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ON RESIDUES OF DIFFERENTIAL FORMS OVER A FIELD OF CHARACTERISTIC p

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Abstract. Let K be a function field over a field k of characteristic $p > 0$ and let R be a discrete valuation ring of K/k. E. Kunz showed that if ω is a closed differential form and $\nu_R(\omega) \geq -1$, then $res_{R,\underline{t}}(\omega)$ does not depend on the choice of parameter $\underline{t} = \{t_1, t_2, \cdots, t_n\}$.

In this paper, we investigate $res_{R,\underline{t}}(\omega)$ in the case where $\nu_R(\omega) \geq -p^m + 1$ for $\omega \in Z_m$.

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§0. Introduction

Let K be a function field of n variables over a field k of characteristic $p > 0$ and let R be a discrete valuation ring of K/k such that the residue field D of R has transcendence degree $n-1$ over k. Y. Suzuki [3] proved the following Theorem A and Corollary B.

Theorem A. *If ω is a differential form in $Z_m\Omega^r(K/k)$ such that $\nu_R(\omega) \geq -p^{m-1}$, then $res_{R,\underline{t}}(\omega)$ is uniquely determined up to addition by differentials in $B_{m-1}\Omega^{r-1}(D/k)$.*

Corollary B. $res_R : Z_\infty\Omega^r(K/k) \longrightarrow Z_\infty\Omega^{r-1}(D/k)/B_\infty\Omega^{r-1}(D/k)$ *is well defined.* (for the definition, see section 1).

His method of proof is the following: First he proved the commutativity of residue map and Cartier operator. Secondly he proved that if $\omega \in Z_m\Omega^r(K/k)$ and $\nu_R(\omega) \geq -p^{m-1}$, then $\nu_{R^{(m)}}(C_K^{(m-1)}(\omega)) \geq -1$ and $C_K^{(m-1)}(\omega)$ is a closed differential, where $C_K^{(m-1)}$ is an iterated Cartier operator. From two results above and a result of E. Kunz (Exercise (1) in §17 of [1]), he proved Theorem A and Corollary B.

On the other hand, our main results are the following:

Theorem 2. *If* ω *is a differential form in* $Z_m\Omega(K/k)$ *such that* $\nu_R(\omega) \geq -p^m + 1$*, then* $res_{R,\underline{t}}(\omega)$ *is uniquely determined up to addition by differentials in* $B_m\Omega(D/k)$.

Corollary. $res_R : Z_\infty\Omega(K/k) \longrightarrow Z_\infty\Omega(D/k)/B_\infty\Omega(D/k)$ *is well defined.*

Our method of proof is quite different from Suzuki's method and our Theorem 2 and Suzuki's Theorem A are independent to each other, that is, Theorem A does not imply Theorem 2 and vice versa. But both Theorem 2 and Theorem A imply the same Corollary.

An advantage of our result is in the following fact. The number $-p^m + 1$ in our Theorem 2 is the best possible (see Example in §2).

§1. Preliminaries

Throughout this paper, K will denote a function field of n variables over a field k of characteristic $p > 0$ and R a discrete valuation ring of rank one of K/k such that the residue field D of R has transcendence degree $n-1$ over k. Furthermore we always assume that K and D are separable over k.

We choose n elements $t_1, t_2, ..., t_n$ in R such that t_1R is the maximal ideal of R and such that $\overline{t_2}, ..., \overline{t_n}$ is a p-basis of D/k, where $\overline{a}$ denotes the canonical image in D of $a \in R$. We will call such a family $\underline{t} = \{t_1, t_2, ..., t_n\}$ a *parameter* of $(K/k, R)$. We put $K_i = kK^{p^i}$, $R_i = kR^{p^i}$ and $\underline{t^{(i)}} = \underline{t^{p^i}} = \{t_1^{p^i}, t_2^{p^i}, \cdots, t_n^{p^i}\}$ $(i = 0, 1, 2, \cdots)$.

Let A be a G-algebra, where G and A are commutative rings, and let $(\Omega(A/G), d_{A/G})$ be the universal differential algebra of A/G. Then we know that $\Omega(A/G) = \bigoplus \Omega^r(A/G)$, $\Omega^r(A/G) = \bigwedge^r \Omega^1(A/G)$ and $\Omega^1(A/G)$ is the module of Kähler differentials of A/G (c.f. §3 in [1]). If there is no confusion, we simply write d, $\Omega, \Omega(D)$ and $\Omega(R)$ instead of $d_{A/G}$, $\Omega(K/k), \Omega(D/k)$ and $\Omega(R/k)$, respectively.

