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REGULARITY OF SOLUTIONS TO THE WAVE EQUATION WITH A NON SMOOTH COEFFICIENT

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Abstract. In this paper, we show that the regularity of solutions to wave equation with a non smooth coefficient propagates through the points at which the coefficient is singular.

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1. Introduction

In this paper, we shall study the regularity of solutions to the wave equation

$$\Box u + a(t,x)u = 0 \tag{1.1}$$

with a non smooth coefficient $a(t,x)$ in an open neighbourhood Ω of the origin in $\mathbf{R}_t \times \mathbf{R}_x^n$, where $\Box = \partial^2/(\partial t)^2 - \triangle_x = \partial^2/(\partial t)^2 - \sum_{i=1}^n \partial^2/(\partial x_i)^2$. We assume that a satisfies the following assumption.

Assumption A. *The coefficient a is in $\mathcal{D}'(\Omega)$ and there exists a positive number s_1 with $\frac{n+1}{2} - 1 < s_1$ and a vector $v \in \mathbf{R}^n$ with $|v| < 1$ such that*

$$(1+\tau^2+|\xi|^2)^{s_1/2}(1+|\tau+v\cdot\xi|^2)^{s_2/2}\widehat{\varphi a}(\tau,\xi) \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$$

for any $s_2 > 0$ and any $\varphi(t,x) \in C_0^\infty(\Omega)$, where $\widehat{\varphi a}$ is the Fourier transform of φa and (τ,ξ) are the dual variables of (t,x) .

We show that if a solution of (1.1) has H^r-regularity in $Char\Box \cap T^*K\backslash 0$ microlocally with a domain K, then the solution has H^r-regularity in $Char\Box \cap T^*\widehat{K}\backslash 0$, where $\widehat{K}$ is a domain in which the value of the solution is determined by the value of the solution in K. (In the following, we call this domain a domain of determine.) To illustrate our results, let us suppose for the moment that a vanishes on $t \le 0$ and $1 \le t$. Our result asserts that if u is smooth in $t < 0$, then u is smooth in $t > 1$. In other words, the regularity of u propagates through the domain where a is singular.

Rauch [8] has studied the propagation of singularities of solutions to semilinear wave equations, $\Box u = f(u)$. He has shown that if a solution is in $H^s(s > \frac{n+1}{2})$ and if the solution is in $H^r(s < r < 2s - \frac{n+1}{2})$ at (x_0, ξ_0) microlocally, then the solution is in H^r on the null bicharcteristic curve starting from (x_0, ξ_0). Bony [2] has had the same result as Rauch[8] for general nonlinear equations. Beals and Reed [1] investigated the propagation of H^r – singularity $(s < r < 2s - \frac{n+1}{2})$ for linear strictly hyperbolic equations assuming that the coefficients are in $H^s(s > \frac{n+1}{2})$. They have shown that if a solution is in $H^s(s > \frac{n+1}{2})$ and if the solutions is in $H^r(s < r < 2s - \frac{n+1}{2})$ at (x_0, ξ_0) microlocally, the solution is in H^r on the null bicharcteristic curve starting from (x_0, ξ_0). Their technique is due to one in Rauch [8] and the commutator estimate. Bony [3][4] and Melrose and Ritter [7] studied H^r-regularity for all $r > s$ for semilinear wave equations. Their technique to get regularity is to use suitable vector fields. In this article, we treat H^r-regularity for all $r > s$ of solutions to wave equations with a non smooth coefficient assuming that the coefficient a is in $H^s(s > \frac{n+1}{2})$. Our technique is Lorentz transformation and multiplication estimate in some Sobolev spaces which is essentially due to Rauch [8].

