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# BOUNDARY INTEGRAL EQUATION FOR NAVIER-STOKES EQUATIONS IN A NON-SMOOTH DOMAIN

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**Abstract.** Boundary integral equations corresponding to the differential equations describing a transient flow of incompressible viscous fluid in three dimensions are considered. Emphasis is put on the treatment of edges and corners. The boundary $\Gamma$ is assumed piecewise Lyapunov surface and the interior solid angle $\Theta(x)$ at the non-smooth boundary point $x$ must satisfy the inequality

$$\lim_{\delta\to 0}\sup_{x\in\Gamma}\frac{1}{2\pi}\left\{\int_{0<|y-x|\le\delta}|d\Theta_x(y)|+|2\pi-\Theta(x)|\right\}<1.$$

Corresponding to the Dirichlet problem of the Navier-Stokes equations, the following series of Volterra integral equations of the first kind for unknown tractions $\sigma_j^{(n)}$ ($j=1,2,3:\ n=0,1,2,\ldots$) is derived.

$$G\sigma_j^{(n)}(x,t)=\int_0^t\int_\Gamma \sigma_i^{(n)}(y,\tau)U_{ij}^*(y,\tau;x,t)\,dS(y)d\tau=b_j^{(n)}(x,t),$$

where $U_{ij}^*$ are components of the Stokes fundamental solution tensor and $b_j^{(n)}$ can be regarded as given functions. The integral $G\sigma_j^{(n)}$ is the single layer potential. The integral involved in the definition of $b_j^{(n)}$ (see the text) is the double layer potential. Those integrals are shown to be weakly singular for the non-smooth domain under consideration. It is proved that, with $\Sigma=\Gamma\times[0,T]$, the operator

$$G: H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)\to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

is coercive;

$$((G\sigma,\sigma))_{L^2(\Sigma)}\ge\beta|||\sigma|||^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}$$

with a constant $\beta>0$, $\sigma=(\sigma_1,\sigma_2,\sigma_3)$.

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## §1. INTRODUCTION

One of the favorable properties of the boundary element method is its high accuracy in the numerical solution for singular problems due to edges and corners of the domain in question. Another favorable property of the method is due to its boundary only formulation. In order to make those properties truly beneficial, it is important to derive boundary integral equations and to show coercivity of the integral operator, for the coercivity property of integral operator plays a crucial role in the convergence and stability of approximate solutions of the boundary integral equations.

In this paper, boundary integral equations corresponding to the Navier-Stokes equations describing the transient viscous fluid flow in non-smooth domain in three dimensions are considered. The non-smoothness is characterized by the existence of edges and corners of some general kind. The Stokes fundamental solution tensor is used as the kernel of the integral operator. Corresponding to the Dirichlet problem of the transient Navier-Stokes equations, a series of Volterra integral equations of the first kind for unknown surface tractions is derived. The integrals involved in the equations are shown to be weakly singular even on the surface having the edges and corners. The unique existence of the solution to the series of boundary integral equations are presented in anisotropic Sobolev space. We show coercivity of the integral operator on the non-smooth surface.

When the domain in question is smooth, the conventional mathematical discussion about constructing the solution in the form of asymptotic expansion is done according to the following process; a) the formal asymptotic series is substituted into the Navier-Stokes equations; b) the differential equation for each term of the series is derived. However, in this paper, we will consider the non-smooth domain. In this case, we must be careful of limiting processes in deriving the differential equation in the step b). To get around the difficulties, unlike the conventional discussion, we will begin with the discussion of the integral representation of solutions for the Navier-Stokes equations.

For a nonstationary viscous flow of compressible fluid, Hebeker and Hsiao [5] showed the coercivity of the corresponding boundary integral operator for a smooth domain. Their method of proof is based on the proof due to Costabel et al. [3] for transient single layer heat potential, the elementary proof is published later in Onishi et al. [9]. As far as the authors are aware, there have been no papers published that are concerned with boundary integral approach for incompressible viscous fluid flow in non-smooth domain.

To be more specific, we describe in §2 an initial-boundary value problem of the Navier-Stokes equations. The non-smoothness of the domain will be characterized by (2.12). We shall derive in §3 the boundary integral representation of the solution in the form (3.1)–(3.3). The integral representation requires knowledge of velocity and traction on the boundary. The velocity on the boundary is given as the Dirichlet data. The traction on the boundary must be determined by boundary integral equations that will be derived in §4 as

**Theorem 1**. *The unknown tractions $\sigma_i^{(n)}$ $(n = 0, 1, 2, \ldots)$ on a non-smooth surface $\Gamma$ characterized by (2.12) are given by solutions of the following linear Volterra integral equations of the first kind on the boundary.*

$$\begin{aligned} C_{ij}\hat{u}_i(x,t) &= \operatorname{Re}\int_\Omega \left(u_i^{(0)}U_{ij}^*\right)_{\tau=0} dV + \int_0^t\int_\Gamma \left(\sigma_i^{(0)}U_{ij}^* - \hat{u}_i\Sigma_{ij}^*\right) dS d\tau \\ &\quad + \int_0^t\int_\Omega f_i U_{ij}^*\, dV d\tau, \end{aligned}$$

$$0 = \int_0^t\int_\Gamma \sigma_i^{(1)}U_{ij}^*\, dSd\tau - \operatorname{Re}\int_0^t\int_\Omega u_k^{(0)}u_i^{(0)}U_{ij}^*\, dV d\tau,$$

*and for* $n = 2, 3, \ldots$ ,

$$0 = \int_0^t\int_\Gamma \sigma_i^{(n)}U_{ij}^*\, dSd\tau - \operatorname{Re}\sum_{l=0}^{n-1}\int_0^t\int_\Omega u_k^{(l)}u_{i,k}^{(n-l-1)}U_{ij}\, dV d\tau.$$

We shall present a jump relation in Theorem 2 for the non-smooth surface. A boundary integral operator is defined by the single layer potential in (4.16). For the coercivity of the integral operator we shall prove

**Theorem 3.** *There exists a constant $\beta > 0$ depending only on $\Sigma$ such that*

$$((G\sigma,\sigma))_{\boldsymbol{L}^2(\Gamma)} \geq \beta|||\sigma|||^2_{\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}$$

in §5. In discussions throughout this paper we shall require rather lengthy but straightforward manipulation of equations, which are gathered in Appendices I, II, III for the main discussions to be made concise.

## §2. NON-SMOOTH DIRICHLET PROBLEM

Let $\Omega$ be an open connected and bounded domain in three-dimensional Euclidean space $E^3$. The boundary of $\Omega$, which is denoted by $\Gamma = \partial\Omega$, is assumed to consist of a finite number of open smooth surface $\Gamma_k$ $(k = 1, 2, \ldots, N)$ so that $\Gamma = \cup_{k=1}^N \overline{\Gamma_k}$. Here $\overline{\Gamma_j}$ denotes the closure of the set $\Gamma_j$.

We consider the unsteady viscous flow of an incompressible Newtonian fluid in $\Omega$. The set of governing equations can be written in dimensionless forms as follows:

**Equations of motion** $(i = 1, 2, 3)$

$$\mathrm{Re}\ (\dot{u}_i + u_j u_{i,j}) = \sigma_{ij,j} + f_i \quad \text{in } \Omega, \tag{2.1}$$

**Continuity equation**

$$u_{i,i} = 0 \quad \text{in } \Omega, \tag{2.2}$$

**Constitutive equations** $(i, j = 1, 2, 3)$

$$\sigma_{ij} = -\mathrm{Re}\, p\delta_{ij} + u_{i,j} + u_{j,i}. \tag{2.3}$$

Here $u_i$ is the component of the flow velocity, $p$ is the pressure, $\sigma_{ij}$ is the $i, j$-component of the Cauchy stress tensor, $f_i$ is the component of the given external force, and Re is the Reynolds number of the fluid motion under consideration. We use Einstein's summation convention on repeated indices. A comma, for example, in $u_{i,j}$ is used to indicate the differentiation for $u_i$ with respect to the corresponding spatial variable $x_j$, a dot in $\dot{u}_i$ indicates the differentiation with respect to the time variable, and $\delta_{ij}$ is the Kronecker symbol.