Lemma 1. *Let* $\underline{t} = \{t_1, t_2, ..., t_n\}$ *be a parameter of* $(K/k, R)$*. Then* $\underline{t} = \{t_1, t_2, ..., t_n\}$ *is a* p*-basis of* R/k.

Proof. From the following exact sequence of vector spaces over D

(c.f. Th. 25.2 in [2]),

$$0 \longrightarrow t_1R/t_1^2R \longrightarrow \Omega^1(R) \otimes_R D \longrightarrow \Omega^1(D) \longrightarrow 0,$$

we get that $\dim(\Omega^1(R) \otimes_R D) = 1 + (n-1) = n$ ($\dim\Omega^1(D) = n-1$ from separability of D/k). It follows from Nakayama's lemma that $\{dt_1, dt_2, ..., dt_n\}$ generates $\Omega^1(R)$ over R. On the other hand, since $\Omega^1 = \Omega^1(R) \otimes_R K$ has dimension n over K, $\{dt_1, dt_2, ..., dt_n\}$ must form a basis of $\Omega^1(R)$ over R.

We will show that $kR^p[t_1, t_2, ..., t_n] = R$. Let $S = kR^p[t_1, t_2, ..., t_n]$. Then S is a local ring with the residue field $kD^p[\overline{t_2}, ..., \overline{t_n}] = D$ (see Remark below). Hence $R = S + t_1R$. Since R is a finite S-module and t_1 is an element of the maximal ideal of S, it follows from Nakayama's lemma that $S = R$. By 5.6 Proposition in [1], we see that $\{t_1, t_2, ..., t_n\}$ is a p-basis of R/k.

Remark. By using the conditions that both K/k and D/k are separable, we observe that kR^p is a discrete valuation ring of rank one with the residue field kD^p and that $\underline{t^p} = \{t_1^p, t_2^p, ..., t_n^p\}$ is a parameter of $(kK^p/k, kR^p)$. In fact, we have $K^p \otimes_{k^p} k = kK^p$ since K^p/k^p is separable, and hence we also get $R^p \otimes_{k^p} k = kR^p$. Thus it follows that $kR^p/(t_1^p) = R^p/(t_1^p) \otimes_{k^p} k = D^p \otimes_{k^p} k$ and that $D^p \otimes_{k^p} k = kD^p$ by separability of D/k. Similarly, we observe that $\{\overline{t_2}^p, ..., \overline{t_n}^p\}$ is a p-basis of kD^p/k and that $\underline{t^{(i)}}$ is a parameter of $(K_i/k, R_i)$ for each i.

We will define a k-linear map of degree -1, $res_{R,\underline{t}} : \Omega \longrightarrow \Omega(D)$. Let $\hat{R}$ be the completion of R. Then there exists a unique coefficient field $E = E_{t_2,...,t_n}$ of $\hat{R}$ such that $\hat{R} = E[[t_1]]$ and $E \supset k(t_2, ..., t_n)$ (c.f. Th. 28.3 in [2]). The quotient field of $\hat{R}$ is the formal power series field $E((t_1))$ and K can be regarded as a subfield of $E((t_1))$. Let ω be a differential form in Ω^r $(r \geq 1)$. Then ω is uniquely expressed in the form

$$\omega = \sum_{1<i_1<\cdots<i_r} g_{i_1\cdots i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r} + \sum_{1<i_2<\cdots<i_r} h_{i_2\cdots i_r} dt_1 \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}$$

where $g_{i_1\cdots i_r}, h_{i_2\cdots i_r} \in K$. Let $h_{i_2\cdots i_r} = \sum_k h_{i_2\cdots i_r,k} t_1^k$ be the formal expression of $h_{i_2\cdots i_r}$ in $\hat{K} = E((t_1))$. We define the residue of ω by

$$res_{R,\underline{t}}(\omega) = \sum_{i_2<\cdots<i_r} \overline{h_{i_2\cdots i_r,-1}} d\overline{t_{i_2}} \wedge \cdots \wedge d\overline{t_{i_r}}$$

where $\bar{a}$ is the canonical image of $a \in \hat{R}$ in D. Thus we can define the map $res_{R,\underline{t}} : \Omega \longrightarrow \Omega(D)$ by linearlity.

