# UNIFORM RESOLVENT CONVERGENCE OF LINEAR OPERATORS UNDER SINGULAR PERTURBATIONS 

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#### Abstract

In this paper, we investigate sufficient conditions in order that a family $T(\varepsilon)=T_{0}+\varepsilon T_{1}$ of closable linear operators with domain $D(T(\varepsilon))=$ $D\left(T_{0}\right) \cap D\left(T_{1}\right)$ converge to $T_{0}$ as $\varepsilon \downarrow 0$ in the sense of uniform and strong resolvent convergence. The obtained abstract results are applied to selfadjoint and nonselfadjoint Schrödinger operators.


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## Introduction

This paper is a continuation of the author's [1]. Our aim is to describe sufficient conditions for resolvent convergence of closed linear operators under singular perturbations in cases of abstract operators in a Hilbert space and Schrödinger operators in $L^{2}\left(\mathbf{R}^{n}\right)$. The term "singular perturbation" means that the domain of the perturbed operator does not necessarily contain the domain of the unperturbed operator (in other words, we do not assume that perturbations are relatively bounded with respect to the unperturbed operators; the relatively bounded case is sufficiently discussed in Kato's book [3]). The problem of determining this convergence is closely connected with the investigations of the stability of eigenvalues under perturbations [3] and the behavior of solutions of singularly perturbed problems [7].

Let $T_{0}$ be closed and $T_{1}$ closable in a Hilbert space. Then the basic inequality in our new sufficient conditions is described as follows:

$$
\operatorname{Re}\left(T_{0} u, T_{1} u\right) \geq-c\|u\|^{2}-a\left\|T_{0} u\right\|\|u\|-b\left\|T_{0} u\right\|^{2}, \quad u \in D\left(T_{0}\right) \cap D\left(T_{1}\right)
$$

where $a, b$ and $c$ are nonnegative constants. This inequality was first introduced by Okazawa [9]. But in [9] he considered only the case of $a=b=c=0$ and $D\left(T_{0}\right) \subset D\left(T_{1}\right)$. For the generalization to the case with $b, c \geq 0$ and $a=0$ see Yoshikawa [15] and Okazawa [10]. The case of $a \neq 0$ was first considered by Kato [4]. On the other hand, the domain inclusion was discarded in Okazawa
[10, Theorem 3.4] and [11, Theorem 2.5]; see also Sohr [13] and Miyajima [8]. More recently, the perturbation theory based on this type of inequalities was investigated by Kato [5], Okazawa [12] and Sohr [14].

In Section 1 we consider abstract selfadjoint and nonselfadjoint operators in a Hilbert space. The sufficient conditions obtained here allow us to consider a new class of perturbations, mainly for nonselfadjoint operators. Lemma 1 below is concerned with holomorphic families of type (A) of closed linear operators and is a result of independent interest.

Section 2 includes applications of the abstract result to Schrödinger operators in $L^{2}\left(\mathbf{R}^{n}\right)$. In case $n=1$ we investigate nonselfadjoint Schrödinger operators with complex-valued potentials.

## 1. Abstract operators in Hilbert spaces

Let $T$ be a linear operator with domain $D(T)$ and range $R(T)$ in a separable Hilbert space $H$. We denote the resolvent set by $\rho(T)$ and the residual spectrum by $\sigma_{\text {res }}(T)$ (i.e., $\lambda \in \sigma_{\text {res }}(T)$ means that $\lambda \in \mathbf{C}$ is not an eigenvalue of $T$ and $R(T-\lambda I)$ is not dense in $H)$. If the operator $T$ is closable, then we denote its closure by $\tilde{T} . C(H)$ is the set of all closed linear operators in $H$. A family $T(\varepsilon) \in C(H)$, defined for $\varepsilon$ in a domain $G \subset \mathbf{C}$, is said to be holomorphic of type (A) if $D(T(\varepsilon))=D$ is independent of $\varepsilon$ and $T(\varepsilon) u$ is holomorphic function of $\varepsilon \in G$ for every $u \in D$.