To state the main theorem precisely, we introduce some notations and function spaces. For $s \in \mathbf{R}$, $H^s(\mathbf{R}^n)$ is the Sobolev space of order s and for a domain $\mathcal{O}$ in $\mathbf{R}^n$, $H^s_{loc}(\mathcal{O}) = \{u \in \mathcal{D}'(\mathcal{O}); \varphi u \in H^s(\mathbf{R}^n) \text{ for any } \varphi \in \mathcal{D}(\mathcal{O})\}$. For $r \in \mathbf{R}$, we say $u \in H^r$ at $(t_0, x_0, \tau_0, \xi_0) \in T^*(\Omega)\backslash 0$ microlocally, if there exist $\varphi(t,x) \in C_0^\infty(\Omega)$ with $\varphi(t_0, x_0) \neq 0$ and a conic neighborhood $\Xi(\tau_0, \xi_0)$ of (τ_0, ξ_0) in $\mathbf{R}^{n+1}_{\tau,\xi}$ such that

$$\iint_{\Xi(\tau_0,\xi_0)} (1 + |\tau|^2 + |\xi|^2)^r |\widehat{\varphi u}|^2 d\tau d\xi < +\infty.$$

$Char\Box = \{(t, x, \tau, \xi) \in T^*\mathbf{R}^{n+1}\backslash 0; \tau^2 - |\xi|^2 = 0\}$. We write for $(t_0, x_0) \in \mathbf{R}_t \times \mathbf{R}^n_x$,

$$C^-_{(t_0,x_0)} = \{(t,x) \in \mathbf{R}^{n+1}; (t - t_0)^2 \geq |x - x_0|^2 \text{ and } t \leq t_0\}.$$

For $w \in \mathbf{R}^n$, we set $T_w = \{(t,x) \in \mathbf{R}^{n+1}_{t,x}; t - w \cdot x = 0\}$. We call an n-dimensional hyperplane T_w is spacelike if $|w| < 1$. For $K \subset T_w$ with $|w| < 1$ and $K \subset \Omega$,

$$\widehat{K} = \{(t,x) \in \Omega; [C^-_{(t,x)} \cap T_w] \subset K \text{ and } [C^-_{(t,x)} \cap \{t - w \cdot x \geq 0\}] \subset \Omega\}$$

is the domain of determine with respect to K in Ω. The main result of this paper is given by the following theorem.

Theorem. *Let Ω be as above and let $K \subset \Omega$ be a subset of a hyperplane $T_w = \{(t,x) \in \mathbf{R}^{n+1}_{t,x}; t - w \cdot x = 0\}$ with $|w| < 1$. Let a satisfy Assumption A and s be a positive real number satisfying $s_1 + s - \frac{n+1}{2} > 0$. Suppose that u satisfies (1.1), $u \in H^s_{loc}(\Omega)$ and*

$$u \in H^r \quad \text{on} \quad (K \times \mathbf{R}^{n+1}_{\tau,\xi} \backslash \{0\}) \cap Char\Box \quad \text{microlocally.}$$

Then

$$(1.2) \qquad (1 + \tau^2 + |\xi|^2)^{s/2} (1 + |\tau + v \cdot \xi|^2)^{(r-s)/2} \widehat{\varphi u}(\tau, \xi) \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$$

for all $\varphi(t,x) \in C_0^\infty(\widehat{K})$ where $\widehat{K}$ is the domain of determine with respect to K in Ω.

Remark 1. A typical example of the coefficient a is given by $a(t,x) = f(x + vt)$ with $f(x) \in H^{s_1}_{loc}(\mathbf{R}^n)$.

Remark 2. The theorem implies, in particular, $u \in H^r$ on $(\widehat{K} \times \mathbf{R}^{n+1}_{\tau,\xi} \backslash \{0\}) \cap Char\Box$ microlocally.

Remark 3. If $a(t,x) \in C^\infty$ in a neighborhood of $(t_0, x_0) \in \widehat{K}$, then $u \in H^r$ in a neighborhood of (t_0, x_0).

The proof of the theorem will be given by a series of lemmas in §2.

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2. Proof of Theorem

We prepare the following three lemmas to prove the theorem.