We observe that $res_{R,\underline{t}}$ has the following property

$$res_{R,\underline{t}} \circ d + d_{D/k} \circ res_{R,\underline{t}} = 0.$$

It follows from this property that $res_{R,\underline{t}}$ maps closed differentials to closed ones and exact differentials to exact ones.

We will denote by $Z(\Omega)$ $(= ker\ d)$, all of closed differentials in Ω and by $B(\Omega)$ $(= im\ d)$, all of exact differentials in Ω. If there is no confusion, we will write Z, B instead of $Z(\Omega), B(\Omega)$, respectively. It follows that Z is a graded kK^p-subalgebra of Ω with $Z^0 = kK^p$ and that B is a two-sided homogeneous ideal of Z.

Definition. For a parameter $\underline{t} = \{t_1, t_2, ..., t_n\}$ of $(K/k, R)$, we define the graded subalgebras $H_m(\underline{t})$ of Z and $I_m(\underline{t})$ of $Z(\Omega(R))$ $(m = 1, 2....)$ as follows;

$$H_m(\underline{t}) := K_m[t_1^{p^m-1}dt_1, t_2^{p^m-1}dt_2, \cdots, t_n^{p^m-1}dt_n],$$

$$I_m(\underline{t}) := R_m[t_1^{p^m-1}dt_1, t_2^{p^m-1}dt_2, \cdots, t_n^{p^m-1}dt_n].$$

We have by Exercise (6) in §5 of [1] that

$$Z = B \bigoplus H_1(\underline{t}), \quad Z(\Omega(R)) = B(\Omega(R)) \bigoplus I_1(\underline{t})$$

for every parameter $\underline{t}$ of $(K/k, R)$ (c.f. Lemma 1).

The Cartier operator $C_{K/k}$ (we denote it by C if there is no confusion) is defined to be a surjective homomorphism of degree zero of graded K_1-algebra $(K_1 = kK^p)$

$$C : Z \longrightarrow \Omega(K_1/k)$$

such that $C(B) = 0$, $C(a) = a$ for any $a \in Z^0 = K_1$ and $C(t_i^{p-1}dt_i) = d_1 t_i^p$ for each i, where d_1 is the differentiation of $\Omega(K_1/k)$ (Exercise (6) in §5 of [1]). It follows that C induces an isomorphism of $H_1(\underline{t})$ on $\Omega(K_1/k)$, but C does not depend on R and *a fortiori* C does not depend on $\underline{t}$. Similarly we can also define Cartier operators $C_{R/k}$, $C_{D/k}$, $C_{K_i/k}$ and $C_{R_i/k}$. We have by Lemma 2 of [3] that

$$C_{D/k} \circ res_{R,\underline{t}} = res_{R_1,\underline{t^p}} \circ C$$

for every parameter $\underline{t}$ of $(K/k, R)$.

The Cartier opreators $C_{K_i/k}$ $(= C_i)$ $(i = 0, 1, 2, ...)$ define the subsets $B_m = B_m(\Omega)$ and $Z_m = Z_m(\Omega)$ of Ω inductively as follows: We first set $B_0(\Omega_i) = 0$,

$Z_0(\Omega_i) = \Omega_i$ for each i, where $\Omega_i = \Omega(K_i/k)$. We note that $C_0 = C$ and $\Omega_0 = \Omega$. Next we set, for every integer $m \geq 0$,

$$B_{m+1}(\Omega_i) = C_i^{-1}(B_m(\Omega_{i+1})), \quad Z_{m+1}(\Omega_i) = C_i^{-1}(Z_m(\Omega_{i+1})).$$

For example, $B_2 = B_2(\Omega)$ is obtained as follows; $B_1(\Omega_1) = C_1^{-1}(0)$ and $B_2(\Omega) = C_0^{-1}(B_1(\Omega_1)) = C_0^{-1}(C_1^{-1}(0))$.

We can easily see that $B_1 = B$, $Z_1 = Z$ and

$$0 = B_0 \subset B_1 \subset \cdots \subset B_m \subset \cdots \subset Z_m \subset \cdots \subset Z_1 \subset Z_0 = \Omega.$$

It follows that Z_m $(m \geq 0)$ is a graded K_m-subalgebra of Ω and that B_m is a two-sided homogeneous ideal of Z_m such that $Z_m/B_m \simeq \Omega_m$. Furthermore, we set $Z_\infty = \bigcap_{m=1}^{\infty} Z_m$ and $B_\infty = \bigcup_{m=1}^{\infty} B_m$.