Now let $T_{0}$ and $T_{1}$ be two linear operators in $H$, with $D:=D\left(T_{0}\right) \cap D\left(T_{1}\right)$ dense in $H: \bar{D}=H$. Then we can define a family of linear operators by $T(\varepsilon):=T_{0}+\varepsilon T_{1}$ with $D(T(\varepsilon)):=D$. Our basic result is the following

Theorem 1. Let $T_{0} \in C(H)$ and $T_{1}$ be closable. Assume that
(i) there are $a, b, c \geq 0$ such that

$$
\begin{equation*}
\operatorname{Re}\left(T_{0} u, T_{1} u\right) \geq-c\|u\|^{2}-a\left\|T_{0} u\right\|\|u\|-b\left\|T_{0} u\right\|^{2}, \quad u \in D \tag{1}
\end{equation*}
$$

Then $T(\varepsilon)$ is closable for $\varepsilon$ in the region $G$ defined by

$$
G:=\left\{\varepsilon \in \mathbf{C} ;|\operatorname{Im} \varepsilon|<\frac{1-b \operatorname{Re} \varepsilon}{2-b \operatorname{Re} \varepsilon} \operatorname{Re} \varepsilon, \quad 0<\operatorname{Re} \varepsilon<b^{-1}\right\}
$$

with closure $\tilde{T}(\varepsilon)=T_{0}+\varepsilon \tilde{T}_{1}$, and hence $\left\{T_{0}+\varepsilon \tilde{T}_{1} ; \varepsilon \in G\right\}$ forms a holomorphic family of type ( $A$ ).

Assume further that
(ii) $0 \in \rho\left(T_{0}\right)$,
(iii) $0 \notin \sigma_{\text {res }}(T(\varepsilon))$ for sufficiently small $\varepsilon>0$.

Then $D:=D\left(T_{0}\right) \cap D\left(T_{1}\right)$ is a core for $T_{0}$ and hence

$$
\begin{equation*}
T_{0}^{-1}=s-\lim _{\varepsilon \downarrow 0} \tilde{T}(\varepsilon)^{-1} \tag{2}
\end{equation*}
$$

In particular, if $T_{0}$ has a compact resolvent, then any number $\lambda \in \rho\left(T_{0}\right)$ also belongs to the set $\rho(\tilde{T}(\varepsilon))$ for sufficiently small $\varepsilon>0$ and

$$
\begin{equation*}
\left\|(\tilde{T}(\varepsilon)-\lambda I)^{-1}-\left(T_{0}-\lambda I\right)^{-1}\right\| \rightarrow 0 \quad \text { as } \varepsilon \downarrow 0 \tag{3}
\end{equation*}
$$

In case of selfadjoint operators this theorem becomes simpler:
Theorem 2. Let $T_{0}$ be a selfadjoint operator with compact resolvent and $T_{1}$ be a symmetric operator. Assume that conditions (i), (ii) in Theorem 1 are satisfied and for some $\varepsilon_{1} \in(0,1 / b)$ the operator $T\left(\varepsilon_{1}\right)$ is essentially selfadjoint. Then the operators $T(\varepsilon), 0 \leq \varepsilon<1 / b$, are essentially selfadjoint, and the uniform resolvent convergence (3) holds.

Remark 1. It is easily seen that Theorem 1 can be applied to the case where $0 \notin \rho\left(T_{0}\right)$. To this end we consider $\left(T_{0}-\lambda_{0} I\right)$ instead of $T_{0}$ for some $\lambda_{0} \in \rho\left(T_{0}\right)$. Theorem 1 will be true if we replace conditions (i)-(iii) by
$\left(\mathbf{i}^{\prime}\right)$ there are $\lambda_{0} \in \mathbf{C}$ and $a, b, c \geq 0$ such that

$$
\operatorname{Re}\left(\left(T_{0}-\lambda_{0} I\right) u, T_{1} u\right) \geq-c\|u\|^{2}-a\left\|T_{0} u\right\|\|u\|-b\left\|T_{0} u\right\|^{2}, \quad u \in D
$$

(ii') $\lambda_{0} \in \rho\left(T_{0}\right)$.
(iii') $\lambda_{0} \notin \sigma_{\text {res }}(T(\varepsilon))$ for sufficiently small $\varepsilon>0$.
Indeed, it follows from (1') that
$\operatorname{Re}\left(\left(T_{0}-\lambda_{0} I\right) u, T_{1} u\right) \geq-c^{\prime}\|u\|^{2}-a^{\prime}\left\|\left(T_{0}-\lambda_{0} I\right) u\right\|\|u\|-b\left\|\left(T_{0}-\lambda_{0} I\right) u\right\|^{2}, u \in D$, for some constants $a^{\prime}, c^{\prime} \geq 0$.

