SUT Journal of Mathematics
Vol. 50, No. 2 (2014), 297–309

Wald-type measure of departure from symmetry for square contingency tables with nominal categories

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(Received December 4, 2013; Revised October 28, 2014)

Abstract. For square contingency tables with nominal categories, Tomizawa (1994), and Tomizawa, Seo and Yamamoto (1998) considered the power-divergence-type measures to represent the degree of departure from symmetry. The present paper proposes the Wald-type measure to represent the degree of departure from symmetry. In sample version the proposed measure is obtained by modifying the Wald test statistic by the upper limit with fixed sample size. The paper also describes the relationship between the proposed measure and asymmetry models, and the relationship between the measure and the bivariate normal distribution. Examples are given.

AMS 2010 *Mathematics Subject Classification.* 62H17.

Key words and phrases. Asymmetry model, bivariate normal distribution, measure, nominal category, square contingency table, symmetry model, Wald statistic.

§1. Introduction

Consider an $r \times r$ square contingency table with *nominal* categories. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). The symmetry model is defined by

$$p_{ij} = p_{ji} \quad (i = 1, \ldots, r; j = 1, \ldots, r; i \neq j).$$

See Bowker (1948), Bishop, Fienberg and Holland (1975, p.282), Tomizawa and Tahata (2007), and Agresti (2013, p.426). Tomizawa (1994) proposed two types of measures to represent the degree of departure from symmetry as follows:

$$\phi_S = \frac{1}{\log 2} \sum_{i \neq j} \sum p^*_{ij} \log \left(\frac{p^*_{ij}}{p^S_{ij}} \right),$$

$$\psi_S = \sum\sum_{i \neq j} \frac{(p_{ij}^* - p_{ij}^S)^2}{p_{ij}^S},$$

where

$$p_{ij}^* = \frac{p_{ij}}{\delta}, \quad \delta = \sum\sum_{i \neq j} p_{ij}, \quad p_{ij}^S = \frac{p_{ij}^* + p_{ji}^*}{2}.$$

The power-divergence type measure of departure from symmetry in Tomizawa, Seo and Yamamoto (1998) is given by

$$\Phi^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)} \quad \text{for} \quad \lambda > -1,$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum\sum_{i \neq j} p_{ij}^* \left[\left(\frac{p_{ij}^*}{p_{ij}^S} \right)^\lambda - 1 \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. We note that λ is a real value that is chosen by the user. Note that $I^{(\lambda)}$ is the power-divergence between $\{p_{ij}^*\}$ and $\{p_{ij}^S\}$. Especially, when $\lambda = 0$ and $\lambda = 1$, the measures $\Phi^{(0)}$ and $\Phi^{(1)}$ are identical to the Kullback-Leibler information type measure ϕ_S and the Pearson chi-squared-type measure ψ_S, respectively.

For testing goodness-of-fit of the symmetry model, we use the likelihood ratio test statistic G_S^2, the Pearson's chi-squared test statistic X_S^2, and the power divergence test statistic $T^{(\lambda)}$ (Read and Cressie, 1988, p.2). See Appendix for these test statistics. Especially when $\lambda = 0$ and $\lambda = 1$, the test statistics $T^{(0)}$ and $T^{(1)}$ are identical to G_S^2 and X_S^2, respectively. In sample version, the estimated measure of $\Phi^{(\lambda)}$ (denoted by $\hat{\Phi}^{(\lambda)}$) is obtained by modifying $T^{(\lambda)}$ by the upper limit with fixed sample size. The estimated measures $\hat{\phi}_S$ and $\hat{\psi}_S$ are obtained by modifying G_S^2 and X_S^2, respectively. For more details, see Tomizawa (1994) and Tomizawa et al. (1998).

We often use the Wald test statistic for testing goodness-of-fit of the symmetry model (denoted by W_S), in addition to G_S^2, X_S^2 and $T^{(\lambda)}$. See Section 3 for the detail of W_S. So we are now interested in proposing the measure of Wald-type to represent the degree of departure from symmetry.

For square contingency tables with *ordered* categories, Tomizawa, Miyamoto and Hatanaka (2001), and Tahata, Yamamoto, Nagatani and Tomizawa (2009) considered the measures to represent the degree of departure from symmetry.

The present paper proposes the Wald-type measure to represent the degree of departure from symmetry for square contingency tables with *nominal* categories.