Let $\underline{t} = \{t_1, t_2, ..., t_n\}$ be a parameter of $(K/k, R)$. Then for every element ω of Ω, we define $\nu_R(\omega)$ as follows;

$$\nu_R(\omega) = \max\{s \in Z |\ t_1^{-s}\omega \in \Omega(R)\}.$$

If $\omega \in \Omega^0 = K$, then $\nu_R(\omega)$ is the valuation value of ω such that $\nu_R(t_1) = 1$. We note that $\nu_R(\omega)$ is dependent on R but not dependent on the parameter $\underline{t}$.

Furthermore we fix a special basis of Ω over K for the parameter $\underline{t}$ named Λ;

$$\Lambda = \{dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r} |\ 0 \leq r \leq n,\ 1 \leq i_1 < \cdots < i_r \leq n\ \}$$

(when $r = 0$, $dt_{i_1} \wedge \cdots \wedge dt_{i_r}$ means 1). Then Λ is also a basis of $\Omega(R)$ over R. Furthermore we see that an element $\omega = \sum a_{i_1,...,i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r}$ belongs to $\Omega(R)$ if and only if all $a_{i_1,...,i_r}$ belong to R.

Lemma 2. *For any parameter $\underline{t}$ of $(K/k, R)$ and for any natural number m,*

$$Z_m = B_m \bigoplus H_m(\underline{t})$$

as K_m-modules (additive groups or k-modules).

Proof. We shall prove this by induction on m, it holding for $m = 1$ (Exercise (6) in §5 in [1]). We assume it for $m - 1 (m \geq 2)$. By using the assumption of induction to the case of the parameter $\underline{t^p}$ of $(K_1/k, R_1)$, we have that

$$Z_{m-1}(\Omega_1) = B_{m-1}(\Omega_1) \bigoplus H_{m-1}(\underline{t^p}),$$

where $H_{m-1}(\underline{t^p}) = kK_1^{p^{m-1}}[(t_1^p)^{p^{m-1}-1}dt_1^p, (t_2^p)^{p^{m-1}-1}dt_2^p, \cdots, (t_n^p)^{p^{m-1}-1}dt_n^p]$. Since $K_m = kK_1^{p^{m-1}}$ and $t_j^{p^{m}-1}dt_j = (t_j^p)^{p^{m-1}-1}t_j^{p-1}dt_j$, it follows that $C(H_m(\underline{t})) = H_{m-1}(\underline{t^p})$. By the definition of Z_m and B_m,

$$Z_m = C^{-1}(Z_{m-1}(\Omega_1)) \text{ and } B_m = C^{-1}(B_{m-1}(\Omega_1)).$$

If $\omega \in B_m \cap H_m(\underline{t})$, then $C(\omega) \in B_{m-1}(\Omega_1) \cap H_{m-1}(\underline{t^p}) = (0)$; hence $\omega \in ker\ C \cap H_m(\underline{t}) \subset B_1 \cap H_1(\underline{t}) = (0)$.

It holds that $Z_m \supset B_m + H_m(\underline{t})$. Conversely, we will prove that $Z_m \subset B_m + H_m(\underline{t})$. Let $\omega \in Z_m$. Then $C(\omega) = x + y$ for some $x \in B_{m-1}(\Omega_1)$ and $y \in H_{m-1}(\underline{t^p})$. Since C is surjective, there exist $\alpha \in B_m$ and $\beta \in H_m(\underline{t})$ such that $C(\alpha) = x$ and $C(\beta) = y$. Hence $\omega - \alpha - \beta \in ker\ C = B = B_1 \subset B_m$. Thus $Z_m = B_m + H_m(\underline{t})$.

Let $\underline{t}$ be a parameter of $(K/k, R)$. Any element $a \neq 0$ of K can be uniquely expressed in the form

$$a = \sum \alpha_{s_1,\cdots,s_n} t_1^{s_1} t_2^{s_2} \cdots t_n^{s_n}, \quad \alpha_{s_1,\cdots,s_n} \in kK^p,$$

where s_i runs over $\{0, 1, \cdots, p-1\}$ for each i. Then we have the following lemma.

Lemma 3. $\nu_R(a) = \min_{s_1,\cdots,s_n} (\nu_R(\alpha_{s_1,\cdots,s_n} t_1^{s_1} \cdots t_n^{s_n}))$.