For the set of governing equations above, we are interested in the following side conditions:

**Boundary condition**.

$$u_i = \widehat{u}_i \quad \text{on } \Gamma, \tag{2.4}$$

**Initial condition**.

$$u_i = u_i^{(0)} \quad \text{at } t = 0, \tag{2.5}$$

where $\hat{u}_i(x, t)$ is the prescribed velocity component, and $u_i^{(0)}$ is the given initial velocity component. We assume that $\widehat{u}_i \in C(\Gamma \times [0, T])$ and $u_i^{(0)} \in C^1(\Omega) \cap C(\overline{\Omega})$. Moreover, we assume that $\widehat{u}_i(x, 0) = u_i^{(0)}(x)$ at $x \in \Gamma$ and that

$$u_{i,i}^{(0)} = 0 \quad \text{in } \Omega \tag{2.6}$$

$$\int_\Gamma \widehat{u}_i n_i \, dS = 0, \tag{2.7}$$

where $n_i$ is $i$th component of the unit outward normal $n(x)$ at $x \in \Gamma$.

We shall confine the geometry of $\Gamma$ as follows: Let each $\Gamma_k$ be a piece of Lyapunov surface so that the Lyapunov condition is satisfied:

$$|\cos\nu| \le L|y-x|^{\kappa} \quad (0<\kappa<1) \tag{2.8}$$

for all $x, y \in \Gamma_k$, where $\nu$ is the angle between the normal $n(x)$ and $(x-y)$, $L$ is a constant depending only on $\Gamma$. The set of points on $\Gamma$, where the surface is not smooth, is denoted by $\delta\Gamma$.

Let $d\Theta_x(y)$ denote an infinitesimal solid angle at any $x \in E^3$ subtending the surface element $dS(y)$ at $y \in \Gamma - \delta\Gamma$:

$$d\Theta_x(y) = -\frac{\partial}{\partial n(y)}(\frac{1}{r})dS(y) \tag{2.9}$$

with $r = |y-x|$. We set $\Theta(x) = \displaystyle\int_\Gamma d\Theta_x(y)$. This is equal to the interior solid angle at the vertex $x$ of the cone, whose side surface is constructed by all the half ray tangential lines to the surface $\Gamma$ radiating from $x$. The cone is assumed to be simply connected. It follows that

$$\sup_{x\in E^3}\int_\Gamma |d\Theta_x(y)| \le A \tag{2.10}$$

with a constant $A>0$. Let us put

$$W_\delta(x) := \frac{1}{2\pi}\left\{\int_{0<|y-x|\le\delta}|d\Theta_x(y)| + |2\pi - \Theta(x)|\right\}, \tag{2.11}$$

and characterize $\delta\Gamma$ so as to satisfy the inequality:

$$\lim_{\delta\to 0}\sup_{x\in\Gamma} W_\delta(x) = \omega < 1 \tag{2.12}$$

with a constant $\omega$. The non-smooth surface characterized by (2.12) was introduced in [14].

As the solution of our initial-boundary value problem (2.1)–(2.5), we seek such $u_j$ and $p$ that $u_j \in C^2(\Omega\times(0,T])\cap C(\overline{\Omega}\times[0,T])$ and $p \in C^1(\Omega\times(0,T])$. However, we cannot expect in general that $\sigma_{ij}$ are continuous on the boundary, because $\Gamma$ has edges and corners. Here we assume that tractions defined by $\sigma_i = \sigma_{ij}n_j$ are $p$th-power summable function on $\Gamma$ with $p>2$; i.e. $\sigma_i(\cdot,t) \in L^p(\Gamma)$,

$$\|\sigma_i(\cdot,t)\|_p := \left\{\int_\Gamma |\sigma_i(x,t)|^p\,dS(x)\right\}^{\frac{1}{p}} < +\infty. \tag{2.13}$$

Moreover, we assume that $\sigma_i(x,t)$ is a function such that

$$\lim_{s\to t}\|\sigma_i(\cdot,s) - \sigma_i(\cdot,t)\|_p = 0 \tag{2.14}$$

for all $t\in[0,T]$. The space of all such functions is denoted by $C(L^p(\Gamma):[0,T])$ equipped with the norm: $|||\sigma_i|||_{C(L^p(\Gamma):[0,T])} := \max_{0\le t\le T}\|\sigma_i(\cdot,t)\|_p$.

## §3. INTEGRAL REPRESENTATION

In this section, we shall derive the successive linear representation of the solution in terms of integrals on the boundary as follows:

$$\begin{aligned} u_j^{(0)}(x,t) &= \mathrm{Re}\int_\Omega \left(u_i^{(0)}U_{ij}^*\right)_{\tau=0} dV \\ &\quad + \int_0^t \int_\Gamma \left(\sigma_i^{(0)}U_{ij}^* - \widehat{u}_i\Sigma_{ij}^*\right) dSd\tau \qquad (3.1) \\ &\quad -\frac{1}{4\pi}\int_\Gamma \widehat{u}_i n_i \frac{\partial}{\partial y_i}\left(\frac{1}{r}\right) dS + \int_0^t\int_\Omega f_i U_{ij}^*\, dV d\tau, \\ u_j^{(1)}(x,t) &= \int_0^t\int_\Gamma \sigma_i^{(1)}U_{ij}^*\, dSd\tau - \mathrm{Re}\int_0^t\int_\Omega u_k^{(0)}u_{i,k}^{(0)}U_{ij}^*\, dVd\tau, \qquad (3.2) \end{aligned}$$

and for $n = 2, 3, \dots$ ,

$$u_j^{(n)}(x,t) = \int_0^t\int_\Gamma \sigma_i^{(n)}U_{ij}^*\, dSd\tau - \mathrm{Re}\int_0^t\int_\Omega \left(\sum_{l=0}^{n-1} u_k^{(l)}u_{i,k}^{(n-l)}\right) U_{ij}^*\, dVd\tau. \qquad (3.3)$$

For this purpose, we consider a sequence of smooth surfaces $\{S_m\}$ $(m = 1, 2, \dots)$ in $\Omega$ such that (i) for each $m$ there exists a one-to-one continuous mapping $\varphi_m$ from $\Gamma$ to $S_m$ such that $\varphi_m(y) \to y$ as $m \to \infty$, and (ii) with the constant $A$ in (2.10) it holds that

$$\int_{S_m} |d\Theta_x(y)| \le A$$

uniformly for all $x \in E^3$ and $m$. The existence of such $\{S_m\}$ is shown in Wendland [14, Hilfssatz 6]. We denote by $\Omega_m$ the open domain enclosed by $S_m$.

As is well-known, see Oseen [10, p. 38, Sec. 5], Ladyzhenskaya [6, p. 78], or Berker [1, p. 276, Sec. 77] for example, the Green formula for $x \in \Omega_m$ with smooth boundary yields

$$\begin{aligned} u_j(x,t) &= \mathrm{Re}\int_{\Omega_m} \left(u_iU_{ij}^*\right)_{\tau=0} dV(y) \\ &\quad + \int_0^t\int_{S_m} \left(\sigma_iU_{ij}^* - u_i\Sigma_{ij}^*\right) dS(y)d\tau \\ &\quad -\frac{1}{4\pi}\int_{S_m} u_in_i\frac{\partial}{\partial y_j}(\frac{1}{r})\, dS(y) \qquad (3.4) \\ &\quad -\mathrm{Re}\int_0^t\int_{\Omega_m} u_ku_{i,k}U_{ij}^*\, dVd\tau + \int_0^t\int_{\Omega_m} f_iU_{ij}^*\, dVd\tau, \end{aligned}$$

where $U_{ij}^*$ are components of the tensor given by the expression:

$$U_{ij}^*(y,\tau;x,t) := -\delta_{ij}\Delta\Phi + \frac{\partial^2\Phi}{\partial y_j\partial y_i}, \qquad (3.5)$$

$$\Phi(y,\tau;x,t) := \frac{1}{r}\int_0^r E(\rho,t-\tau)\,d\rho H(t-\tau) \tag{3.6a}$$

$$= \frac{1}{4\pi\mathrm{Re}}\frac{1}{r}\mathrm{Erf}\left(\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t-\tau}}\right)H(t-\tau) \tag{3.6b}$$

with Heaviside step function $H(\cdot)$. The function $E(\cdot,\cdot)$ and the Gauss error function $\mathrm{Erf}(\cdot)$ are defined by

$$E(r,t-\tau) := \frac{1}{2\pi\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi\,(t-\tau)}\right)^{\frac{1}{2}} e^{-\frac{\mathrm{Re}\,r^2}{4(t-\tau)}}, \tag{3.7a}$$

$$\mathrm{Erf}\,(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-\zeta^2}\,d\zeta, \tag{3.7b}$$

respectively. Moreover $\Sigma^*_{ij}$ is the pseudo-traction defined by the expression:

$$\Sigma^*_{ij}\,(y,\tau;x,t) := \left(U^*_{ij,k} + U^*_{kj,i}\right)n_k. \tag{3.8}$$

The equations (3.4) are derived in Appendix I, in which we will follow Oseen [10, Sec. 5], Kupradze [4], Tosaka [12], and Tosaka and Kakuda [13] for the way of the derivation. Essentially the same equations are presented in Oseen [10, p. 44].