Lemma 1. *If the theorem holds when $v = 0$, so does it for general $|v| < 1$.*

Proof. Without loss of generality, we may assume $v = (v_1, 0, \ldots, 0)$. By the Lorentz transformation $t = \frac{t' + v_1 x_1'}{\sqrt{1 - v_1^2}}, x_1 = \frac{x_1' + t' v_1}{\sqrt{1 - v_1^2}}, x_2 = x_2', \ldots, x_n = x_n'$, the equation (1.1) is transformed to

$$\Box \widetilde{u}(t,x) + \widetilde{a}(t,x) \widetilde{u}(t,x) = 0,$$

where

$$\widetilde{u}(t,x) = u\left(\frac{t + v_1 x_1}{\sqrt{1 - v_1^2}}, \frac{x_1 + t v_1}{\sqrt{1 - v_1^2}}, x_2, \ldots, x_n\right)$$

and

$$\widetilde{a}(t,x) = a\left(\frac{t + v_1 x_1}{\sqrt{1 - v_1^2}}, \frac{x_1 + t v_1}{\sqrt{1 - v_1^2}}, x_2, \ldots, x_n\right).$$

We denote the image of Ω, K and $\widehat{K}$ under the Lorentz trasformation by $\widetilde{\Omega}$, $\widetilde{K}$ and $\widetilde{\widehat{K}}$ respectively. Since Lorentz trasformation maps spacelike hyperplanes to spacelike hyperplanes, $\widetilde{K}$ is spacelike. Obviously, $\widetilde{u} \in H^s_{loc}(\widetilde{\Omega})$ and $\widetilde{u} \in H^r$ on $(\widetilde{K} \times \mathbf{R}^{n+1}_{\tau,\xi} \backslash \{0\}) \cap Char\Box$ microlocally.

Note that $\widetilde{a}(t,x)$ satisfies Assumption A with $v = 0$. Indeed, for any $\varphi(t,x) \in C_0^\infty(\widetilde{\Omega})$, we have, with the same notation for $\widetilde{\varphi}$ as above,

(2.1)

$$\begin{aligned}
&\iint (1+\tau^2+|\xi|^2)^{s_1}(1+\tau^2)^{s_2}|\widehat{\varphi\widetilde{a}}(\tau,\xi)|^2 d\tau d\xi \\
=&\iint (1+\tau^2+|\xi|^2)^{s_1}(1+\tau^2)^{s_2}|\widehat{\widetilde{\varphi}a}(\frac{\tau - v_1\xi_1}{\sqrt{1-v_1^2}}, \frac{\xi_1 - \tau v_1}{\sqrt{1-v_1^2}}, \xi_2, \dots, \xi_n)|^2 d\tau d\xi \\
=&\iint \left(1 + \frac{(\tau+v_1\xi_1)^2}{1-v_1^2} + \frac{(\xi_1+\tau v_1)^2}{1-v_1^2} + \xi_2^2 + \dots + \xi_n^2\right)^{s_1} \\
&\qquad\qquad\qquad \times \left(1 + \frac{(\tau+v_1\xi_1)^2}{1-v_1^2}\right)^{s_2} |\widehat{\widetilde{\varphi}a}(\tau,\xi)|^2 d\tau d\xi \\
\le& C \iint (1+\tau^2+|\xi|^2)^{s_1}(1+|\tau+v\cdot\xi|^2)^{s_2}|\widehat{\widetilde{\varphi}a}(\tau,\xi)|^2 d\tau d\xi < +\infty,
\end{aligned}$$

where we made the change of variables $\tau = \frac{\tau' + v_1\xi_1'}{\sqrt{1-v_1^2}}, \xi_1 = \frac{\xi_1' + \tau' v_1}{\sqrt{1-v_1^2}}, \xi_2 = \xi_2', \dots,$ $\xi_n = \xi_n'$ in the second step. If the statement of the theorem for the case $v = 0$ is valid, we have $(1+\tau^2+|\xi|^2)^{s/2}(1+\tau^2)^{(s-r)/2}|\widehat{\varphi\widetilde{u}}(\tau,\xi)| \in L^2(\mathbf{R}^n_{\tau,\xi})$ for all $\varphi(t,x) \in C_0^\infty(\widetilde{\widetilde{K}})$. Hence by the argument similar to (2.1), we obtain