§2. Wald-type measure

Consider an $r \times r$ square contingency tables with nominal categories. Assume that $\{p_{ij} + p_{ji} \neq 0\}$. We propose the measure defined by

$$\Psi = \frac{(1-\delta)\gamma}{\delta(1-\gamma)},$$

where

$$\gamma = \sum\sum_{i<j} \frac{(p_{ij} - p_{ji})^2}{p_{ij} + p_{ji}}.$$

Using Tomizawa's (1994) measure ψ_S, Ψ is also expressed as

$$\Psi = \frac{(1-\delta)\psi_S}{1-\delta\psi_S}.$$

We obtain the following theorem.

Theorem 1.

(1) $0 \leq \Psi \leq 1$,

(2) *the symmetry model holds if and only if* $\Psi = 0$,

(3) *the degree of departure from symmetry is the largest in the sense that* $p_{ij} = 0$ *(then* $p_{ji} \neq 0$*) or* $p_{ji} = 0$ *(then* $p_{ij} \neq 0$*) for all* $i \neq j$*, if and only if* $\Psi = 1$.

Proof. We see

$$\begin{aligned} 0 \leq \gamma &\leq \sum\sum_{i<j} \frac{p_{ij}^2 + p_{ji}^2}{p_{ij} + p_{ji}} \\ &\leq \sum\sum_{i<j} \frac{(p_{ij} + p_{ji})^2}{p_{ij} + p_{ji}} \\ &= \delta, \end{aligned} \tag{2.1}$$

where the equality holds if and only if $p_{ij}p_{ji} = 0$ for all $i < j$. Thus we see

$$1 \leq \frac{1}{1-\gamma} \leq \frac{1}{1-\delta}. \tag{2.2}$$

From (2.1) and (2.2), we obtain (1) and (3). Noting that $\delta \neq 1$, it is easily seen that (2) holds. □

§3. Test statistic and estimated measure

3.1. Wald test statistic for symmetry

Consider the $r \times r$ contingency tables. Let x_{ij} be the observed frequency in (i,j)th cell on a multinomial variate with cell probability p_{ij}, where $\sum\sum x_{ij} = n$.

We shall consider the Wald test statistic for testing goodness-of-fit of the symmetry model. Let $\boldsymbol{x} = (\boldsymbol{x}_U, \boldsymbol{x}_L, \boldsymbol{x}_D)'$ where

$$\boldsymbol{x}_U = (x_{12}, x_{13}, \ldots, x_{1r}, x_{23}, x_{24}, \ldots, x_{2r}, \ldots, x_{r-1,r});\ \text{the } 1 \times \tfrac{r(r-1)}{2} \text{ vector},$$

$$\boldsymbol{x}_L = (x_{21}, x_{31}, \ldots, x_{r1}, x_{32}, x_{42}, \ldots, x_{r2}, \ldots, x_{r,r-1});\ \text{the } 1 \times \tfrac{r(r-1)}{2} \text{ vector},$$

$$\boldsymbol{x}_D = (x_{11}, x_{22}, \ldots, x_{rr});\ \text{the } 1 \times r \text{ vector},$$

and let $\boldsymbol{p}$ be the vector defined by the same way as $\boldsymbol{x}$. Note that "$\boldsymbol{\prime}$" denote the transpose. Then $\sqrt{n}(\boldsymbol{x}/n - \boldsymbol{p})$ has asymptotically a normal distribution with mean zero vector and covariance matrix $D - \boldsymbol{p}\boldsymbol{p}'$ where D denotes a diagonal matrix with the ith element of $\boldsymbol{p}$ as the ith diagonal element (see, e.g., Bishop et al., 1975, p.469; Agresti, 2013, p.590).

Now we define the $r(r-1)/2 \times r^2$ matrix C by

$$C = (I_{r(r-1)/2},\ \ -I_{r(r-1)/2},\ O_{r(r-1)/2,r}),$$

where I_m and $O_{s,t}$ denote the identity matrix of order m and the $s \times t$ zero matrix, respectively. Then $\boldsymbol{y}$ defined by $\boldsymbol{y} = C\boldsymbol{x}$ has asymptotically a normal distribution with mean vector $nC\boldsymbol{p}$ and covariance matrix $nC(D - \boldsymbol{p}\boldsymbol{p}')C'$. Then we can see

$$nC(D - \boldsymbol{p}\boldsymbol{p}')C' = n(E - \boldsymbol{b}\boldsymbol{b}'),$$

where

$$E = \begin{pmatrix} p_{12}+p_{21} & 0 & \cdots & 0 \\ 0 & p_{13}+p_{31} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{r-1,r}+p_{r,r-1} \end{pmatrix},$$

and

$$\boldsymbol{b}' = (p_{12} - p_{21}, p_{13} - p_{31}, \ldots, p_{r-1,r} - p_{r,r-1}).$$