**Remarks**. The expression (3.6b) is more convenient than (3.6a) for the numerical evaluation of $\Phi$. See, e.g. Yamauchi et al. [15, Chap. 9].

We shall derive an integral representation of $u_j(x,t)$ for $x \in \Omega$ by letting (3.4) in the limit as $m \to \infty$. To this end, it is sufficient to show that next two integrals are uniformly bounded for any $m$.

$$I_1 := \int_0^t \int_{S_m} U^*_{ij}(y,\tau;x,t)\,dS(y)d\tau, \tag{3.9}$$

$$I_2 := \int_0^t \int_{S_m} \Sigma^*_{ij}(y,\tau;x,t)\,dS(y)d\tau. \tag{3.10}$$

**Lemma 3.1.** *$U^*_{ij}(y,\tau : x,t)$ is weakly singular at $y = x$, $\tau = t$: Namely, there exist two positive constants $G_1$ and $G_2$ such that*

$$|U^*_{ij}| \le \frac{G_1}{(t-\tau)^{\mu}}\frac{1}{r^{3-2\mu}} + \frac{G_2}{(t-\tau)^{\nu-1}}\frac{1}{r^{5-2\nu}}$$

*with any $\mu\,(\frac{1}{2} < \mu < 1)$, $\nu\,(\frac{3}{2} < \nu < 2)$. The integral (3.9) is absolutely convergent.*

The lemma can be proved by the combination of ideas in Oseen [10, p. 69] and Pogorzelski [11, p. 353]. The proof is given in Appendix II.

**Lemma 3.2.** *The integral (3.10) is absolutely convergent for any $x \in \Omega$ and it is uniformly bounded for any $m$.*

*Proof.* We shall show in Appendix III that

$$\begin{aligned} U^*_{ij,k} + U^*_{kj,i} &= -\frac{\delta_{ij} r_k + \delta_{kj} r_i}{r} \left(\frac{\mathrm{Re}}{2}\right)^2 \frac{rE(r, t-\tau)}{(t-\tau)^2} \\ &+ \frac{\delta_{ij} r_k + \delta_{jk} r_i + \delta_{ki} r_j}{r} \left(\frac{\mathrm{Re}}{2}\right)^2 r \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^3}\, ds \\ &- \frac{r_i r_j r_k}{r^3} \left(\frac{\mathrm{Re}}{2}\right)^3 r^3 \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^4}\, ds, \end{aligned} \tag{3.11}$$

where $r_i = y_i - x_i$. From (3.7a) we can see that

$$\begin{aligned} &\left(\frac{\mathrm{Re}}{2}\right)^2 \frac{rE(r, t-\tau)}{(t-\tau)^2} \\ &= \frac{1}{2^{2\mu-1} \pi^{\frac{3}{2}} \mathrm{Re}^{1-\mu}} \frac{1}{(t-\tau)^{\mu}} \frac{1}{r^{4-2\mu}} \left\{\frac{\mathrm{Re}\, r^2}{4(t-\tau)}\right\}^{\frac{5}{2}-\mu} \exp\left[-\frac{\mathrm{Re}\, r^2}{4(t-\tau)}\right] \\ &\le \frac{G_3}{(t-\tau)^{\mu}} \frac{1}{r^{4-2\mu}} \end{aligned}$$

with $0 < \dfrac{5}{2} - \mu = \alpha_3$. Here we put $G_3(\mu) = \dfrac{\alpha_3 e^{-\alpha_3}}{\left(2^{2\mu-1} \pi^{\frac{3}{2}} \mathrm{Re}^{1-\mu}\right)}$. We shall restrict $\mu$ as to satisfy $\mu < 1$. Similarly we can see that

$$\begin{aligned} &\left(\frac{\mathrm{Re}}{2}\right)^2 r \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^3}\, ds \\ &= \int_{-\infty}^{\tau} \frac{1}{2^{3-2\nu} \pi^{\frac{3}{2}} \mathrm{Re}^{2-\nu}} \frac{1}{(t-s)^{\nu}} \frac{1}{r^{6-2\nu}} \left\{\frac{\mathrm{Re}\, r^2}{4(t-s)}\right\}^{\frac{7}{2}-\nu} \exp\left[-\frac{\mathrm{Re}\, r^2}{4(t-s)}\right] ds \\ &\le \frac{\mathrm{Re}^{\nu-2}}{2^{3-2\nu} \pi^{\frac{3}{2}}} \frac{\nu - 1}{(t-\tau)^{\nu-1}} \frac{1}{r^{6-2\nu}} \alpha_4^{\alpha_4} e^{-\alpha_4} \\ &= \frac{G_4}{(t-\tau)^{\nu-1}} \frac{1}{r^{6-2\nu}} \end{aligned}$$

with $0 < \frac{7}{2} - \nu = \alpha_4$, $G_4(\nu) = (\nu - 1)\frac{\mathrm{Re}^{\nu-2}\alpha_4^{\alpha_4}e^{-\alpha_4}}{2^{3-2\nu}\pi^{\frac{3}{2}}}$, and that

$$\begin{aligned}
&\left(\frac{\mathrm{Re}}{2}\right)^3 r^3 \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^4}\, ds \\
&\quad = \int_{-\infty}^{\tau} \frac{\mathrm{Re}^{\lambda-2}}{2^{4-2\lambda}\pi^{\frac{3}{2}}} \frac{1}{(t-s)^{\lambda}} \frac{1}{r^{6-2\lambda}} \left\{\frac{\mathrm{Re}\, r^2}{4(t-s)}\right\}^{\frac{9}{2}-\lambda} \exp\left[-\frac{\mathrm{Re}\, r^2}{4(t-s)}\right] ds \\
&\quad \le \frac{\mathrm{Re}^{\lambda-2}}{2^{3-2\nu}\pi^{\frac{3}{2}}} \frac{\lambda-1}{(t-\tau)^{\lambda-1}} \frac{1}{r^{6-2\lambda}} \alpha_5^{\alpha_5} e^{-\alpha_5} \\
&\frac{G_5}{(t-\tau)^{\lambda-1}} \frac{1}{r^{6-2\lambda}}
\end{aligned}$$

with $0 < \frac{9}{2} - \lambda = \alpha_5$, $G_5(\lambda) = (\lambda - 1)\frac{\mathrm{Re}^{\lambda-2}\alpha_5^{\alpha_5}e^{-\alpha_5}}{2^{3-2\nu}\pi^{\frac{3}{2}}}$. We shall restrict $\nu$ and $\lambda$ further as to satisfy $0 < \nu - 1 < 1$ and $0 < \lambda - 1 < 1$. Note that $\left|\frac{r_i}{r}\right| \le 1$. From (3.8) and (3.11) we have

$$\begin{aligned}
&\int_0^t \int_{S_m} \left|\Sigma_{ij}^*\right| dS(y)d\tau \\
&\quad \le \int_0^t \int_{S_m} \left\{\frac{2G_3}{(t-\tau)^{\mu}} \frac{1}{r^{4-2\mu}} + \frac{3G_4}{(t-\tau)^{\nu-1}} \frac{1}{r^{6-2\nu}} + \frac{G_5}{(t-\tau)^{\lambda-1}} \frac{1}{r^{6-2\lambda}}\right\} dSd\tau \\
&\quad \le \int_0^t \frac{2G_3}{(t-\tau)^{\mu}}\, d\tau \sup_{\substack{0<\tau<t \\ m}} \int_{S_m} \frac{dS(y)}{r^{4-2\mu}} + \int_0^t \frac{3G_4}{(t-\tau)^{\nu-1}}\, d\tau \sup_{\substack{0<\tau<t \\ m}} \int_{S_m} \frac{dS(y)}{r^{6-2\nu}} \\
&\quad + \int_0^t \frac{G_5}{(t-\tau)^{\lambda-1}}\, d\tau \sup_{\substack{0<\tau<t \\ m}} \int_{S_m} \frac{dS(y)}{r^{6-2\lambda}}.
\end{aligned}$$