$$\iint (1+\tau^2+|\xi|^2)^{s}(1+|\tau+v\cdot\xi|^2)^{r-s}|\widehat{\widetilde{\varphi}u}(\tau,\xi)|^2 d\tau d\xi < +\infty. \quad \Box$$

Lemma 2. *Let $0 \le s, t \le \frac{n}{2}$ with $s + t - \frac{n}{2} > 0$ and suppose that $u \in H^s_{loc}(\Omega)$ and $v \in H^t_{loc}(\Omega)$. Then*

$$uv \quad \in \quad H^{s+t-(n/2)-\epsilon}_{loc}(\Omega)$$

for any $\epsilon > 0$.

Proof. Replacing u and v by φu and φv respectively with $\varphi \in C_0^\infty(\Omega)$, it suffices to show $uv \in H^{s+t-(n/2)-\epsilon}(\mathbf{R}^n)$ when $u \in H^s(\mathbf{R}^n)$ and $v \in H^t(\mathbf{R}^n)$. Write $\sqrt{1+|\xi|^2} = \langle\xi\rangle$. Then

$$\begin{aligned}
&\langle\xi\rangle^{s+t-(n/2)-\epsilon}|\widehat{uv}(\xi)| \\
=& C\langle\xi\rangle^{s+t-(n/2)-\epsilon}\left|\int \widehat{u}(\xi-\eta)\widehat{v}(\eta)d\eta\right|
\end{aligned}$$

$$
\begin{aligned}
\le & C\langle\xi\rangle^{s+t-(n/2)-\epsilon}\Big(\int_{D1}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta+\int_{D2}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta \\
& \qquad +\int_{D3}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta+\int_{D4}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta\Big) \\
= & I_1+I_2+I_3+I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
D1 &= \{\eta\in\mathbf{R}^n; |\xi-\eta|\ge\frac{1}{2}|\xi| \quad \text{and} \quad \frac{1}{2}|\xi|\ge|\eta|\}, \\
D2 &= \{\eta\in\mathbf{R}^n; |\xi-\eta|\le\frac{1}{2}|\xi| \quad \text{and} \quad \frac{1}{2}|\xi|\le|\eta|\}, \\
D3 &= \{\eta\in\mathbf{R}^n; |\xi-\eta|\ge|\eta|\ge\frac{1}{2}|\xi|\}, \\
D4 &= \{\eta\in\mathbf{R}^n; |\eta|\ge|\xi-\eta|\ge\frac{1}{2}|\xi|\}.
\end{aligned}
$$

As $s>0$ and $t-\frac{n}{2}-\epsilon<0$,

$$
\begin{aligned}
I_1 \le & C\int_{D1}\langle\xi-\eta\rangle^s|\widehat{u}(\xi-\eta)|\langle\eta\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\eta)|d\eta \\
\le & C\int_{\mathbf{R}^n}\langle\xi-\eta\rangle^s|\widehat{u}(\xi-\eta)|\langle\eta\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\eta)|d\eta.
\end{aligned}
$$