We note that the symmetry model can be expressed as $C\boldsymbol{p} = 0$. Therefore by approximating $\boldsymbol{p}$ by $\boldsymbol{x}/n$, Wald test statistic for symmetry is given by

$$W_S = \frac{1}{n}\boldsymbol{y}'(\hat{E} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}')^{-1}\boldsymbol{y},$$

where $\hat{E}$ and $\hat{\boldsymbol{b}}$ denote E and $\boldsymbol{b}$, respectively, with $\boldsymbol{p}$ replaced by $\boldsymbol{x}/n$ and

$$(E - \boldsymbol{b}\boldsymbol{b}')^{-1} = \frac{1}{1 - \boldsymbol{b}'E^{-1}\boldsymbol{b}}(E^{-1}\boldsymbol{b})(E^{-1}\boldsymbol{b})' + E^{-1}.$$

Then we can simplify the test statistic as follows:

$$W_S = \frac{X_S^2}{1 - X_S^2/n},$$

where

$$X_S^2 = \sum\sum_{s<t} \frac{(x_{st} - x_{ts})^2}{x_{st} + x_{ts}}.$$

Under the symmetry model, W_S has an asymptotic chi-squared distribution with $r(r-1)/2$ degrees of freedom. Note that X_S^2 is the Pearson's test statistic and also Bowker's test statistic.

3.2. Relation between estimated measure and test statistic

The sample version of Ψ, i.e., $\hat{\Psi}$, is given by Ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = x_{ij}/n$. The Wald test statistic for testing goodness-of-fit of the symmetry model is given by

$$W_S = \frac{n\hat{\gamma}}{1 - \hat{\gamma}},$$

where $\hat{\gamma}$ is given by γ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. [Note that the Pearson's chi-squared statistic is given by $X_S^2 = n\hat{\gamma}$.] Therefore we see that

$$\hat{\Psi} = \frac{n - \sum\sum_{s\neq t} x_{st}}{n\sum\sum_{s\neq t} x_{st}} W_S,$$

namely

$$0 \leq W_S \leq \frac{n(1 - \sum_{s=1}^{r} \hat{p}_{ss})}{\sum_{s=1}^{r} \hat{p}_{ss}}.$$

Therefore, when n is fixed, the range of the test statistic W_S depends on the diagonal proportions, however, the measure $\hat{\Psi}$ is always in the range between 0 and 1, without depending on the diagonal proportions. Note that the symmetry model is saturated on diagonal cells (i, i) for $i = 1, \ldots, r$.

We also note that the measure Ψ may be expressed as

$$\Psi = \frac{1 - \delta}{\delta}(C\boldsymbol{p})'(E - \boldsymbol{b}\boldsymbol{b}')^{-1}C\boldsymbol{p}.$$

§4. Relationship between measure and asymmetry models

As an asymmetry model, McCullagh (1978) considered the conditional symmetry model, which is defined by

$$p_{ij} = \Delta p_{ji} \quad (i < j),$$

where Δ is unspecified. A special case of this model obtained by putting $\Delta = 1$ is the symmetry model. As another asymmetry model, Agresti (1983) considered the linear diagonals-parameter symmetry model, which is defined by

$$p_{ij} = \theta^{j-i} p_{ji} \quad (i < j),$$

where θ is unspecified. A special case of this model obtained by putting $\theta = 1$ is the symmetry model.

We shall consider the relationship between the measure Ψ and the conditional symmetry model (the linear diagonals-parameter symmetry model). The measure Ψ may be expressed as

$$\Psi = \frac{(1-\delta)\psi_S}{1-\delta\psi_S},$$

where

$$\psi_S = \sum\sum_{i<j} (p^*_{ij} + p^*_{ji})(p^c_{ij} - p^c_{ji})^2$$

and

$$p^c_{st} = \frac{p_{st}}{p_{st} + p_{ts}}.$$

Therefore, if there is a structure of conditional symmetry in the table, then the measure Ψ can be simply expressed as

$$\Psi = \frac{(1-\delta)\psi_S}{1-\delta\psi_S},$$

where

$$\psi_S = \left(\frac{\Delta - 1}{\Delta + 1}\right)^2.$$

Thus when $\Delta = 1$ (i.e., when the symmetry model holds), $\Psi = 0$. Also, Ψ approaches the maximum value 1 as Δ approaches infinity (or zero). This property would be natural.