Proof. It is easy to see that $\nu_R(\alpha_{s_1,\cdots,s_n})$ are multiples of p, $\nu_R(t_i) = 0$ for $i \geq 2$ and $\nu_R(t_1) = 1$. Therefore the values of valuation ν_R of the following p elements are distinct to each other except $\infty = \infty$;

$$\sum \alpha_{0,s_2,\cdots,s_n} t_2^{s_2} \cdots t_n^{s_n}, (\sum \alpha_{1,s_2,\cdots,s_n} t_2^{s_2} \cdots t_n^{s_n}) t_1, \cdots, (\sum \alpha_{p-1,s_2,\cdots,s_n} t_2^{s_2} \cdots t_n^{s_n}) t_1^{p-1}.$$

Therefore it holds that

$$\nu_R(a) = \min_{i=0,1,\cdots,p-1} (\ \nu_R(\sum \alpha_{i,s_2,\cdots,s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n})\ ).$$

Let $\min_{s_2,\cdots,s_n} (\nu_R(\alpha_{i,s_2,\cdots,s_n})) = r_i p$ $(r_i \in Z)$ and $\alpha_{i,s_2,\cdots,s_n} = t_1^{r_i p} \alpha'_{i,s_2,\cdots,s_n}$ $(\alpha'_{i,s_2,\cdots,s_n} \in kR^p)$ for each i. Then $\sum \alpha'_{i,s_2,\cdots,s_n} t_2^{s_2} \cdots t_n^{s_n}$ is an element of R and its image $\sum \overline{\alpha'_{i,s_2,\cdots,s_n}}\, \overline{t_2}^{s_2} \cdots \overline{t_n}^{s_n}$ in D is not zero, because $\{\overline{t_2}, \cdots, \overline{t_n}\}$ is a p-basis of D/kD^p and at least one of the elements $\{\overline{\alpha'_{i,s_2,\cdots,s_n}}\}$ is not zero. Therefore, for each i, it holds that

$$\nu_R(\sum \alpha_{i,s_2,\cdots,s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n}) = \min_{s_2,\cdots,s_n} (\nu_R(\alpha_{i,s_2,\cdots,s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n})).$$

This completes the proof.

Lemma 4. *Let $\alpha, \beta \in H_1(\underline{t})$. If $\nu_R(\alpha) = \nu_R(\beta)$, then $\nu_{kR^p}(C(\alpha)) = \nu_{kR^p}(C(\beta))$.*

Proof. Since $\alpha, \beta \in H_1(\underline{t}) = kK^p[t_1^{p-1}dt_1, \cdots, t_n^{p-1}dt_n]$, it follows that $\nu_R(\alpha) = \nu_R(\beta) = mp$, or $\nu_R(\alpha) = \nu_R(\beta) = mp + p - 1$, for some integer m. Since kR^p is a discrete valuation ring with a prime element t_1^p, $\nu_{kR^p}(t_1^p) = 1$ and since $C(t_i^{p-1}dt_i) = dt_i^p$ for each i, we obtain that $\nu_{kR^p}(C(\alpha)) = \nu_{kR^p}(C(\beta)) = m$.

§2. Main theorems

Let ω be an element of Z_m $(m \geq 1)$. Then we have by Lemma 2 that ω is uniquely expressed in the form $\omega_1 + \omega_2$, where $\omega_1 \in B_m,\ \omega_2 \in H_m(\underline{t})$.

Theorem 1. *Let ω, ω_1, and ω_2 be as above. Then we have* $\nu_R(\omega) = \min(\nu_R(\omega_1), \nu_R(\omega_2))$.

Proof. If $\nu_R(\omega_1) \neq \nu_R(\omega_2)$, then we have $\nu_R(\omega) = \min(\nu_R(\omega_1), \nu_R(\omega_2))$. Therefore we may assume that $\nu_R(\omega_1) = \nu_R(\omega_2) = s$. Then it is enough to show that $\nu_R(\omega) = s$. We prove this by induction on m.

First we prove the case of $m = 1$. Using the base Λ of Ω over K, we can express ω_1 and ω_2 as follows ;

$\omega_1 = \cdots + xdt_{i_1} \wedge \cdots \wedge dt_{i_r} + \cdots$

$\omega_2 = \cdots + ydt_{i_1} \wedge \cdots \wedge dt_{i_r} + \cdots$.