Since $\langle\xi\rangle^s|\widehat{u}(\xi)|\in L^2$ and $\langle\xi\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\xi)|\in L^1$, Hausdorff-Young's inequality implies that $I_1\in L^2(\mathbf{R}^n_\xi)$. Using the same argument as above, we see I_2 also belongs to $L^2(\mathbf{R}^n_\xi)$. As $s+t-\frac{n}{2}-\epsilon>0$ and $t-\frac{n}{2}-\epsilon<0$,

$$
\begin{aligned}
I_3 \le & C\int_{D3}\langle\xi-\eta\rangle^{s+t-(n/2)-\epsilon}|\widehat{u}(\xi-\eta)\widehat{v}(\eta)|d\eta \\
\le & C\int_{D3}\langle\xi-\eta\rangle^s|\widehat{u}(\xi-\eta)|\langle\eta\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\eta)|d\eta \\
\le & C\int_{\mathbf{R}^n}\langle\xi-\eta\rangle^s|\widehat{u}(\xi-\eta)|\langle\eta\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\eta)|d\eta.
\end{aligned}
$$

Since $\langle\xi\rangle^s|\widehat{u}(\xi)|\in L^2$ and $\langle\xi\rangle^{t-(n/2)-\epsilon}|\widehat{v}(\xi)|\in L^1$, Hausdorff-Young's inequality implies that $I_3\in L^2(\mathbf{R}^n_\xi)$. Using the same argument as above, we see $I_4\in L^2(\mathbf{R}^n_\xi)$. Hence,

$$
\langle\xi\rangle^{s+t-(n/2)-\epsilon}|\widehat{uv}(\xi)| \in L^2(\mathbf{R}^n_\xi). \quad \square
$$

Definition. *For $s, s' \in \mathbf{R}$, we say $u \in H^{s,s'}(\mathbf{R}^{n+1}_{t,x})$ if $u \in \mathcal{S}'(\mathbf{R}^{n+1}_{t,x})$ and*

$$(1+\tau^2+|\xi|^2)^{s/2}(1+\tau^2)^{s'/2}\widehat{u}(\tau,\xi) \in L^2(\mathbf{R}^{n+1}_{\tau,\xi}).$$

$H^{s,s'}_{loc}(\Omega) = \{\varphi u \in \mathcal{D}'(\Omega); \varphi u \in H^{s,s'}(\mathbf{R}^{n+1}_{t,x})$ *and for any* $\varphi \in C_0^\infty(\Omega)\}$.

Remark. The $H^{s,s'}_{loc}(\Omega)$ in this definition is slightly different from the one in Hörmander's book[5].

Lemma 3. *Let $0 < s \le \frac{n+1}{2}$ and $\frac{n+1}{2} - 1 < s_1 \le \frac{n+1}{2}$ with $s + s_1 - \frac{n+1}{2} > 0$ and let $r > s$. Suppose that $u \in H^{s,r-s}_{loc}(\Omega)$ and $v \in H^{s_1,s_2}_{loc}(\Omega)$ for all $s_2 > 0$, then*

$$uv \in H^{t_1,t_2}_{loc}(\Omega),$$

where $t_1 = s + s_1 - \frac{n+1}{2} - \epsilon$ and $t_2 = r - s$ for any $\epsilon > 0$.

Proof. Replacing u and v by φu and φv respectively with $\varphi \in C_0^\infty(\Omega)$, it suffices to show $uv \in H^{t_1,t_2}(\mathbf{R}^{n+1}_{t,x})$ for $u \in H^{s,r-s}(\mathbf{R}^{n+1}_{t,x})$ and $v \in H^{s_1,s_2}(\mathbf{R}^{n+1}_{t,x})$. We denote $(1+\tau^2+|\xi|^2)^{1/2}$ and $(1+\tau^2)^{1/2}$ by $\langle\tau,\xi\rangle$ and $\langle\tau\rangle$ respectively. We set $\zeta = (\tau,\xi)$. We show that $\langle\tau,\xi\rangle^{t_1}\langle\tau\rangle^{t_2}|\widehat{uv}(\tau,\xi)| \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$.