The following lemma is interesting by itself. The result is concerned with holomorphic families of closed linear operators.
Lemma 1. Under condition (i) in Theorem 1, the family $\left\{T_{0}+\varepsilon \tilde{T}_{1} ; \varepsilon \in G\right\}$ forms a holomorphic family of type (A).
Proof. Let us fix an $\varepsilon_{1} \in\left(0, b^{-1}\right)$ arbitrarily. Applying [12, Lemma 1.1] to the pair and $A:=T_{0}$ and $B:=\varepsilon_{1} T_{1}$, we see that both operators $T_{0}$ and $\varepsilon_{1} T_{1}$ are $\left(T_{0}+\varepsilon_{1} T_{1}\right)$-bounded. In particular, we have

$$
\left\|\varepsilon_{1} T_{1} u\right\| \leq \frac{2-\varepsilon_{1} b}{1-\varepsilon_{1} b}\left\|\left(T_{0}+\varepsilon_{1} T_{1}\right) u\right\|+K\left(\varepsilon_{1}\right)\|u\|
$$

where $K\left(\varepsilon_{1}\right)$ is a positive constant depending on $\varepsilon_{1}$. Since $T_{0}$ is closed and $\varepsilon_{1} T_{1}$ is closable, it follows that $T_{0}+\varepsilon_{1} T_{1}$ is also closable, with closure given by

$$
\left(T_{0}+\varepsilon_{1} T_{1}\right)^{\sim}=T_{0}+\varepsilon_{1} \tilde{T}_{1} .
$$

Furthermore, we see from [3, Theorem IV.1.1] that the operators $T_{0}+$ $\varepsilon_{1} T_{1}+\varepsilon\left(\varepsilon_{1} T_{1}\right)$ are closable for $\varepsilon$ with $|\varepsilon|<\frac{1-\varepsilon_{1} b}{2-\varepsilon_{1} b}$. This means that the family $\left(T_{0}+\varepsilon T_{1}\right) \tilde{u}=T_{0} u+\varepsilon \tilde{T}_{1} u, \quad u \in D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)$, is holomorphic with respect to $\varepsilon$ in the open circle with center $\varepsilon_{1}$ and radius $\frac{1-\varepsilon_{1} b}{2-\varepsilon_{1} b} \varepsilon_{1}$.

Since the number $\varepsilon_{1} \in(0,1 / b)$ is arbitrary, the assertion is proved.
Remark 2. Using $\tilde{T}(\varepsilon)=T_{0}+\varepsilon \tilde{T}_{1}$ for small $\varepsilon>0$, it is easy to show that inequality (1) holds also for $u \in D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)$ :

$$
\operatorname{Re}\left(T_{0} u, \tilde{T}_{1} u\right) \geq-c\|u\|^{2}-a\left\|T_{0} u\right\|\|u\|-b\left\|T_{0} u\right\|^{2}
$$

Lemma 2. Under conditions (i)-(iii) in Theorem 1 the set $D=D\left(T_{0}\right) \cap D\left(T_{1}\right)$ is a core for the operator $T_{0}$.

Proof. By the condition (ii), $0 \in \rho\left(T_{0}\right)$. Put

$$
\begin{equation*}
\varepsilon_{0}:=2^{-1}\left(c\left\|T_{0}^{-1}\right\|^{2}+a\left\|T_{0}^{-1}\right\|+b\right)^{-1} \tag{4}
\end{equation*}
$$

Then it follows from (1) that for every $\varepsilon>0$

$$
\begin{align*}
\|T(\varepsilon) u\|^{2} & \geq\left\|T_{0} u\right\|^{2}+2 \varepsilon \operatorname{Re}\left(T_{0} u, T_{1} u\right)  \tag{5}\\
& \geq\left\|T_{0} u\right\|^{2}-2 \varepsilon\left(c\|u\|^{2}+a\|u\|\left\|T_{0} u\right\|-b\left\|T_{0} u\right\|^{2}\right) \\
& \geq\left(1-2 \varepsilon\left(c\left\|T_{0}^{-1}\right\|^{2}+a\left\|T_{0}^{-1}\right\|+b\right)\right)\left\|T_{0} u\right\|^{2} \\
& \geq\left(1-\frac{\varepsilon}{\varepsilon_{0}}\right)\left\|T_{0}^{-1}\right\|^{-2}\|u\|^{2}, \quad u \in D,
\end{align*}
$$

and hence $\tilde{T}(\varepsilon)$ is invertible for $0<\varepsilon<\varepsilon_{0}$, with

$$
\left\|\tilde{T}(\varepsilon)^{-1}\right\| \leq \sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}\left\|T_{0}^{-1}\right\|
$$

Therefore we see from condition (iii) that $R(\tilde{T}(\varepsilon))=H$, that is, $0 \in \rho(\tilde{T}(\varepsilon))$ for sufficiently small $0<\varepsilon<\varepsilon_{0}$.

