\psi_{jt} - \sum\sum\sum\sum_{(i,s),(j,t)\in S_k^*} it\mu_k^2\delta_i\delta_j\gamma_s\gamma_t\psi_{is}\psi_{jt}\\
&= \sum\sum\sum\sum_{(i,s),(j,t)\in S_k^*} (j-i)t\mu_k^2\delta_i\delta_j\gamma_s\gamma_t\psi_{is}\psi_{jt}\\
&= \sum_s\sum_t 2kt\mu_k^2\delta_{-k}\delta_k\gamma_s\gamma_t\psi_{-k,s}\psi_{kt} + \sum_s\sum_t(-2k)t\mu_k^2\delta_k\delta_{-k}\gamma_s\gamma_t\psi_{ks}\psi_{-k,t}\\
&= 2k\mu_k^2\delta_k\delta_{-k}\sum_s\sum_t t\gamma_s\gamma_t(\psi_{-k,s}\psi_{kt}-\psi_{ks}\psi_{-k,t})\\
&= 2k\mu_k^2\delta_k\delta_{-k}\Big(\sum\sum_{s<t} t\gamma_s\gamma_t(\psi_{-k,s}\psi_{kt}-\psi_{ks}\psi_{-k,t})\\
&\qquad\qquad\qquad + \sum\sum_{s>t} t\gamma_s\gamma_t(\psi_{-k,s}\psi_{kt}-\psi_{ks}\psi_{-k,t})\Big)\\
&= 2k\mu_k^2\delta_k\delta_{-k}\Big(\sum\sum_{s<t} t\gamma_s\gamma_t(\psi_{-k,s}\psi_{kt}-\psi_{ks}\psi_{-k,t})\\
&\qquad\qquad\qquad + \sum\sum_{s<t} s\gamma_s\gamma_t(\psi_{-k,t}\psi_{ks}-\psi_{kt}\psi_{-k,s})\Big)\\
&= 2k\mu_k^2\delta_k\delta_{-k}\sum\sum_{s<t}(t-s)\gamma_s\gamma_t(\psi_{-k,s}\psi_{kt}-\psi_{ks}\psi_{-k,t})\\
&= 2k\sum\sum_{s<t}(t-s)(\mu_k\delta_k\gamma_s\psi_{ks})(\mu_k\delta_{-k}\gamma_t\psi_{-k,t})\left(\frac{\psi_{-k,s}\psi_{kt}}{\psi_{ks}\psi_{-k,t}}-1\right)\\
&= 2k\sum\sum_{s<t}(t-s)p^*_{ks(k)}p^*_{-k,t(k)}(\theta^*_{(-k<k;s<t)}-1).
\end{aligned}
$$

The proof is completed. □

From Lemma 1, if the UADD model holds, then the covariance $\mathrm{Cov}(U, V \mid$

$|U| = k)$ is expressed as

$$\begin{aligned}
&\mathrm{Cov}(U, V \mid |U| = k) \\
&= 2k \sum_{s<t}\sum (t-s) p^*_{ks(k)} p^*_{-k,t(k)} \left(\frac{\phi^{-ks}\phi^{kt}}{\phi^{ks}\phi^{-kt}} - 1 \right) \\
&= 2k \sum_{s<t}\sum (t-s) p^*_{ks(k)} p^*_{-k,t(k)} (\phi^{2k(t-s)} - 1),
\end{aligned}$$

for $k = 1, \ldots, R-2$. If $\phi = 1$ in the UADD model, we see $\mathrm{Cov}(U, V \mid |U| = k) = 0$ for $k = 1, \ldots, R-2$. If $\phi > 1$ in the UADD model, we see $\mathrm{Cov}(U, V \mid |U| = k) > 0$ for $k = 1, \ldots, R-2$. If $\phi < 1$ in the UADD model, we see $\mathrm{Cov}(U, V \mid |U| = k) < 0$ for $k = 1, \ldots, R-2$. Therefore, for a fixed k ($k = 1, \ldots, R-2$), when the UADD model holds and the covariance $\mathrm{Cov}(U, V \mid |U| = k)$ equals zero, we obtain $\phi = 1$. Namely, the DD model holds. We obtain the following theorems.

Theorem 2.2. *The DD model holds if and only if the weighted covariance* $\mathrm{Cov}(U, V \mid |U|) = 0$ *and the UADD model holds.*

Theorem 2.3. *For a fixed k ($k = 1, \ldots, R-2$), the DD model holds if and only if the covariance* $\mathrm{Cov}(U, V \mid |U| = k) = 0$ *and the UADD model holds.*