$$\begin{aligned}
&\langle\tau,\xi\rangle^{t_1}\langle\tau\rangle^{t_2}|\widehat{uv}(\tau,\xi)| \\
=&\langle\tau,\xi\rangle^{t_1}\langle\tau\rangle^{t_2}\left|\iint \widehat{u}(\tau-\tau',\xi-\xi')\widehat{v}(\tau',\xi')d\tau'd\xi'\right| \\
\le&\langle\tau,\xi\rangle^{t_1}\langle\tau\rangle^{t_2}\sum_{i=1}^{8}\iint_{Di}|\widehat{u}(\tau-\tau',\xi-\xi')\widehat{v}(\tau',\xi')|d\tau'd\xi' = \sum_{i=1}^{8} J_i,
\end{aligned}$$

where the domain of the integrations Di are as follows:

$$\text{(D1)} \qquad |\zeta-\zeta'| \ge \frac{1}{2}|\zeta| \ge |\zeta'|\,, \qquad |\tau-\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D2)} \qquad |\zeta-\zeta'| \ge \frac{1}{2}|\zeta| \ge |\zeta'|\,, \qquad |\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D3)} \qquad |\zeta-\zeta'| \le \frac{1}{2}|\zeta| \le |\zeta'|\,, \qquad |\tau-\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D4)} \qquad |\zeta-\zeta'| \le \frac{1}{2}|\zeta| \le |\zeta'|\,, \qquad |\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D5)} \qquad \frac{1}{2}|\zeta| \le |\zeta-\zeta'| \le |\zeta'|\,, \qquad |\tau-\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D6)} \qquad \frac{1}{2}|\zeta| \le |\zeta-\zeta'| \le |\zeta'|\,, \qquad |\tau'| \ge \frac{1}{2}|\tau|\,;$$

$$\text{(D7)} \qquad \frac{1}{2}|\zeta| \le |\zeta'| \le |\zeta-\zeta'|\,, \qquad |\tau-\tau'| \ge \frac{1}{2}|\tau|\,;$$

(D8) $$\frac{1}{2}|\zeta| \leq |\zeta'| \leq |\zeta - \zeta'|\,, \qquad |\tau'| \geq \frac{1}{2}|\tau|\,.$$

First we estimate J_1.

$$J_1 \leq C \iint_{\mathbf{R}^{n+1}} \langle \tau - \tau', \xi - \xi' \rangle^s \langle \tau - \tau' \rangle^{r-s} |\widehat{u}(\tau - \tau', \xi - \xi')| \times \langle \tau', \xi' \rangle^{s_1 - \frac{n+1}{2} - \epsilon} |\widehat{v}(\tau', \xi')| d\tau' d\xi'.$$

Since $\langle \tau, \xi \rangle^s \langle \tau \rangle^{r-s} |\widehat{u}(\tau, \xi)| \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$ and $\langle \tau, \xi \rangle^{s_1 - \frac{n+1}{2} - \epsilon} |\widehat{v}(\tau, \xi)| \in L^1(\mathbf{R}^{n+1}_{\tau,\xi})$,

Housdorff-Young's inequality yields that $J_1 \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$. Using the same argument as above, we see that J_2, J_3 and J_4 are also in $L^2(\mathbf{R}^{n+1}_{\tau,\xi})$. Next we estimate J_5. Note that $\langle \tau, \xi \rangle^{t_1} \leq C \langle \tau', \xi' \rangle^{t_1} \leq C \langle \tau - \tau', \xi - \xi' \rangle^{s - \frac{n+1}{2} - \epsilon} \langle \tau', \xi' \rangle^{s_1}$ in $D5$. Hence,

$$J_5 \leq C \iint_{\mathbf{R}^{n+1}} \langle \tau - \tau', \xi - \xi' \rangle^{s - \frac{n+1}{2} - \epsilon} \langle \tau - \tau' \rangle^{r-s} |\widehat{u}(\tau - \tau', \xi - \xi')| \times \langle \tau', \xi' \rangle^{s_1} |\widehat{v}(\tau', \xi')| d\tau' d\xi'.$$

Since $\langle \tau, \xi \rangle^{s - \frac{n+1}{2} - \epsilon} \langle \tau \rangle^{r-s} |\widehat{u}(\tau, \xi)| \in L^1(\mathbf{R}^{n+1}_{\tau,\xi})$ and $\langle \tau, \xi \rangle^{s_1} |\widehat{v}(\tau, \xi)| \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$,

Hausdorff-Young's inequality proves $J_5 \in L^2(\mathbf{R}^{n+1}_{\tau,\xi})$. Using the same argument as above, we see that J_6, J_7 and J_8 are also in $L^2(\mathbf{R}^{n+1}_{\tau,\xi})$.