The integrations with respect to the variable $\tau$ are convergent. Since $x$ is in $\Omega$, we can choose $\delta(x) > 0$ and an integer $M(x) \in \boldsymbol{N}$ such that $r = |y - x| \ge \delta$ for any $y \in S_m$ and $m \ge M$. This completes the proof. □

From Lemmas 3.1 and 3.2, the integrals (3.9) and (3.10) have the corresponding finite value as $m \to \infty$. Therefore, we have

$$\lim_{m\to\infty} \int_0^t \int_{S_m} \left(\sigma_i U_{ij}^* - u_i \Sigma_{ij}^*\right) dS(y)d\tau = \int_0^t \int_{\Gamma} \left(\sigma_i U_{ij}^* - u_i \Sigma_{ij}^*\right) dS(y)d\tau.$$

Similarly we can see

$$\lim_{m\to\infty} \int_{S_m} u_i n_i \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) dS(y) = \int_{\Gamma} u_i n_i \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) dS(y).$$

Hence, from (3.4) as $m \to \infty$, we have

$$
\begin{aligned}
u_j(x,t) = \mathrm{Re} &\int_\Omega \left(u_i U_{ij}^*\right)_{\tau=0} dV(y) \\
&+ \int_0^t \int_\Gamma \left(\sigma_i U_{ij}^* - u_i \Sigma_{ij}^*\right) dS(y) d\tau \\
&- \frac{1}{4\pi} \int_\Gamma u_i n_i \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) dS(y) - \mathrm{Re} \int_0^t \int_\Omega u_k u_{i,k} U_{ij}^* \, dV(y) d\tau \\
&+ \int_0^t \int_\Omega f_i U_{ij}^* \, dV(y) d\tau
\end{aligned}
\tag{3.12}
$$

with $x \in \Omega$. This is a representation formula for $u_j(x,t)$. However it involves the volume integral of the nonlinear term $u_k u_{i,k}$. In order to linearize the formula, we introduce a parameter $\lambda$ in (3.12) according to Oseen [10, p. 71]. This leads to the equation:

$$
\begin{aligned}
u_j(x,t) = \mathrm{Re} &\int_\Omega \left(u_i U_{ij}^*\right)_{\tau=0} dV \\
&+ \int_0^t \int_\Gamma \left(\sigma_i U_{ij}^* - u_i \Sigma_{ij}^*\right) dS d\tau \\
&- \frac{1}{4\pi} \int_\Gamma u_i n_i \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) dS - \lambda \mathrm{Re} \int_0^t \int_\Omega u_k u_{i,k} U_{ij}^* \, dV d\tau \\
&+ \int_0^t \int_\Omega f_i U_{ij}^* \, dV d\tau.
\end{aligned}
\tag{3.13}
$$

We try to find the solution corresponding to this equation in the form:

$$
u_j(x,t) := \sum_{n=0}^{\infty} \lambda^n u_j^{(n)}(x,t), \tag{3.14}
$$

$$
p(x,t) := \sum_{n=0}^{\infty} \lambda^n p^{(n)}(x,t), \tag{3.15}
$$

respectively. We impose here that $u_i^{(n)}(x,0) = 0 \, (n \geq 1)$ in $\Omega$. We define

$$
\sigma_{ij}(x,t) := \sum_{n=0}^{\infty} \lambda^n \sigma_{ij}^{(n)}(x,t) \tag{3.16}
$$

with

$$
\sigma_{ij}^{(n)}(x,t) := -\mathrm{Re}\, p^{(n)} \delta_{ij} + u_{i,j}^{(n)} + u_{j,i}^{(n)} \quad \text{in } \Omega, \tag{3.17}
$$

$$
\sigma_i^{(n)} := \sigma_{ij}^{(n)} n_j \quad \text{on } \Gamma, \tag{3.18}
$$

if the series are absolutely convergent. Substituting (3.14) and (3.15) into (3.13), and equating the like powers of $\lambda$, we obtain (3.1)–(3.3). These relations are successive linear representations of $u_j(x,t)$ at $x \in \Omega$ in terms of velocities $\widehat{u}_i$ and tractions $\sigma_i^{(0)}$ on the boundary. æ

## §4. BOUNDARY INTEGRAL EQUATIONS

In this section we shall transform the initial-boundary value problem in the non-smooth domain described in section 2 into a series of boundary integral equations of the first kind.

**Theorem 1**. *The unknown tractions $\sigma_i^{(n)}$ $(n = 0, 1, 2, \ldots)$ on a non-smooth surface $\Gamma$ characterized by (2.12) are given by solutions of the following linear Volterra integral equations of the first kind on the boundary.*

$$\begin{aligned} C_{ij}\hat{u}_i(x,t) &= \mathrm{Re}\int_\Omega \left(u_i^{(0)}U_{ij}^*\right)_{\tau=0} dV + \int_0^t\int_\Gamma \left(\sigma_i^{(0)}U_{ij}^* - \hat{u}_i\Sigma_{ij}^*\right) dSd\tau \\ &\quad + \int_0^t\int_\Omega f_i U_{ij}^*\, dV d\tau, \end{aligned} \tag{4.1}$$

$$0 = \int_0^t\int_\Gamma \sigma_i^{(1)}U_{ij}^*\, dSd\tau - \mathrm{Re}\int_0^t\int_\Omega u_k^{(0)}u_i^{(0)}U_{ij}^*\, dVd\tau, \tag{4.2}$$

*and for* $n = 2, 3, \ldots$ ,

$$0 = \int_0^t\int_\Gamma \sigma_i^{(n)}U_{ij}^*\, dSd\tau - \mathrm{Re}\sum_{l=0}^{n-1}\int_0^t\int_\Omega u_k^{(l)}u_{i,k}^{(n-l-1)}U_{ij}\, dVd\tau. \tag{4.3}$$

To begin with, let us define potential functions:

**Single layer potential**

$$G\sigma_j(x,t) := \int_0^t\int_\Gamma \sigma_i(y,\tau)U_{ij}^*(y,\tau;x,t)\, dS(y)d\tau, \tag{4.4}$$

**Double layer potential**

$$Hu_j(x,t) := \int_0^t\int_\Gamma u_i(y,\tau)\Sigma_{ij}^*(y,\tau;x,t)\, dS(y)d\tau. \tag{4.5}$$

About the continuity of (4.4) we have

**Lemma 4.1.** *Under the assumption (2.13), the single layer potential $G\sigma_j(x,t)$ is continuous in $E^3 \times [0,T]$.*

*Proof.* From (I.30) and (II.4), the kernel $U^*_{ij}$ can be written in the form:

$$U^*_{ij} = \frac{\mathrm{Re}}{2}\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\frac{E(r,t-\tau)}{t-\tau} - \frac{\mathrm{Re}}{4}\left(\delta_{ij} - 3\frac{r_i r_j}{r^2}\right)\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^2}\,ds. \tag{4.6}$$

We first show that

$$g_1(x,t) := \int_0^t \int_\Gamma \sigma_j(y,\tau)\frac{E(r,t-\tau)}{t-\tau}\,dS(y)d\tau$$

is continuous. Since the integral is absolutely convergent from Lemma 3.1, we can transform the multiple integral into iterated integrals. The variable transformation: $\tau \mapsto \sigma = \frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t-\tau}}$ yields

$$g_1(x,t) = \frac{1}{\pi^{\frac{3}{2}}\mathrm{Re}}\int_\Gamma \frac{1}{r}\left\{\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} e^{-\sigma^2}\sigma_j\left(y, t - \frac{\mathrm{Re}\, r^2}{4\sigma^2}\right)d\sigma\right\}dS(y).$$

Let us put the integral in $\{\cdots\}$ as $\Phi_1(y;x,t)$. This $\Phi_1$, as a function of $y \in \Gamma$, is $p$th-power summable with $p > 2$: In fact,

$$\begin{aligned}\int_\Gamma |\Phi_1|^p\,dS(y) &\le \int_\Gamma \left|\int_0^\infty e^{-\sigma^2}\sup_\tau |\sigma_j(y,\tau)|\,d\sigma\right|^p dS(y)\\ &= \int_\Gamma \sup_\tau |\sigma_j(y,\tau)|^p\,dS(y)\left(\frac{\sqrt{\pi}}{2}\right)^p.\end{aligned}$$

From (2.13) we know that $\sup_\tau |\sigma_j(y,\tau)|$ is also in $L^p(\Gamma)$. This implies that $\Phi_1(\cdot;x,t) \in L^p(\Gamma)$.