To prove the assertion, we first note that $D$ is a core for $\tilde{T}(\varepsilon)$. Since $T_{0}$ is $\tilde{T}(\varepsilon)$-bounded, $D$ is dense in $D(\tilde{T}(\varepsilon))=D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)$ with respect to the graph norm of $T_{0}$. Therefore, it suffices to show that $D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)$ is a core for $T_{0}$. To this end, we shall show that $T_{0}\left[D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)\right]$ is dense in $H$ [3, Problem III.5.19]. Now let $h \in H$ be orthogonal to $T_{0}\left[D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)\right]$ :

$$
\begin{equation*}
\left(h, T_{0} u\right)=0 \quad \forall u \in D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right) \tag{6}
\end{equation*}
$$

We shall show that $h=0$. Since $0 \in \rho(\tilde{T}(\varepsilon))$ for small $\varepsilon>0$, there is a family $\left\{u_{\varepsilon}\right\}$ in $D(\tilde{T}(\varepsilon))=D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)$ such that

$$
h=\tilde{T}(\varepsilon) u_{\varepsilon}=T_{0} u_{\varepsilon}+\varepsilon \tilde{T}_{1} u_{\varepsilon} .
$$

It follows from (6) that

$$
\left(\tilde{T}_{1} u_{\varepsilon}, T_{0} u_{\varepsilon}\right)=-\frac{1}{\varepsilon}\left\|T_{0} u_{\varepsilon}\right\|^{2} \leq-\frac{1}{2 \varepsilon}\left\|T_{0} u_{\varepsilon}\right\|^{2}-\frac{1}{2 \varepsilon\left\|T_{0}^{-1}\right\|^{2}}\left\|u_{\varepsilon}\right\|^{2} .
$$

Since we can take $\varepsilon>0$ as small as we want, we see that the last inequality and (1) (see Remark 2) can be true simultaneously only for $u_{\varepsilon}=0$. Consequently, we obtain $h=\tilde{T}(\varepsilon) u_{\varepsilon}=0$.
Proof of Theorem 1. Let $\varepsilon_{0}$ be as defined in (4). First we shall show that $\tilde{T}_{1} T_{0}^{-1}$ is a densely defined and closed linear operator in $H$ such that $I+\varepsilon \tilde{T}_{1} T_{0}^{-1}$ is boundedly invertible, that is, $\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{-1}$ exists and $R\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)=$ $H$ for small $0<\varepsilon<\varepsilon_{0}$, with

$$
\begin{equation*}
\left\|\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{-1}\right\| \leq \sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}, \quad 0<\varepsilon<\varepsilon_{0} \tag{7}
\end{equation*}
$$

Noting that $T_{0} D$ is contained in $D\left(\tilde{T}_{1} T_{0}^{-1}\right)=D_{1}:=\left\{T_{0} u ; u \in D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)\right\}$, we see from Lemma 2 that $\tilde{T}_{1} T_{0}^{-1}$ is densely defined. Since the closedness of $\tilde{T}_{1} T_{0}^{-1}$ is clear, it remains to prove (7). It follows from (5) that

$$
\left\|\left(T_{0}+\varepsilon T_{1}\right) u\right\|^{2} \geq\left(1-\frac{\varepsilon}{\varepsilon_{0}}\right)\left\|T_{0} u\right\|^{2}, \quad u \in D .
$$

Since $D$ is a core for both $\tilde{T}(\varepsilon)=T_{0}+\varepsilon \tilde{T}_{1}$ and $T_{0}$, we have

$$
\left\|\left(T_{0}+\varepsilon \tilde{T}_{1}\right) u\right\| \geq \sqrt{\varepsilon_{0}^{-1}\left(\varepsilon_{0}-\varepsilon\right)}\left\|T_{0} u\right\|, \quad u \in D\left(T_{0}\right) \cap D\left(\tilde{T}_{1}\right)
$$

and hence

$$
\begin{equation*}
\left\|\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right) v\right\| \geq \sqrt{\varepsilon_{0}^{-1}\left(\varepsilon_{0}-\varepsilon\right)}\|v\|, \quad v \in D_{1} . \tag{8}
\end{equation*}
$$