Proof of the theorem. By virtue of the lemma 1, it suffices to prove the theorem for the case $v = 0$. We devide the proof of the theorem into two steps. We shall show in the first step that $u \in H^{\frac{n+1}{2}}_{loc}(\widehat{K})$ by using the lemma 2, and in the second step $u \in H^{s,r-s}_{loc}(\widehat{K})$ by using the lemma 3.

(First Step) Let $(t_0, x_0, \tau_0, \xi_0) \in T^*\widehat{K} \backslash 0 \cap Char\square$. Since $\widehat{K}$ is the domain of determine with respect to K in Ω, there exists a point $(\widetilde{t_0}, \widetilde{x_0}) \in K$ such that the null bicharacteristic curve starting from the point $(\widetilde{t_0}, \widetilde{x_0}, \tau_0, \xi_0)$ passes through $(t_0, x_0, \tau_0, \xi_0)$. The assumption A implies $a \in H^{s_1}_{loc}(\Omega)$ and u is in $H^s_{loc}(\Omega)$. Hence the lemma 2 yields $u\, a \in H^{s+s_1-(n+1)/2-\epsilon}_{loc}(\Omega)$ for any $\epsilon > 0$. Thus $\square\, u = -au \in H^{s+s_1-(n+1)/2-\epsilon}_{loc}(\Omega)$ and $u \in H^r$ at $(\widetilde{t_0}, \widetilde{x_0}, \tau_0, \xi_0)$ microlocally. It follows by
Hörmander's theorem for propagation of singularities (e.g. Taylor[6]) that

$$u \in H^{min(s+\delta,r)} \text{ at } (t_0, x_0, \tau_0, \xi_0) \text{ microlocally with } \delta = s_1 - \frac{n+1}{2} + 1 - \epsilon.$$

If $(t_0, x_0, \tau_0, \xi_0) \in T^*\widehat{K} \backslash 0 \cap (Char\square)^c$ where $(Char\square)^c$ is the complement of $Char\square$ in $T^*\widehat{K} \backslash 0$, then $\square$ is elliptic at $(t_0, x_0, \tau_0, \xi_0)$ microlocally. Thus

$$u \in H^{min(s+\delta+1,r)} \text{ at } (t_0, x_0, \tau_0, \xi_0) \text{ microlocally.}$$

Hence, since $(t_0, x_0) \in \widehat{K}$ is chosen arbitrarily, we have $u \in H^{min(s+\delta,r)}(\widehat{K})$. Repeating the same argument as above $(m-1)$-times until $s + m\delta$ becomes greater than $\frac{n+1}{2}$, we have

$$u \in H^{min(s+m\delta,r)}(\widehat{K}), \quad s + (m-1)\delta \leq \frac{n+1}{2} < s + m\delta. \tag{2.2}$$

Note that if $s + m\delta > \frac{n+1}{2}$, the argument as above does not work as the lemma 2 does not apply to this case. If $r \leq s + m\delta$, we are done, since (2.2) implies (1.2).