Using the theorem in Wendland [14, Hilfssatz 2.3.2] we know that

$$g_1(x,t) = \frac{1}{\pi^{\frac{3}{2}}\mathrm{Re}}\int_\Gamma \frac{\Phi_1(y;x,t)}{r}\,dS(y)$$

is continuous in $E^3 \times [0,T]$. The continuity at $t = 0$ is understood in the sense: $g_1(x,t) \to 0$ as $t \to 0$.

Secondly we show that

$$g_2(x,t) := \int_0^t \int_\Gamma \sigma_j(y,\tau)\left\{\frac{\mathrm{Re}}{4}\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^2}\,ds\right\}dS(y)d\tau$$

is continuous. This can be shown in the similar way as in (4.6): By $s \mapsto \zeta = \frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t-s}}$, we have

$$\frac{\mathrm{Re}}{4}\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^2}\,ds = \frac{1}{\pi^{\frac{3}{2}}\mathrm{Re}}\frac{1}{r^3}\int_0^{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t-\tau}}}\zeta^2 e^{-\zeta^2}\,d\zeta. \tag{4.7}$$

Therefore we can see that

$$\begin{aligned} g_2(x,t) &= \frac{1}{\pi^{\frac{3}{2}}\mathrm{Re}} \int_0^t \int_\Gamma \sigma_j(y,\tau) \frac{1}{r^3} \int_0^{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{(t-\tau)}}} \zeta^2 e^{-\zeta^2}\, d\zeta dS(y) d\tau \\ &= \frac{1}{2\pi^{\frac{3}{2}}} \int_\Gamma \frac{1}{r} \left\{ \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^\infty \frac{1}{\sigma^3} \int_0^\sigma \zeta^2 e^{-\zeta^2}\, d\zeta \sigma_j(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) d\sigma \right\} dS(y). \end{aligned}$$

Let us put the integral in $\{\cdots\}$ as $\Phi_2(y;x,t)$. We now show that $\Phi_2(\cdot;x,t) \in L^p(\Gamma)$ with $p > 2$: In fact,

$$\begin{aligned} \int_\Gamma |\Phi_2|^p\, dS(y) &\le \int_\Gamma \left| \int_0^\infty \frac{1}{\sigma^3} \int_0^\sigma \zeta^2 e^{-\zeta^2}\, d\zeta \sup_\tau |\sigma_j(y,\tau)|\, d\sigma \right|^p dS(y) \\ &= \int_\Gamma \sup_\tau |\sigma_j(y,\tau)|^p\, dS(y) (\frac{\sqrt{\pi}}{4})^p. \end{aligned}$$

The last equality follows from the relation

$$\int_0^\infty \frac{1}{\sigma^3} \int_0^\sigma \zeta^2 e^{-\zeta^2}\, d\zeta d\sigma = \frac{\sqrt{\pi}}{4}.$$

Hence $g_2(x,t)$ is continuous in $E^3 \times [0,T]$.

Moreover, we can see that the following $\widehat{g}_1$, $\widehat{g}_2$ are also continuous in $E^3 \times [0,T]$:

$$\begin{aligned} \widehat{g}_1(x,t) &:= \int_0^t \int_\Gamma \sigma_j(y,\tau) \frac{r_i r_j}{r^2} \frac{E(r,t-\tau)}{t-\tau}\, dS(y) d\tau \\ &= \frac{1}{\pi^{\frac{3}{2}}\mathrm{Re}} \int_\Gamma \frac{\Phi_1(y;x,t)}{r} \frac{r_i r_j}{r^2}\, dS(y), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \widehat{g}_2(x,t) &:= \int_0^t \int_\Gamma \sigma_j(y,\tau) \frac{r_i r_j}{r^2} \left\{ \frac{\mathrm{Re}}{4} \int_{-\infty}^\tau \frac{E(r,t-s)}{(t-s)^2}\, ds \right\} dS(y) d\tau \\ &= \frac{1}{2\pi^{\frac{3}{2}}} \int_\Gamma \frac{\Phi_2(y;x,t)}{r} \frac{r_i r_j}{r^2}\, dS(y), \end{aligned} \tag{4.9}$$

because $|\Phi_k \frac{r_i r_j}{r^2}| \le |\Phi_k|$ $(k = 1,2)$ and they are $p$th-power summable. Therefore, the lemma is proved. □

Next lemma shows that the double layer potential (4.5) with continuous density satisfies a jump relation on the boundary.

**Lemma 4.2.** *Suppose that $u_j \in C(\overline{\Omega} \times [0,T])$ with $j = 1,2,3$. Then, as $x \in \Omega$ approaches a boundary point $z \in \Gamma$, at which $\Gamma$ is smooth, the double layer potential $Hu_j(x,t)$ satisfies*

$$\lim_{x \to z} Hu_j(x,t) = -\frac{1}{2} u_j(z,t) + Hu_j(z,t). \tag{4.10}$$

Instead of giving the proof of this well-known lemma, we consider another limit than (4.10): Let $x$ be on the boundary $\Gamma$. Here, $x$ may be the point at edges and corners. Let $K_\delta(x)$ be a sphere of radius $\delta$ with the center $x$; $K_\delta(x) = \{y \,|\, |y-x| \le \delta\}$. Define $\Omega_\delta = \Omega - K_\delta(x)$. The boundary of $\Omega_\delta$ consists of two parts; $S_\delta = \Omega \cap \partial K_\delta(x)$ and $\Gamma_\delta = \Gamma - \overline{S}_\delta$. As $\delta \to 0$, we see that $\Gamma_\delta \to \Gamma$. If $\delta$ is sufficiently small, $S_\delta$ is simply connected. In this case, since $x$ is an exterior point of $\Omega_\delta$, we have the Green formula:

$$\begin{aligned} 0 \;=\; & \mathrm{Re}\int_{\Omega_\delta}\left(u_i U^*_{ij}\right)_{\tau=0} dV + \int_0^t\int_{\partial\Omega_\delta}\left(\sigma_i U^*_{ij} - u_i \Sigma^*_{ij}\right) dS d\tau \\ & + \int_0^t\int_{\Omega_\delta}\left(f_i - \mathrm{Re} u_k u_{i,k}\right) U^*_{ij}\, dV d\tau, \end{aligned} \tag{4.11}$$

which corresponds (3.12).

**Theorem 2.** *For any $u_i \in C^2(\Omega\times[0,T]) \cap C(\overline{\Omega}\times[0,T])$ and $x \in \Gamma$, it holds that*

$$\lim_{\delta\to 0}\int_0^t\int_{\partial\Omega_\delta} u_i(y,\tau)\Sigma^*_{ij}(y,\tau;x,t)\, dS(y) d\tau = C_{ij}u_i(x,t) + Hu_j(x,t), \tag{4.12}$$

*where $C_{ij}$ is given by the expression:*

$$C_{ij} = \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_\delta}\left(\frac{r_j}{r}n_i - 3\frac{r_i r_j r_k}{r^3}n_k\right)\frac{1}{r^2}\, dS(y). \tag{4.13}$$

*Proof.* We divide the integral in (4.12) into two parts:

$$\int_0^t\int_{\partial\Omega_\delta} u_i\Sigma^*_{ij}\, dS d\tau = \int_0^t\int_{\Gamma_\delta} + \int_0^t\int_{S_\delta}. \tag{4.14}$$