This implies that $I+\varepsilon \tilde{T}_{1} T_{0}^{-1}$ is invertible. Furthermore, since $R\left(T_{0}+\varepsilon \tilde{T}_{1}\right)=R(\tilde{T}(\varepsilon))=H$, we see that

$$
R\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)=R\left(T_{0}+\varepsilon \tilde{T}_{1}\right)=H
$$

Therefore, (7) follows from (8).
As its consequence we have

$$
\begin{align*}
& I=s-\lim _{\varepsilon \downarrow 0}\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{-1}  \tag{9}\\
& I=s-\lim _{\varepsilon \downarrow 0}\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{-1 *}
\end{align*}
$$

To prove (10), let $\nu \in D\left(T_{1}^{*}\right)$. Then

$$
\begin{aligned}
\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{*-1} \nu-\nu & =\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{*-1}\left\{I-\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{*}\right\} \nu \\
& =\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{*-1}(-\varepsilon) T_{0}^{-1^{*}} T_{1}^{*} \nu
\end{aligned}
$$

we see from (7) that

$$
\left\|\left(I+\varepsilon \tilde{T}_{1} T_{0}^{-1}\right)^{*-1} \nu-\nu\right\| \leq \varepsilon \sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}\left\|T_{0}^{*-1} T_{1} \nu\right\| \rightarrow 0 \quad \text { as } \quad \varepsilon \rightarrow 0
$$

Since $D\left(T_{1}^{*}\right)$ is dense in $H$, we obtain (10) by the Banach - Steinhaus theorem. The proof of (9) is simpler than that of (10). Hence we obtain (2).

Suppose now that $T_{0}^{-1}$ and hence $T_{0}^{-1^{*}}$ are compact. Therefore the wellknown Schmidt decomposition [2] is true: $T_{0}^{-1^{*}}=\sum_{i=1}^{\infty} s_{i}\left(\cdot, z_{i}\right) y_{i}$, where $\left\{y_{i}\right\}_{i=1}^{\infty}$ and $\left\{z_{i}\right\}_{i=1}^{\infty}$ are orthonormal systems of eigenvectors of the operators $T_{0} T_{0}^{*}$ and $T_{0}^{*} T_{0}$, respectively, and $\left\{s_{i}^{-2}\right\}_{i=1}^{\infty}$ are the sequence of corresponding eigenvalues enumerated in the increasing order. Denote by $B_{N}$ the orthogonal projector of the space $H$ onto the linear hull of the vectors $\left\{z_{i}\right\}_{i=1}^{N}$.

To prove the theorem, it is required to show that for any $\alpha>0$ there exists $\beta=\beta(\alpha)>0$ such that

$$
\left\|\tilde{T}^{-1}(\varepsilon)-T_{0}^{-1}\right\| \leq \alpha \quad \text { for } \quad \forall \varepsilon \in(0, \beta)
$$

Fix an arbitrary number $\alpha>0$. Select $N$ such that

$$
s_{i}<\alpha / 2\left(\sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}+1\right) \quad \text { for } \quad i>N
$$

We have

$$
\begin{equation*}
\left\|T_{0}^{-1 *}\left(I-B_{N}\right)\right\|<\frac{\alpha}{2\left(\sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}+1\right)} \tag{11}
\end{equation*}
$$

Next (10) implies a uniform convergence of the operators

$$
\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1 *} T_{0}^{-1 *} B_{N} \rightarrow T_{0}^{-1 *} B_{N} \quad \text { as } \quad \varepsilon \downarrow 0
$$

Select $\beta>0$ such that

$$
\begin{equation*}
\left\|\left\{\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1 *}-I\right\} T_{0}^{-1 *} B_{N}\right\|<\frac{\alpha}{2}, \quad \varepsilon \in(0, \beta) \tag{12}
\end{equation*}
$$