Also if there is a structure of linear diagonals-parameter symmetry in the table, then the measure Ψ can be expressed as

$$\Psi = \frac{(1-\delta)\psi_S}{1-\delta\psi_S},$$

where

$$\psi_S = \sum\sum_{i<j} (p^*_{ij} + p^*_{ji}) \left(\frac{\theta^{j-i} - 1}{\theta^{j-i} + 1} \right)^2 .$$

Thus when $\theta = 1$ (i.e., when the symmetry model holds), $\Psi = 0$. Also, noting that $\sum\sum_{i<j}(p^*_{ij} + p^*_{ji}) = 1$, Ψ approaches the maximum value 1 as θ approaches infinity (or zero). This property also would be natural.

Therefore the measure Ψ would be appropriate to measure the degree of departure from symmetry.

§5. Approximate confidence interval for measure

We obtain the following theorem.

Theorem 2. $\sqrt{n}(\hat{\Psi} - \Psi)$ *has asymptotically a normal distribution with mean zero and variance* σ^2, *where*

$$\sigma^2 = \sum\sum_{i \neq j} p_{ij} \Gamma^2_{ij} - \left(\frac{\delta \psi_S (\psi_S - 1)}{(1 - \delta \psi_S)^2} \right)^2$$

with

$$\Gamma_{ij} = \frac{1}{(1 - \delta \psi_S)^2} \left\{ \left(\frac{(p_{ij} - p_{ji})(p_{ij} + 3p_{ji})}{(p_{ij} + p_{ji})^2} - \psi_S \right) \frac{1 - \delta}{\delta} + \psi_S (\psi_S - 1) \right\} .$$

Proof. The vector $\sqrt{n}(\boldsymbol{x}/n - \boldsymbol{p})$ has asymptotically a normal distribution with mean zero vector and covariance matrix $D - \boldsymbol{p}\boldsymbol{p}'$. The estimated measure $\hat{\Psi}$ is expressed as

$$\hat{\Psi} = \Psi + \left[\frac{\partial \Psi}{\partial \boldsymbol{p}'} \right] \left(\frac{\boldsymbol{x}}{n} - \boldsymbol{p} \right) + o \left(\left|\left| \frac{\boldsymbol{x}}{n} - \boldsymbol{p} \right|\right| \right) .$$

Using the delta method (Bishop et al., 1975, Section 14.6), we can see that $\sqrt{n}(\hat{\Psi} - \Psi)$ has asymptotically a normal distribution with mean zero and variance

$$\left[\frac{\partial \Psi}{\partial \boldsymbol{p}'} \right] (D - \boldsymbol{p}\boldsymbol{p}') \left[\frac{\partial \Psi}{\partial \boldsymbol{p}'} \right]' .$$

The proof is completed. □

Note that the asymptotic distribution of $\sqrt{n}(\hat{\Psi} - \Psi)$ is not applicable when $\Psi = 0$ and $\Psi = 1$ because then $\sigma^2 = 0$. Let $\hat{\sigma}^2$ denote σ^2 with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Then $\hat{\sigma}/\sqrt{n}$ is an estimated approximate standard error of $\hat{\Psi}$, and

$\hat{\Psi} \pm z_{\alpha/2}\hat{\sigma}/\sqrt{n}$ is an approximated $100(1-\alpha)$ percent confidence interval for Ψ, where $z_{\alpha/2}$ is the percentage point from the standard normal distribution corresponding to a two-tail probability equal to α.

§6. Examples

Consider the data in Tables 1a and 1b, which were earlier analyzed by Andersen (1980, p.328). As reported by Andersen, these data are the results of three consecutive opinion polls held in August 1971, October 1971 and December 1973, which were held in connection with the Danish referendum on whether or not to join the European Common Market.