The first part converges to $Hu_j(x,t)$ in the sense of Cauchy's principal value as $\delta \to 0$. We know that $\Sigma_{ij}$ as in (3.11) consists of three terms. To examine the limit of the second part, we consider corresponding three integrals:

$$\begin{aligned} I_1 \;:=\; & \int_0^t u_i(y,\tau)\left(\frac{\mathrm{Re}}{2}\right)^2 \frac{rE(r,t-\tau)}{(t-\tau)^2}\, d\tau \\ =\; & \frac{1}{\pi^{\frac{3}{2}} r^2}\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2})\sigma^2 e^{-\sigma^2}\, d\sigma, \\ I_2 \;:=\; & \int_0^t u_i(y,\tau)\left(\frac{\mathrm{Re}}{2}\right)^2 r\left\{\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^3}\, ds\right\} d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi^{\frac{3}{2}} r^2} \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) \frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\, d\zeta}{\sigma^3}\, d\sigma, \\
I_3 &:= \int_0^t u_i(y,\tau) \left(\frac{\mathrm{Re}}{2}\right)^3 r^3 \left\{ \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^4}\, ds \right\} d\tau \\
&= \frac{2^2}{\pi^{\frac{3}{2}} r^2} \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) \frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\, d\zeta}{\sigma^3}\, d\sigma.
\end{aligned}
$$

Let $t > 0$, which is arbitrary, be fixed. We can choose $r$ so small that the inequality: $\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}} < \frac{\sqrt{\mathrm{Re} r}}{2}$ is satisfied: In fact, $r < t$ is sufficient. We divide the integral involved in $I_1$ into two parts and consider that

$$
\begin{aligned}
&\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) \sigma^2 e^{-\sigma^2}\, d\sigma = \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re} r}}{2}} + \int_{\frac{\sqrt{\mathrm{Re} r}}{2}}^{\infty} \\
&= \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re} r}}{2}} u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) \sigma^2 e^{-\sigma^2}\, d\sigma + u_i(x,t) \int_{\frac{\sqrt{\mathrm{Re} r}}{2}}^{\infty} \sigma^2 e^{-\sigma^2}\, d\sigma \\
&\quad + \int_{\frac{\sqrt{\mathrm{Re} r}}{2}}^{\infty} \left\{ u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) - u_i(x,t) \right\} \sigma^2 e^{-\sigma^2}\, d\sigma.
\end{aligned}
$$

Since $u_i$ is bounded on $\overline{\Omega} \times [0, T]$, we can write $\max_{x,t,i} |u_i(x,t)| \le M$ for some constant $M$. The first integral on the most right hand side converges to zero with the order $O(r^{\frac{3}{2}})$: This can be shown as follows:

$$
\begin{aligned}
\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re} r}}{2}} |u_i|\, \sigma^2 e^{-\sigma^2}\, d\sigma &\le M \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re} r}}{2}} \sigma^2 \left(1 + O(\sigma^2)\right) d\sigma \\
&= M \left[ \frac{\sigma^3}{3} + O(\sigma^5) \right]_{\sigma = \frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re} r}}{2}} \\
&= M \frac{\mathrm{Re}^{\frac{3}{2}}}{24} r^{\frac{3}{2}} + O(r^{\frac{5}{2}}) \quad \text{as } r \to 0.
\end{aligned}
$$

Since $|y - x| = r$ ($= \delta$ with $y \in S_\delta$) and $0 < \frac{\mathrm{Re} r^2}{4\sigma^2} < r$ in the last integral, we have for arbitrary $\varepsilon > 0$ that

$$
\max_{y,\sigma} |u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) - u_i(x.t)| < \varepsilon
$$

with sufficiently small $\delta$. Therefore we see that

$$
\int_{\frac{\sqrt{\mathrm{Re} r}}{2}}^{\infty} |u_i(y, t - \frac{\mathrm{Re} r^2}{4\sigma^2}) - u_i(x,t)|\, \sigma^2 e^{-\sigma^2}\, d\sigma < \varepsilon \int_0^\infty \sigma^2 e^{-\sigma^2}\, d\sigma = \varepsilon \frac{\sqrt{\pi}}{4}.
$$

Next, we consider the integral involved in $I_2$ and divide it into two parts as before: Namely,

$$\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma = \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re}r}}{2}} + \int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}$$
$$= \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re}r}}{2}} u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma + u_i(x,t)\int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma$$
$$+\int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}\left\{u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})-u_i(x,t)\right\}\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma.$$

The first integral on the most right hand side converges to zero with the order of $O(r^{\frac{3}{2}})$: This can be seen from the estimate:

$$\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re}r}}{2}}\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma = \frac{\mathrm{Re}^{\frac{3}{2}}}{120}r^{\frac{3}{2}}+O(r^{\frac{5}{2}}) \quad \text{as} \quad r\to 0.$$

The last integral can be made arbitrary small as we can see that

$$\int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}|u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})-u_i(x,t)|\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma$$
$$<\varepsilon\int_0^\infty\frac{1}{\sigma^3}\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta d\sigma = \varepsilon\frac{\sqrt{\pi}}{8}.$$

Similarly we consider the integral involved in $I_3$ as follows:

$$\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty} u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma$$
$$= \int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\frac{\sqrt{\mathrm{Re}r}}{2}} u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma + u_i(x,t)\int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}\frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma$$
$$+\int_{\frac{\sqrt{\mathrm{Re}r}}{2}}^{\infty}\left\{u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})-u_i(x,t)\right\}\frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma.$$

The first integral on the right hand side is the order of $O(r^2)$ as $r\to 0$. The absolute value of the last integral is bounded by $\varepsilon\dfrac{3\sqrt{\pi}}{16}$.

From (3.8) and (3.11) we can write the last integral in (4.14) as follows:

$$\int_0^t\int_{S_\delta} u_i\Sigma_{ij}^*\,dSd\tau = \frac{1}{\pi^{\frac{3}{2}}}\int_{S_\delta}\{-\frac{\delta_{ij}r_k+\delta_{kj}r_i}{r}\frac{1}{r^2}\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty}u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\sigma^2 e^{-\sigma^2}\,d\sigma$$
$$+\frac{\delta_{ij}r_k+\delta_{jk}r_i+\delta_{ki}r_j}{r}\frac{2}{r^2}\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty}u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^4 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma$$
$$-\frac{r_ir_jr_k}{r^3}\frac{4}{r^2}\int_{\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t}}}^{\infty}u_i(y,t-\frac{\mathrm{Re}r^2}{4\sigma^2})\frac{\int_0^\sigma \zeta^6 e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma\}\,n_k\,dS.$$

From the estimates so far for each integration with respect to $\sigma$, we can see that

$$\begin{aligned}
\lim_{\delta\to 0}\int_0^t\int_{S_\delta} u_i\Sigma_{ij}^*\,dSd\tau &= \lim_{\delta\to 0}\frac{1}{\pi^{\frac{3}{2}}}\int_{S_\delta}\{-\frac{\delta_{ij}r_k+\delta_{kj}r_i}{r}\frac{u_i(x,t)}{r^2}\int_{\frac{\sqrt{\mathrm{Re}}r}{2}}^{\infty}\sigma^2e^{-\sigma^2}\,d\sigma\\
&\quad+\frac{\delta_{ij}r_k+\delta_{jk}r_i+\delta_{ki}r_j}{r}\frac{2u_i(x,t)}{r^2}\int_{\frac{\sqrt{\mathrm{Re}}r}{2}}^{\infty}\frac{\int_0^\sigma\zeta^4e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma\\
&\quad-\frac{r_ir_jr_k}{r^3}\frac{4u_i(x,t)}{r^2}\int_{\frac{\sqrt{\mathrm{Re}}r}{2}}^{\infty}\frac{\int_0^\sigma\zeta^6e^{-\zeta^2}\,d\zeta}{\sigma^3}\,d\sigma\}_{r=\delta}\,n_k\,dS(y)\\
&= \frac{u_i(x,t)}{\pi^{\frac{3}{2}}}\lim_{\delta\to 0}\int_{S_\delta}\{-\frac{\delta_{ij}r_k+\delta_{kj}r_i}{r}\frac{1}{r^2}\frac{\sqrt{\pi}}{4}\\
&\quad+\frac{\delta_{ij}r_k+\delta_{jk}r_i+\delta_{ki}r_j}{r}\frac{2}{r^2}\frac{\sqrt{\pi}}{8}-\frac{r_ir_jr_k}{r^3}\frac{4}{r^2}\frac{3\sqrt{\pi}}{16}\}_{r=\delta}\,n_k\,dS\\
&= C_{ij}u_i(x,t).
\end{aligned}$$

Hence we put $C_{ij}$ as

$$\begin{aligned}
C_{ij} &= \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_\delta}\left\{-\frac{\delta_{ij}r_k+\delta_{kj}r_i}{r}+\frac{\delta_{ij}r_k+\delta_{jk}r_i+\delta_{ki}r_j}{r}-3\frac{r_ir_jr_k}{r^3}\right\}\frac{n_k}{r^2}\,dS\\
&= \frac{1}{4\pi}\lim_{\delta\to 0}\int_{S_\delta}\left\{\frac{\delta_{ki}r_j}{r}-3\frac{r_ir_jr_k}{r^3}\right\}\frac{n_k}{r^2}\,dS,
\end{aligned}$$

which completes the proof of the theorem. □

**Remarks**. Coefficients $C_{ij}$ depends only on the geometry of the boundary $\Gamma$ at $x$. When $\Gamma$ is smooth at $x$, we have $C_{ij}=\frac{1}{2}\delta_{ij}$.