It follows from (7), (11), (12) that

$$
\begin{aligned}
& \left\|\left(\varepsilon \tilde{T}_{1}+T_{0}\right)^{-1}-T_{0}^{-1}\right\|=\left\|T_{0}^{-1}\left\{\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1}-I\right\}\right\| \\
& =\left\|\left\{\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1 *}-I\right\} T_{0}^{-1 *}\right\| \\
& \leq\left\|\left\{\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1 *}-I\right\} T_{0}^{-1 *} B_{N}\right\| \\
& \quad+\left\|\left\{\left(\varepsilon \tilde{T}_{1} T_{0}^{-1}+I\right)^{-1 *}-I\right\} T_{0}^{-1 *}\left(I-B_{N}\right)\right\| \\
& \leq \frac{\alpha}{2}+\left(\sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}+1\right) \frac{\alpha}{2\left(\sqrt{\varepsilon_{0}\left(\varepsilon_{0}-\varepsilon\right)^{-1}}+1\right)}=\alpha
\end{aligned}
$$

Thus we have proved convergence (3) for $\lambda=0$. We have (3) for any number $\lambda \in \rho\left(T_{0}\right)$ from [3, Theorem IV.2.25]. This completes the proof of Theorem 1.

Remark 3. Condition (iii) in Theorem 1 and Lemma 2 can be replaced by the condition
(iii' $\left.{ }^{\prime \prime}\right) 0 \notin \sigma_{\text {res }}\left(T\left(\varepsilon_{1}\right)\right)$ for some $\varepsilon_{1} \in\left(0, \varepsilon_{0}\right)$,where the constant $\varepsilon_{0}$ is defined by (4).
In fact, since bounded invertability is stable under relatively bounded small perturbation[3, Theorem IV.1.16], $0 \in \rho\left(\tilde{T}\left(\varepsilon_{1}\right)\right)$ implies that $0 \in \rho(\tilde{T}(\varepsilon))$ for $\forall \varepsilon \in\left(0, \varepsilon_{0}\right)$ (see also [12, Proposition 1.6]).

Now it is easy to prove Theorem 2. In fact, Lemma 1 and [3, Section VII.3] imply that $\tilde{T}(\varepsilon), 0<\varepsilon<b^{-1}$, is a selfadjoint holomorphic family of type (A). Since the residual spectrum is empty for selfadjoint operators, the norm convergence (3) follows from Theorem 1.

## 2. Applications

As an application of the obtained result, consider the following operators in $L_{2}\left(\mathbf{R}^{n}\right)$ :

$$
\begin{equation*}
T(\varepsilon) u=T_{0} u+\varepsilon T_{1} u=-\Delta u+V(x) u+\varepsilon V_{1}(x) u, \quad \varepsilon>0 \tag{13}
\end{equation*}
$$ where $D\left(T_{i}\right)=\left\{u: T_{i} u, u \in L_{2}\left(\mathbf{R}^{n}\right)\right\}, i=0,1 ; \quad V, V_{1} \in C^{1}\left(\mathbf{R}^{n}\right)$.

Assume also that if $n \geq 2$, then the functions $V, V_{1}$ are real-valued; if $n=1$ then the functions are complex-valued.

Theorem 3. Let either

$$
\lim _{|x| \rightarrow \infty} \operatorname{ReV}(x)=\infty \quad \text { or } \quad \lim _{|x| \rightarrow \infty} \operatorname{Im} V(x)=\infty(-\infty)
$$

(the last for $n=1$ ). Assume that there are constants $b, c>0$ and $M \in R$ such that the following two inequalities hold:

$$
\begin{gather*}
\operatorname{Re}\left(V_{1}+2 b V\right)>M>-\infty  \tag{14}\\
4 \operatorname{Re}\left(V_{1}+2 b V-M\right)\left(\operatorname{Re}\left(V \overline{V_{1}}\right)+b|V|^{2}+c\right) \geq\left|\nabla\left(V_{1}+2 b V\right)\right|^{2} \tag{15}
\end{gather*}
$$

Then $\tilde{T}(\varepsilon)$ converge to $T_{0}$ as $\varepsilon \downarrow 0$ in the sense of uniform resolvent convergence (3).

Proof. Under the imposed assumptions it is known that the following are true [6, pp. 56-65]: the operator $T_{0}$ possesses a compact resolvent; the set of functions $C_{0}^{\infty}\left(\mathbf{R}^{n}\right)$ is a core for the operators $T(\varepsilon), \varepsilon>0 ; \rho(T(\varepsilon)) \neq \emptyset$, $\sigma_{\text {res }}(T(\varepsilon)) \neq \emptyset$.