(Second Step) Suppose that $r > s + m\delta$. Note that if $b > 0$, $H^{s-b,s'+b} \subset H^{s,s'}$. Hence (2.2) shows

$$u \in H^{s+(m-1)\delta,\delta}_{loc}(\widehat{K}). \tag{2.3}$$

By virtue of (2.3) and the assumption A, the lemma 3 implies that

$$au \in H^{t_1,\delta}_{loc}(\widehat{K}), \tag{2.4}$$

with $t_1 = s + (m-1)\delta + s_1 - \frac{n+1}{2} - \epsilon$. We use the same notation as in the first step. Let $(t_0, x_0, \tau_0, \xi_0) \in T^*\widehat{K}\backslash 0 \cap Char\Box$. The same argument as in the first step guarantees the existence of the null bicharacteristic curve Γ starting from the point $(\widetilde{t_0}, \widetilde{x_0}, \tau_0, \xi_0)$ and passing through $(t_0, x_0, \tau_0, \xi_0)$. Recalling the definition of $H^{s,s'}$, we immediately have from (2.4) that

$$au \in H^{s+m\delta+s_1-\frac{n+1}{2}-\epsilon} \text{ on } \Gamma \cap T^*\widehat{K}\backslash 0 \text{ microlocally.} \tag{2.5}$$

From (2.5) and the assumption that $u \in H^r$ at $(\widetilde{t_0}, \widetilde{x_0}, \tau_0, \xi_0)$, Hörmander's theorem for propagation of singularities implies $u \in H^{min(s+(m+1)\delta,r)}$ at $(t_0, x_0, \tau_0, \xi_0)$ microlocally. We set $\Sigma_{\epsilon_1} = \{(\tau, \xi) \in \mathbf{R}^{n+1}; \tau^2 \geq (1+\epsilon_1)|\xi|^2 \text{ or } \tau^2 \leq (1-\epsilon_1)|\xi|^2\}$. Since $(t_0, x_0, \tau_0, \xi_0)$ is chosen arbitrarily in $T^*\widehat{K}\backslash 0 \cap Char\Box$, we have

$$u \in H^{s+(m+1)\delta} \quad \text{on} \quad \widetilde{K} \times \Sigma^c_{\epsilon_1} \qquad \text{microlocally,} \tag{2.6}$$

for sufficiently small $\epsilon_1 > 0$ where $\Sigma^c_{\epsilon_1}$ is the complement of Σ_{ϵ_1} in $\mathbf{R}^{n+1}_{\tau,\xi}\backslash\{0\}$. We take and fix $\varphi(t,x) \in C_0^\infty(\widehat{K})$ and we define $F(t,x)$ by

$$\Box(\varphi u) = \frac{\partial\varphi}{\partial t}\frac{\partial u}{\partial t} + \frac{\partial^2\varphi}{\partial t^2}u - 2\nabla\varphi\cdot\nabla u - (\triangle\varphi)u - \varphi a u =: F(t,x). \tag{2.7}$$

From (2.3), (2.4) and the fact that $s_1 - \frac{n+1}{2} - \epsilon > -1$, we have $F(t,x) \in H^{s+(m-1)\delta-1,\delta}(\widehat{K})$. Taking the Fourier trasformation of both sides of (2.7), we have $(\tau^2 - |\xi|^2)\widehat{\varphi u}(\tau, \xi) = \widehat{F}(\tau, \xi)$. From the fact $\left|\frac{\tau^2+|\xi|^2}{\tau^2-|\xi|^2}\right| \le C \quad$ on Σ_{ϵ_1}, we obtain

$$(1 + \tau^2 + |\xi|^2)^{s+(m-1)\delta+1}(1 + \tau^2)^{\delta}|\widehat{\varphi u}(\tau, \xi)|^2 \in L^1(\Sigma_{\epsilon_1}). \tag{2.8}$$

Hence we have from (2.6), (2.8) and the fact that $\delta = s_1 - \frac{n+1}{2} + 1 - \epsilon < 1$, $u \in H^{min(s+m\delta, r-\delta),\delta}_{loc}(\widehat{K})$. Repeating the same argument as above $(l-1)$-times until $r \le s + m\delta + l\delta$, we obtain $u \in H^{r-l\delta,l\delta}_{loc}(\widehat{K})$. Since $s < r - l\delta$, this implies (1.2). $\square$

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