In the case $\nu_R(x) = \nu_R(y) = s$, it will be enough to show $\nu_R(x+y) = s$. Since $\omega_2 \in H_1(\underline{t})$, y is of the form $\alpha t_{i_1}^{p-1} \cdots t_{i_r}^{p-1}$ $(\alpha \in kK^p)$. Since $\omega_1 \in B$, $\omega_1 = d\omega_0$ for some $\omega_0 \in \Omega$. Since any element a of K is uniquely written in the form

$$a = \sum_{i_1, \cdots, i_r = 0}^{p-1} \alpha_{i_1 \cdots i_n} t_1^{i_1} \cdots t_n^{i_n} \quad (\alpha_{i_1 \cdots i_n} \in kK^p),$$

hence the definition of da, the definition of $d\omega_0$ and Lemma 3 show that $\nu_R(x+y) = s$.

Next we assume that this theorem is true for $1, 2, \cdots, m-1$ $(m \geq 2)$. We may assume that $\nu_R(\omega_1) = \nu_R(\omega_2) = s$. Since $B_m \subset Z_{m-1}$, it follows that $B_m = B_m \cap Z_{m-1} = B_m \cap (B_{m-1} + H_{m-1}) = B_{m-1} + B_m \cap H_{m-1}$ (direct sum). Therefore $\omega_1 \in B_m$ is uniquely written in the form $\omega_{11} + \omega_{12}$, where $\omega_{11} \in B_{m-1}$ and $\omega_{12} \in B_m \cap H_{m-1}$. Since $\omega_{11} \in B_{m-1}$ and $\omega_{12} \in H_{m-1}$, we get by the assumption of induction that

$$s = \nu_R(\omega_1) = \min(\nu_R(\omega_{11}), \nu_R(\omega_{12})).$$

Case I. $\nu_R(\omega_{11}) = s$. It is easy to see that $\omega_{12} + \omega_2 \in H_{m-1} + H_m = H_{m-1}$ and $\nu_R(\omega_{11} + \omega_2) \geq s$. Since $\omega_{11} \in B_{m-1}$, we get by the assumption of induction on m that

$$\begin{aligned}\nu_R(\omega) &= \nu_R(\omega_1 + \omega_2) = \nu_R(\omega_{11} + (\omega_{12} + \omega_2)) \\ &= \min(\nu_R(\omega_{11}), \nu_R(\omega_{12} + \omega_2)) = s.\end{aligned}$$

Case II. $\nu_R(\omega_{12}) = s$ and $\nu_R(\omega_{11}) > s$. In this case, we have that $\omega_{12} \in B_m \cap H_{m-1} \subset H_1$, $\omega_2 \in H_m \subset H_1$ and $\nu_R(\omega_{12}) = \nu_R(\omega_2) = s$, where $s = mp$, or $s = mp + p - 1$ for some integer m (see Lemma 4). By Lemma 4, $\nu_{kR^p}(C(\omega_{12})) = \nu_{kR^p}(C(\omega_2)) = m$. On the other hand, since $\omega_{12} \in B_m$ and $\omega_2 \in H_m$, we have $C(\omega_{12}) \in B_{m-1}(\Omega_1)$ and $C(\omega_2) \in H_{m-1}(\underline{t^p})$. By the assumption of induction on m, we get that

$$\begin{aligned}\nu_{kR^p}(C(\omega_{12} + \omega_2)) &= \nu_{kR^p}(C(\omega_{12}) + C(\omega_2)) \\ &= \min(\nu_{kR^p}(C(\omega_{12})), \nu_{kR^p}(C(\omega_2))) = m.\end{aligned}$$

It then follows that $\nu_R(\omega_{12} + \omega_2) = mp$ or $mp + p - 1$ (c.f. Lemma 4). Furthermore one can observe that $\nu_R(\omega_{12} + \omega_2) = s$ (c.f. Lemma 3). Since $\nu_R(\omega_{11}) > s$, we get that $\nu_R(\omega) = \nu_R(\omega_{11} + \omega_{12} + \omega_2) = s$, as desired.