For the data in Table 1a, the value of measure $\hat{\Psi}$ is 0.031 with standard error 0.021. The 95 percent confidence interval for Ψ is $(-0.010,\ 0.071)$. Since this includes zero, this would indicate that there is a structure of symmetry in Table 1a, or, if this is not the case, then it indicates that the degree of departure from symmetry is slight.

For the data in Table 1b, the value of $\hat{\Psi}$ is 0.191 with standard error 0.051. The 95 percent confidence interval of Ψ is $(0.091,\ 0.291)$. Thus this indicates that there is not a structure of symmetry in Table 1b.

Using the confidence interval of Ψ, when we compare the degrees of departure from symmetry in Tables 1a and 1b, the degree of departure in Table 1b would be stronger than that in Table 1a.

§7. Relationship between measure and normal distribution

Tomizawa et al. (2001) discussed the relationship between the measure and the bivariate normal distribution for square contingency tables with ordered categories. We shall consider the relationship between the proposed measure Ψ and the bivariate normal distribution for square contingency tables with nominal categories.

Consider random variables U and V having a joint bivariate normal distribution, with means $E(U) = \mu_1$ and $E(V) = \mu_2$, variances $Var(U) = Var(V) = \sigma^2$, and correlation $corr(U, V) = \rho$. When we denote the probability density function of (U, V) by $f(u, v)$, we see

$$\frac{f(u,v)}{f(v,u)} = \exp\left[\frac{(v-u)(\mu_2-\mu_1)}{(1-\rho)\sigma^2}\right] \quad (u < v).$$

Therefore we see that if $\mu_1 < \mu_2$, then $f(u,v)/f(v,u) > 1$ for $u < v$, and $f(u,v)/f(v,u)$ increases as the correlation ρ increases for u, v, μ_1, μ_2, σ^2 fixed; namely, the degree of departure from symmetry of density function increases as the correlation ρ increases.

In terms of simulation studies, Table 2, which is taken from Tomizawa et al. (2001), gives the 4×4 tables of sample size 10000, formed by using cut points for each variable at μ_1, $\mu_1 \pm 0.6\sigma$, for an underlying bivariate normal distribution with the conditions $\mu_2 = \mu_1 + 0.4$, $\sigma_1^2 = \sigma_2^2 (= \sigma^2)$ and $\rho =0$, 0.3, 0.6, and 0.9. For Tables 2a, 2b, 2c and 2d, the values of estimated measure $\hat{\Psi}$ are 0.025 (for $\rho = 0$), 0.046 (for $\rho = 0.3$), 0.103 (for $\rho = 0.6$), and 0.472 (for $\rho = 0.9$). Therefore the measure $\hat{\Psi}$ increases as ρ increases. From Table 2 we see that as ρ approaches 1, the square table with an underlying bivariate normal distribution with equality of two variances tends to have zero observations in one or more of the off-diagonals. Thus it would be natural to assume that the degree of departure from symmetry for the obtained square contingency tables approaches the maximum as the correlation ρ approaches 1. Therefore, the proposed measure $\hat{\Psi}$ would be appropriate for measuring the degree of departure from symmetry for a square contingency table formed from an underlying bivariate normal distribution with equal variances.

§8. Concluding remarks

As described in Section 3.2, when n is fixed, the range of the test statistic W_S depends on the diagonal proportions, however, the proposed measure $\hat{\Psi}$ is always in the range between 0 and 1, without depending on the diagonal proportions. Therefore $\hat{\Psi}$ rather than W_S (or W_S/n) would be useful for comparing the degrees of departure from symmetry in several square tables, especially, in several square tables with different sample sizes.

As described in Sections 4 and 7, from the relationship between the measure Ψ and asymmetry models and from the relationship between Ψ and the bivariate normal distribution, the measure Ψ would be appropriate to represent the degree of departure from symmetry.

Finally we note that the proposed measure Ψ would be suitable to be applied to square tables with *nominal* categories because it is invariant under arbitrary similar permutations of rows and columns categories.

Acknowledgements

The authors would like to thank the editor and an anonymous referee for their helpful comments and suggestions.