First we prove Theorem 3 in case $0 \in \rho\left(T_{0}\right)$. We need to check (1) or the equivalent inequality

$$
\begin{equation*}
\operatorname{Re}\left(T_{0} u, T_{1} u\right)+c\|u\|^{2}+b\left\|T_{0} u\right\|^{2} \geq 0 \tag{16}
\end{equation*}
$$

with some constants $c, b \geq 0$. Let $u \in C_{0}^{\infty}\left(\mathbf{R}^{n}\right)$. Then the left-hand side of (16) is written as

$$
\begin{aligned}
& \operatorname{Re} \int_{\mathbf{R}^{n}}(-\triangle u+V u) \overline{V_{1} u} d x+c \int_{\mathbf{R}^{n}}|u|^{2} d x+b \int_{\mathbf{R}^{n}}(-\Delta u+V u)(\overline{-\Delta u+V u}) d x \\
& =\operatorname{Re} \int_{\mathbf{R}^{n}}(-\Delta u) \overline{\left(V_{1}+2 b V\right) u} d x+b \int_{\mathbf{R}^{n}}|\Delta u|^{2} d x \\
& \quad+\int_{\mathbf{R}^{n}}\left(\operatorname{Re}\left(V \overline{V_{1}}\right)+c+b|V|^{2}\right)|u|^{2} d x \\
& \geq b \int_{\mathbf{R}^{n}}|\Delta u|^{2} d x+M \int_{\mathbf{R}^{n}}|\nabla u|^{2} d x+\int_{\mathbf{R}^{n}} \mathcal{Q}_{x}(|u|,|\nabla u|) d x
\end{aligned}
$$

where
$\mathcal{Q}_{x}(s, t):=\operatorname{Re}\left(V_{1}+2 b V-M\right) s^{2}-\left|\nabla\left(V_{1}+2 b V\right)\right| s t+\left(\operatorname{Re}\left(V \overline{V_{1}}\right)+b|V|^{2}+c\right) t^{2}$.
The form $\mathcal{Q}_{x}(s, t)$ is nonnegative if (14) and (15) hold. Noting further that

$$
b \int_{\mathbf{R}^{n}}|\Delta u|^{2} d x+M \int_{\mathbf{R}^{n}}|\nabla u|^{2} d x \geq b\|\Delta u\|^{2}-|M| \delta\|\Delta u\|^{2}-|M| \frac{1}{4 \delta}\|u\|^{2}
$$

we can obtain (16). The assertion is proved in case $0 \in \rho\left(T_{0}\right)$.
Next, let us consider the general case. Since the spectrum of $T_{0}$ is discrete, we can take some $\lambda_{0} \in \rho\left(T_{0}\right) \cap \mathbf{R}, \lambda_{0}<0$. Set

$$
S(\varepsilon):=T(\varepsilon)-\lambda_{0} I=-\Delta u+\left(V-\lambda_{0}\right) u+\varepsilon V_{1} u
$$

Note that the norm convergence (3) and

$$
\begin{equation*}
S(\varepsilon)^{-1}=\left(T(\varepsilon)-\lambda_{0} I\right)^{-1} \rightarrow\left(T_{0}-\lambda_{0} I\right)^{-1}=S(0)^{-1}, \quad \varepsilon \downarrow 0 \tag{17}
\end{equation*}
$$

are equivalent. So if we prove (17), then we obtain Theorem 3.
It is easily seen that $0 \in \rho(S(0))$. We have already proved that if (14) and (15) with $V$ replaced by $V-\lambda_{0}$ holds then we have the assertion. So we need to prove that (14) and (15) yield the following inequalities for some constant $c_{s} \geq 0$ :

$$
\begin{gather*}
\operatorname{Re}\left(V_{1}+2 b\left(V-\lambda_{0}\right)\right)>M  \tag{18a}\\
4 \operatorname{Re}\left(V_{1}-M+2 b\left(V-\lambda_{0}\right)\right)\left(\operatorname{Re}\left\{\left(V-\lambda_{0}\right) \overline{V_{1}}\right\}+b\left|V-\lambda_{0}\right|^{2}+c_{s}\right)  \tag{18b}\\
\geq\left|\nabla\left(V_{1}+2 b\left(V-\lambda_{0}\right)\right)\right|^{2}=\left|\nabla\left(V_{1}+2 b V\right)\right|^{2}
\end{gather*}
$$

Since $\lambda_{0}<0$, (18a) is obvious. Next we estimate two factors on the left-hand side of $(18 b)$ separately. Since $\lambda_{0}<0$,

$$
\begin{equation*}
\operatorname{Re}\left(V_{1}-M+2 b\left(V-\lambda_{0}\right)\right) \geq \operatorname{Re}\left(V_{1}+2 b V-M\right) \tag{19}
\end{equation*}
$$