From Lemmas 3.1 and 4.1, we know that all integrals involving $U_{ij}^*$ in (4.11) are continuous in $E^3\times[0,T]$. Therefore, the formula (4.11) yields the equation:

$$\begin{aligned}
C_{ij}u_i(x,t) &= \mathrm{Re}\int_\Omega\left(u_iU_{ij}^*\right)_{\tau=0}dV+\int_0^t\int_\Gamma\left(\sigma_iU_{ij}^*-u_i\Sigma_{ij}^*\right)dSd\tau\\
&\quad+\int_0^t\int_\Omega\left(f_i-\mathrm{Re}u_ku_{i,k}\right)U_{ij}^*\,dVd\tau
\end{aligned}\tag{4.15}$$

with $x\in\Gamma$. We introduce the parameter $\lambda$ to the nonlinear term as in (3.13). Corresponding to (3.1)–(3.3), we then have the series of boundary integral equations (4.1)–(4.3). These equations are Volterra integral equations of the first kind for unknown $\sigma_i^{(n)}$ $(n = 0, 1, 2, \ldots)$. They have the common form:

$$\begin{aligned} G\sigma_j(x,t) &:= \int_0^t \int_\Gamma \sigma_i(y,\tau) U_{ij}^*(y,\tau;x,t)\, dS(y) d\tau \\ &= b_j(x,t). \end{aligned} \tag{4.16}$$

For (4.1), $b_j$ has the form:

$$\begin{aligned} b_j^{(0)}(x,t) &= C_{ij}\hat{u}_i(x,t) + \int_0^t \int_\Gamma \hat{u}_i \Sigma_{ij}^*\, dS d\tau \\ &\quad - \mathrm{Re} \int_\Omega \left( u_i^{(0)} U_{ij}^* \right)_{\tau=0} dV - \int_0^t \int_\Omega f_i U_{ij}^*\, dV d\tau, \end{aligned} \tag{4.17}$$

and for (4.2), (4.3), it has the form:

$$b_j^{(n)}(x,t) = \mathrm{Re} \sum_{l=0}^{n-1} \int_0^t \int_\Omega u_k^{(l)} u_{i,k}^{(n-l-1)} U_{ij}^*\, dV d\tau \tag{4.18}$$

with $n = 1, 2, 3, \ldots$ .

## §5. COERCIVITY OF THE INTEGRAL OPERATOR

We shall show the existence of the solution to the boundary integral equation (4.16). The way of arguments will proceed in parallel with the one used in Onishi [8].

We consider the properties of the integral operator $G$ in the space $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and in its dual space $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ with $\Sigma = \Gamma \times [0,T]$, introduced by Lions and Magenes [7, p. 10 and p. 44]:

$$H^{\frac{1}{2},\frac{1}{4}}(\Sigma) := L^2\left([0,T]; H^{\frac{1}{2}}(\Gamma)\right) \cap H^{\frac{1}{4}}\left([0,T]; L^2(\Gamma)\right)$$

equipped with the norm:

$$|||w|||^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} = \int_0^T \|w(\cdot,t)\|^2_{H^{\frac{1}{2}}(\Gamma)}\, dt + \int_0^T \int_0^T \frac{\|w(\cdot,t) - w(\cdot,s)\|^2_{L^2(\Gamma)}}{|t-s|^{\frac{3}{2}}}\, dsdt$$

even for our non-smooth $\Gamma$. We shall use also the Banach space

$$H^{1,\frac{1}{2}}(Q) := L^2\left([0,T]; H^1(\Omega)\right) \cap H^{\frac{1}{2}}\left([0,T]; L^2(\Omega)\right)$$

with $Q = \Omega \times [0,T]$.

For three component vector function $w = (w_1, w_2, w_3)$, the product spaces are defined by

$$\boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma) := H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

and

$$\boldsymbol{H}^{1,\frac{1}{2}}(Q) := H^{1,\frac{1}{2}}(Q) \times H^{1,\frac{1}{2}}(Q) \times H^{1,\frac{1}{2}}(Q)$$

with the norm:

$$|||w|||^2_{\boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)} = \sum_{j=1}^{3} |||w_j|||^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}.$$

The space $\boldsymbol{H}^{\frac{1}{2},0}(\Sigma)$ is similarly defined. Let $\boldsymbol{L}^2(\Sigma) := L^2(\Sigma) \times L^2(\Sigma) \times L^2(\Sigma)$. We denote by $((\cdot,\cdot))_0$ the scalar product:

$$((v, w))_{\boldsymbol{L}^2(\Sigma)} := \sum_{j=1}^{3} \int_0^T (v_j(\cdot, t), w_j(\cdot, t))_{L^2(\Gamma)}\, dt.$$

Then, we have

**Lemma 5.1.** *There exists a constant $C > 0$ such that*

$$|||G\sigma|||^2_{\boldsymbol{H}^{\frac{1}{2},0}(\Sigma)} \le C((G\sigma, \sigma))_{\boldsymbol{L}^2(\Sigma)}$$

*for any $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ in $\boldsymbol{L}^2(\Sigma)$.*

The proof is done by the direct extension of the proof for heat equation in Onishi et al. [9, Lemma 1]. Next lemma essentially due to Lions and Magenes [7] for heat equation implies the unique existence of the solution $\sigma$ to the equation (4.16) in $\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$.

**Lemma 5.2.** *The operator*

$$G : \boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to \boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

*is an isomorphism.*

From the lemma, we know that there exists a constant $\alpha > 0$ depending only on $\Sigma$ such that

$$\alpha^{-1}|||\sigma|||_{\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \le |||G\sigma|||_{\boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \le \alpha|||\sigma|||_{\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}. \tag{5.1}$$

Moreover, in a similar way as in the proof of Theorem 1 in Onishi et al. [9] it can be proved that $G$ is coercive.

**Theorem 3.** *There exists a constant $\beta > 0$ depending only on $\Sigma$ such that*

$$((G\sigma, \sigma))_{\boldsymbol{L}^2(\Gamma)} \geq \beta |||\sigma|||^2_{\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}.$$

*Proof of Theorem 3.* Let $C$ denote a generic constant. From Lemma 5.1 and from the continuous dependence of solutions on Dirichlet data, we can see

$$\begin{aligned} ((G\sigma, \sigma))_{\boldsymbol{L}^2(\Sigma)} &\geq C|||G\sigma|||^2_{\boldsymbol{H}^{\frac{1}{2},0}(\Sigma)} \\ &\geq C|||G\sigma|||^2_{\boldsymbol{H}^{1,0}(Q)}. \end{aligned}$$

By the extension of the result in Lions and Magenes [7] for heat equations, we know that the trace operator

$$\gamma_0 : \boldsymbol{H}^{1,\frac{1}{2}}(Q) \to \boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

is bounded. Namely, there exists a constant $C$ $(> 0)$ such that

$$|||G\sigma|||_{\boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq C|||G\sigma|||_{\boldsymbol{H}^{1,\frac{1}{2}}(Q)}.$$

From Costabel [2, Lemma 2.15] we know that

$$|||G\sigma|||_{\boldsymbol{H}^{1,\frac{1}{2}}(Q)} \leq C|||G\sigma|||_{\boldsymbol{H}^{1,0}(Q)}$$

holds for some constant $C$ $(> 0)$. Therefore we have

$$\begin{aligned} ((G\sigma, \sigma))_{\boldsymbol{L}^2(\Sigma)} &\geq C|||G\sigma|||^2_{\boldsymbol{H}^{1,0}(Q)} \\ &\geq C|||G\sigma|||^2_{\boldsymbol{H}^{1,\frac{1}{2}}(Q)} \\ &\geq C|||G\sigma|||^2_{\boldsymbol{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \\ &\geq C|||\sigma|||^2_{\boldsymbol{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)}. \end{aligned}$$