Appendix

For testing goodness-of-fit of the symmetry model, the likelihood ratio test statistic G^2 is defined by

$$G^2 = 2\sum_{i=1}^{r}\sum_{j=1}^{r} x_{ij} \log\left(\frac{2x_{ij}}{x_{ij}+x_{ji}}\right),$$

where x_{ij} is the observed frequency in (i,j)th cell in the $r \times r$ table. The Pearson's chi-squared test statistic is defined by

$$X_S^2 = \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} \frac{(x_{ij}-x_{ji})^2}{x_{ij}+x_{ji}}.$$

The power-divergence test statistic is defined by

$$T^{(\lambda)} = \frac{2}{\lambda(\lambda+1)}\sum_{i=1}^{r}\sum_{j=1}^{r} x_{ij}\left[\left(\frac{2x_{ij}}{x_{ij}+x_{ji}}\right)^{\lambda} - 1\right] \quad \text{for } -\infty < \lambda < \infty,$$

where the values at $\lambda = -1$ and $\lambda = 0$ are taken to be the limits as $\lambda \to -1$ and $\lambda \to 0$, respectively.

References

[1] Agresti, A. (1983). A simple diagonals-parameter symmetry and quasi-symmetry model. *Statistics and Probability Letters*, **1**, 313-316.

[2] Agresti, A. (2013). *Categorical Data Analysis*, 3rd edition. Wiley, Hoboken, New Jersey.

[3] Andersen, E. B. (1980). *Discrete Statistical Models with Social Science Applications*. North-Holland, Amsterdam.

[4] Bishop, Y. M. M., Fienberg, S. E. and Holland, P. W. (1975). *Discrete Multivariate Analysis: Theory and Practice*. The MIT Press, Cambridge, Massachusetts.

[5] Bowker, A. H. (1948). A test for symmetry in contingency tables. *Journal of the American Statistical Association*, **43**, 572-574.

[6] McCullagh, P. (1978). A class of parametric models for the analysis of square contingency tables with ordered categories. *Biometrika*, **65**, 413-418.

[7] Read, T. R. C. and Cressie, N. (1988). *Goodness-of-Fit Statistics for Discrete Multivariate Data*. Springer, New York.

[8] Tahata, K., Yamamoto, K., Nagatani, N. and Tomizawa, S. (2009). A measure of departure from average symmetry for square contingency tables with ordered categories. *Austrian Journal of Statistics*, **38**, 101-108.

[9] Tomizawa, S. (1994). Two kinds of measures of departure from symmetry in square contingency tables having nominal categories. *Statistica Sinica*, **4**, 325-334.

[10] Tomizawa, S., Miyamoto, N. and Hatanaka, Y. (2001). Measure of asymmetry for square contingency tables having ordered categories. *Australian and New Zealand Journal of Statistics*, **43**, 335-349.

[11] Tomizawa, S., Seo, T. and Yamamoto, H. (1998). Power-divergence-type measure of departure from symmetry for square contingency tables that have nominal categories. *Journal of Applied Statistics*, **25**, 387-398.

[12] Tomizawa, S. and Tahata, K. (2007). The analysis of symmetry and asymmetry: Orthogonality of decomposition of symmetry into quasi-symmetry and marginal symmetry for multi-way tables. *Journal de la Société Française de Statistique*, **148**, 3-36.

Table 1: The results from three consecutive opinion polls on the question: Do you think Denmark should join the European Common Market? (Andersen, 1980, p. 328)

(a) The results of the first and second polls

	Poll II			
Poll I	Yes	No	Undecided	Total
Yes	176	33	40	249
No	21	94	32	147
Undecided	21	33	43	97
Total	218	160	115	493

(b) The results of the second and third polls

	Poll III			
Poll II	Yes	No	Undecided	Total
Yes	167	36	15	218
No	19	131	10	160
Undecided	45	50	20	115
Total	231	217	45	493

Table 2: The 4×4 tables of sample size 10000, formed by using cutpoints for each variable at μ_1, $\mu_1 \pm 0.6\sigma$, from an underlying bivariate normal distribution with the conditions $\mu_2 = \mu_1 + 0.4$, $\sigma_1^2 = \sigma_2^2 (= \sigma^2)$, and $\rho = 0$, 0.3, 0.6, and 0.9 (from Tomizawa et al., 2001).

(a) $\rho = 0$				(b) $\rho = 0.3$			
428	526	671	1174	696	666	678	785
358	416	561	951	384	436	587	836
374	405	544	875	269	388	554	1008
405	509	658	1145	216	366	615	1516
(c) $\rho = 0.6$				(d) $\rho = 0.9$			
1017	787	620	383	1432	974	328	21
330	488	686	720	129	693	1073	347
162	379	630	1098	4	179	868	1241
56	202	498	1944	0	10	165	2536

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