From (14) we obtain

$$
\begin{align*}
\operatorname{Re} & \left\{\left(V-\lambda_{0}\right) \overline{V_{1}}\right\}+b\left|V-\lambda_{0}\right|^{2}+c_{s}  \tag{20}\\
& =\operatorname{Re}\left(V \overline{V_{1}}\right)-\lambda_{0} \operatorname{Re} V_{1}+b|V|^{2}-2 b \lambda_{0} \operatorname{Re} V+b \lambda_{0}^{2}+c_{s} \\
& \geq \operatorname{Re}\left(V \overline{V_{1}}\right)+b|V|^{2}-\lambda_{0} M+b \lambda_{0}^{2}+c_{s}-c+c \\
& \geq \operatorname{Re}\left(V \overline{V_{1}}\right)+b|V|^{2}+c .
\end{align*}
$$

Here we have chosen the constant $c_{s} \geq 0$ such that $b \lambda_{0}^{2}-\lambda_{0} M+c_{s}-c \geq 0$. From (15), (19) and (20) we have (18b). This completes the proof of Theorem 3.

Remark 4. It is clear that for any $V, V_{1} \in C^{1}\left(\mathbf{R}^{n}\right)$, (14), (15) holds for $x \in K$, where $K$ is any compact set. Hence it remains to check (14), (15) for large $|x|$.

Example 1. Let us consider the operator (13) for $V(x)=|x|^{2}$ :

$$
T(\varepsilon) u=-\Delta u+|x|^{2} u+\varepsilon V_{1} u
$$

CASE 1: $\operatorname{Re} V_{1}(x)$ is bounded below: $\operatorname{Re} V_{1}(X)>-M$. To check (15) for sufficiently large $|x|$, we should have

$$
4\left(2 b|x|^{2}+\operatorname{Re} V_{1}\right)\left(|x|^{2} \operatorname{Re} V_{1}+b|x|^{4}+c\right) \geq\left|\nabla V_{1}+4 b x\right|^{2} \quad(M=0)
$$

or equivalently for large $|x|$,

$$
\begin{equation*}
|x|^{4} \operatorname{Re} V_{1}+|x|^{6}+|x|^{2}\left(\operatorname{Re} V_{1}\right)^{2} \geq C_{1}\left|\nabla V_{1}\right|^{2} \tag{21}
\end{equation*}
$$

where $C_{1}>0$ is a constant.
Now assume that

$$
\begin{equation*}
|x|^{3}+|x| \operatorname{Re} V_{1} \geq C\left|\nabla V_{1}\right| \quad \text { for large }|x| \tag{22}
\end{equation*}
$$

where $C>0$ is a constant. Then we have

$$
\left(|x|^{3}+|x| \operatorname{Re} V_{1}\right)^{2} \geq C^{2}\left|\nabla V_{1}\right|^{2} \quad \text { for large }|x|
$$

and hence (21).
The inequality (21) holds, for example, if
a) $V_{1}=|x|^{\alpha_{1}}, \quad \forall \alpha_{1}>1$; or
b) $V_{1}=|x|^{\alpha_{1}} \pm i|x|^{\beta_{1}}$ if $\left\{\alpha_{1}>1,1<\beta_{1}<4\right\}$ or $\alpha_{1}>\beta_{1}-2$.

Case 2: $\operatorname{Re} V_{1} \rightarrow-\infty$ as $|x| \rightarrow \infty$. The conclusions of Theorem 3 are true if

$$
\operatorname{Re} V_{1}=o\left(|x|^{2}\right), \quad\left|\nabla V_{1}\right|=O\left(|x|^{3}\right) \quad \text { as } \quad|x| \rightarrow \infty
$$

Example 2. Let the operator (13) is given by

$$
T(\varepsilon) u=-u^{\prime \prime}+\left(-|x|^{\alpha} \pm i|x|^{\beta}\right) u+\varepsilon\left(|x|^{\alpha_{1}} \pm i|x|^{\beta_{1}}\right) \quad(n=1)
$$

The conclusion of Theorem 3 is true if $\alpha_{1}>\alpha>1$ and

$$
\max \left\{\alpha_{1}+\beta+\beta_{1} ; 2 \beta+\alpha_{1}\right\}>\max \left\{\alpha+2 \alpha_{1} ; 2 \alpha_{1}-2\right\}
$$

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