§3. Goodness-of-fit test

Let n_{ij} denote the observed frequency in the (i, j)th cell of the original table for $i = 1, \ldots, R; j = 1, \ldots, R$ with $n = \sum\sum n_{ij}$. Assume that a multinomial distribution is applied to the original $R \times R$ table. The maximum likelihood estimates of expected frequencies $\{m_{ij}\}$ under the DD and UADD models and the structure of $\mathrm{Cov}(U, V \mid |U|) = 0$ could be obtained using the Newton-Raphson method in the log-likelihood equation. Each model and structure can be tested for goodness-of-fit by the likelihood ratio chi-squared statistic (defined by G^2) with the corresponding degrees of freedom. The test statistic G^2 is given by

$$G^2 = 2 \sum_{i=1}^{R} \sum_{j=1}^{R} n_{ij} \log \left(\frac{n_{ij}}{\hat{m}_{ij}} \right),$$

where $\hat{m}_{ij}$ is the maximum likelihood estimate of expected frequency m_{ij} under the given model. The numbers of degrees of freedom for testing the goodness-of-fit of the DD and UADD models and the structure of $\mathrm{Cov}(U, V \mid |U|) = 0$ are $(R-2)^2$, $(R-3)(R-1)$ and 1, respectively.

§4. An Example

The data in Table 1, taken from Stuart (1953), are constructed from unaided distance vision of 3242 men in Britain. Table 2 gives the 7×7 table of the diamond shape formed by rotating the data in Table 1 forty-five degrees.

The DD model fits the data poorly yielding $G^2 = 53.69$ with 4 degrees of freedom. Also, the UADD model fits poorly yielding $G^2 = 51.61$ with 3 degrees of freedom. However, the structure of $\mathrm{Cov}(U, V \mid |U|) = 0$ using equally scores (i.e., $w_1 = w_2 = 1/2$) fits very well yielding $G^2 = 2.33$ with 1 degrees of freedom. From Theorem 2.2, we see that the poor fit of the DD model is caused by the influence of lack of structure of the UADD model (not the lack of the structure of $\mathrm{Cov}(U, V \mid |U|) = 0$).

Table 1. Unaided distance vision of 3242 men in Britain; from Stuart (1953). The parentheses values are maximum likelihood estimates of expected frequencies under the hypothesis that $\mathrm{Cov}(U, V \mid |U|) = 0$.

Right eye grade	Left eye grade Best (1)	Second (2)	Third (3)	Worst (4)	Total
Best (1)	821	112	85	35	1053
	(821.00)	(108.61)	(80.40)	(35.00)	
Second (2)	116	494	145	27	782
	(119.47)	(494.00)	(145.26)	(31.60)	
Third (3)	72	151	583	87	893
	(76.56)	(150.80)	(583.00)	(90.13)	
Worst (4)	43	34	106	331	514
	(43.00)	(29.44)	(102.73)	(331.00)	
Total	1052	791	919	480	3242

Table 2. The 7×7 table of the diamond shape formed by rotating the data in Table 1 forty-five degrees.

Right eye grade minus left eye grade	Right eye grade plus left eye grade 2	3	4	5	6	7	8
−3	*	*	*	35	*	*	*
−2	*	*	85	*	27	*	*
−1	*	112	*	145	*	87	*
0	821	*	494	*	583	*	331
1	*	116	*	151	*	106	*
2	*	*	72	*	34	*	*
3	*	*	*	43	*	*	*

§5. Conclusion

When the DD model fits the data poorly, Theorem 2.2 may be useful for seeing the reason for the poor fit, namely, which of the lack of structure that the weighted covariance $\mathrm{Cov}(U, V \mid |U|)$ equals zero and lack of the UADD model influences strongly.

§6. Discussion

Many readers may think that the decomposition of the DD model using the structure of $\mathrm{Cov}(U, V)$ equals zero, where

$$\begin{aligned}\mathrm{Cov}(U,V) &= \mathrm{E}(UV) - \mathrm{E}(U)\mathrm{E}(V)\\ &= \sum\sum_{(s,t)\in S^*} stp^*_{st} - \Big(\sum\sum_{(s,t)\in S^*} sp^*_{st}\Big)\Big(\sum\sum_{(s,t)\in S^*} tp^*_{st}\Big).\end{aligned}$$

However, when the DD model holds, the structure of $\mathrm{Cov}(U, V) = 0$ does not always hold. Under the DD model, the probabilities p_{11}, p_{1R}, p_{R1} and p_{RR} are unrestricted. On the other hand, under the structure of $\mathrm{Cov}(U, V) = 0$, these probabilities are restricted. Thus, it is difficult to consider the decomposition of the DD model using the the structure of $\mathrm{Cov}(U, V) = 0$. Therefore, in this paper, we consider the decomposition of the DD model using the weighted covariance $\mathrm{Cov}(U, V \mid |U|)$ and covariance $\mathrm{Cov}(U, V \mid |U| = k)$ in Section 2.

When we express $\mathrm{P}(U = s, V = t \mid \{V = R + 1 - k\} \cup \{V = R + 1 + k\})$ as $p^{**}_{st(k)}$ for $(s, t) \in S^*$ and $k = 1, \ldots, R - 1$, we can consider another weighted covariance and similar decomposition of the DD model using another conditional probabilities $\{p^{**}_{st(k)}\}$, although the detail is omitted.

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