Theorem 2. *Let ω be an element of Z_m such that $\nu_R(\omega) \geq -p^m + 1$. Let $\underline{t} = \{t_1, \cdots, t_n\}$ and $\underline{u} = \{u_1, \cdots, u_n\}$ be two parameters of $(K/k, R)$. Then $res_{R,\underline{t}}(\omega) - res_{R,\underline{u}}(\omega)$ is an element of $B_m\Omega(D/k)$. In other words, $res_{R,\underline{t}}(\omega)$ is uniquely determined by R up to addition by differentials in $B_m\Omega(D/k)$.*

Proof. By Lemma 2 we have $\omega = \omega_1 + \omega_2$, where $\omega_1 \in B_m$ and $\omega_2 \in H_m(\underline{t})$. Theorem 1 says that $\nu_R(\omega_2) \geq -p^m + 1$. On the other hand, since $H_m(\underline{t}) = K_m[t_1^{p^m-1}dt_1, \cdots, t_n^{p^m-1}dt_n]$, we get $\nu_R(\omega_2) \equiv 0, -1 \pmod{p^m}$. Hence it follows that $\nu_R(\omega_2) \geq -1$. From E. Kunz (Exercise (1) in §17 of [1]), we have $res_{R,\underline{t}}(\omega_2) = res_{R,\underline{u}}(\omega_2)$. Since both $res_{R,\underline{t}}$ and $res_{R,\underline{u}}$ map B_m to $B_m\Omega(D/k)$, we get that

$$res_{R,\underline{t}}(\omega) - res_{R,\underline{u}}(\omega) = res_{R,\underline{t}}(\omega_1) - res_{R,\underline{u}}(\omega_1) \in B_m\Omega(D/k).$$

From this theorem, we can define the residue map res_R, which is independent from the choice of a parameter $\underline{t}$.

Corollary. *$res_R : Z_\infty \longrightarrow Z_\infty\Omega(D/k)/B_\infty\Omega(D/k)$ is well defined.*

We will show an example which asserts that the number $-p^m + 1$ in Theorem 2 is the best possible. In fact, we can find a function field K/k, a valuation ring R of K/k, two parameters $\underline{t}$ and $\underline{u}$ of $(K/k, R)$ and a differential form $\omega \in Z_m$ such that $\nu_R(\omega) = -p^m$, $res_{R,\underline{u}}(\omega) = 0$ and such that $res_{R,\underline{t}}(\omega) \notin Z_{m+1}(\Omega(D))$. So the difference $res_{R,\underline{t}}(\omega) - res_{R,\underline{u}}(\omega)$ does not belong to $B_\infty(\Omega(D))$ because $B_\infty(\Omega(D)) \subset Z_{m+1}(\Omega(D))$.

Example. Let $K = k(x, y, z)$ be the rational function field of 3 variables x, y, z over k and let $R = k(y, z)[x]_{(x)}$. Then $\underline{t} = \{x, y, z\}$ is a parameter of $(K/k, R)$. If we set $y_1 = y - x$, then $\underline{u} = \{x, y_1, z\}$ is also a parameter of $(K/k, R)$. We note that $R = k(y_1, z)[x]_{(x)}$ and that $\widehat{R} = k(y, z)[[x]] = k(y_1, z)[[x]]$.

Let $\omega = (x^{-1}y)^{p^m} y^{p^m-1} z^{p^m-1} dy \wedge dz$. It follows that $\omega \in H_m(\underline{t}) \subset Z_m$ and $res_{R,\underline{t}}(\omega) = 0$. On the other hand,

$$\omega = (1 + x^{-1}y_1)^{p^m} (x + y_1)^{p^m-1} z^{p^m-1} \{(dx + dy_1) \wedge dz\}.$$

From this, it follows that

$res_{R.\underline{u}}(\omega) = \overline{y_1}^{p^m} \overline{z}^{p^m-1} d\overline{z}$ and $C^m_{D/k}(\overline{y_1}^{p^m} \overline{z}^{p^m-1} d\overline{z}) = \overline{y_1}^{p^m} d\overline{z}^{p^m}$,

where $C^m_{D/k} = C_{D_{m-1}/k} \circ \cdots \circ C_{D/k}$ $(D_i = kD^{p^i})$.

Since $\{\overline{y_1}^{p^m}, \overline{z}^{p^m}\}$ is a p-basis of D_m/k, we have

$$d(\overline{y_1}^{p^m} d\overline{z}^{p^m}) = d\overline{y_1}^{p^m} \wedge d\overline{z}^{p^m} \neq 0.$$

Thus we get that $\overline{y_1}^{p^m} d\overline{z}^{p^m} \notin Z(\Omega(D_m))$ and $res_{R,\underline{u}}(\omega) \notin Z_{m+1}(\Omega(D))$.

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