The last inequality follows from (5.1). □

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## APPENDIX I

In this appendix, we shall transform the set of differential equations (2.1)-(2.3) into the set of integro-differential equations (3.4). To this end we write (2.1)-(2.3) formally in the matrix form:

$$\begin{pmatrix} -\mathrm{Re}D_t + \Delta + D_1^2 & D_1D_2 & D_1D_3 & D_1 \\ D_2D_1 & -\mathrm{Re}D_t + \Delta + D_2^2 & D_2D_3 & D_2 \\ D_3D_1 & D_3D_2 & -\mathrm{Re}D_t + \Delta + D_3^2 & D_3 \\ D_1 & D_2 & D_3 & 0 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ -\mathrm{Re}p \end{pmatrix} = \begin{pmatrix} \mathrm{Re}u_ju_{1,j} - f_1 \\ \mathrm{Re}u_ju_{2,j} - f_2 \\ \mathrm{Re}u_ju_{3,j} - f_3 \\ 0 \end{pmatrix}, \tag{I.1}$$

where $D_t = \frac{\partial()}{\partial t}$, $D_j = \frac{\partial()}{\partial x_j}$, and $\Delta$ is the Laplacian in three dimensions. We denote (I.1) simply by the expression:

$$L_{IJ}U_J = B_I \quad (I, J = 1, 2, 3, 4), \tag{I.2}$$

where we put $U_i = u_i$ $(i = 1, 2, 3)$, $U_4 = -\mathrm{Re}p$, $B_i = \mathrm{Re}u_ju_{i,j} - f_i$, and $B_4 = 0$.

**Remarks**. We use two kinds of indices. The indices with upper case letters run from 1 to 4, the indices with lower case letters run from 1 to 3.

We assume that the solution $U_J$ of (I.2) is sufficiently smooth. Then the coefficient matrix $[L_{IJ}]$ becomes symmetric. In order to determine four unknowns $U_J$ $(J = 1, 2, 3, 4)$ we require corresponding four sets of linearly independent fundamental solutions associated with $L_{IJ}$ in general. Let $U^*_{IL}$ $(L = 1, 2, 3, 4)$ be such fundamental solutions, that are assumed to be admissible in the Galerkin form:

$$\int_0^t \int_\Omega (L_{IJ}U_J - B_I)\, U^*_{IL}\, dV(y)d\tau = 0. \tag{I.3}$$

After integration by parts and using the relation:

$$\frac{\partial}{\partial t}\int_\Omega u_iU^*_{iL}\, dV = \int_\Omega \frac{\partial}{\partial t}\left(u_iU^*_{iL}\right)\, dV, \tag{I.4}$$

we can obtain the Green formula:

$$
\begin{aligned}
&\int_0^t \int_\Omega \left\{ (L_{IJ} U_J)\, U^*_{IJ} - U_I\, (L^*_{IJ} U^*_{JL}) \right\} \, dV d\tau \\
&\qquad = \int_0^t \int_\Gamma (\sigma_i U^*_{iL} - u_i \Sigma^*_{iL}) \, dS d\tau - \mathrm{Re} \left[ \int_\Omega u_i U^*_{iL} \, dV \right]_{\tau=0}^t ,
\end{aligned}
\tag{I.5}
$$

in which $\Sigma^*_{iL} = \left( -U^*_{4L}\delta_{ij} + U^*_{iL,j} + U^*_{jL,i} \right) n_j$ and the adjoint operators $L^*_{IJ}$ are given as follows:

$$
[L^*_{IJ}] = \begin{pmatrix}
\mathrm{Re}D_t + \Delta + D_1^2 & D_1 D_2 & D_1 D_3 & -D_1 \\
D_1 D_2 & \mathrm{Re}D_t + \Delta + D_2^2 & D_2 D_3 & -D_2 \\
D_1 D_3 & D_2 D_3 & \mathrm{Re}D_t + \Delta + D_3^2 & -D_3 \\
-D_1 & -D_2 & -D_3 & 0
\end{pmatrix}.
\tag{I.6}
$$

We consider the fundamental solution tensor $U^*_{JL}$ satisfying the equation:

$$
L^*_{IJ} U^*_{JL} = -\delta_{IL}\delta(x)\delta(t),
\tag{I.7}
$$

where $\delta(\cdot)$ is the Dirac function. In order to find the explicit form of the solution, we assume that $U^*_{JL}$ can be derived from the expression: $U^*_{JL} = M_{JL}\varphi^*$ with a scalar function (often called stress function) $\varphi^*$ in such a way that $M_{JL}$ satisfies the relation:

$$
L^*_{IJ} M_{JL} = \det\, [L^*_{IJ}]\, \delta_{IL}.
\tag{I.8}
$$

This implies that $M_{JL}$ is the formal cofactor of $L^*_{IJ}$. From (I.6) the cofactors are given by

$$
\begin{aligned}
&[M_{IJ}] = (\mathrm{Re}D_t + \Delta) \\
&\times \begin{pmatrix}
-(D_2^2 + D_3^2) & D_1 D_2 & D_1 D_3 & D_1(\mathrm{Re}D_t + \Delta) \\
D_1 D_2 & -(D_3^2 + D_1^2) & D_2 D_3 & D_2(\mathrm{Re}D_t + \Delta) \\
D_1 D_3 & D_2 D_3 & -(D_1^2 + D_2^2) & D_3(\mathrm{Re}D_t + \Delta) \\
D_1(\mathrm{Re}D_t + \Delta) & D_2(\mathrm{Re}D_t + \Delta) & D_3(\mathrm{Re}D_t + \Delta) & -(\mathrm{Re}D_t + \Delta)(\mathrm{Re}D_t + 2\Delta)
\end{pmatrix}.
\end{aligned}
\tag{I.9}
$$

If we put $M_{IJ}$ as $M_{IJ} = (\mathrm{Re}D_t + \Delta)\, M'_{IJ}$, then $M'_{IJ}$ are expressed as follows:

$$\begin{aligned} M'_{ij} &= -\Delta\delta_{ij} + D_i D_j, && \text{(I.10a)}\\ M'_{i4} &= D_i\left(\mathrm{Re}D_t + \Delta\right), && \text{(I.10b)}\\ M'_{44} &= -\left(\mathrm{Re}D_t + \Delta\right)\left(\mathrm{Re}D_t + 2\Delta\right). && \text{(I.10c)} \end{aligned}$$

The determinant calculated formally is given by

$$\det\,[L^*_{IJ}] = -\Delta\left(\mathrm{Re}D_t + \Delta\right)^2. \tag{I.11}$$

Therefore, $\varphi^*(y,\tau;x,t)$ as a function of $y$ and $\tau$ with parameters $x$ and $t$ must satisfy the equation:

$$\Delta_y\left(\mathrm{Re}D_t + \Delta\right)^2 \varphi^* = \delta(x)\delta(t). \tag{I.12}$$

We require explicit forms of all $U^*_{JL}$. Since each $M_{IJ}$ contains the factor $(\mathrm{Re}D_t + \Delta)$, it is sufficient to determine an unknown $\Phi(y,\tau;x,t)$ satisfying

$$\Delta_y\left(\mathrm{Re}D_\tau + \Delta_y\right)\Phi = \delta(x)\delta(t) \tag{I.13}$$

with $\Phi = (\mathrm{Re}D_\tau + \Delta_y)\,\varphi^*$. The solution with the spherical symmetry around $x$ takes the form:

$$\Phi = \frac{1}{r}\int_0^r E(\rho, t-\tau)\,d\rho H(t-\tau). \tag{I.14}$$

We notice that $(\mathrm{Re}D_\tau + \Delta_y)\,\Phi = 0$. Therefore, the fundamental solution tensor is given as follows:

$$[U^*_{IL}] = \left(\begin{array}{ccc|c} & & & 0\\ & U^*_{ij} & & 0\\ & & & 0\\ \hline 0 & 0 & 0 & 0 \end{array}\right) \tag{I.15}$$

with

$$\begin{aligned} U^*_{ij}\,(y,\tau;x,t) &= M'_{ij}\Phi\\ &= \delta_{ij}\frac{H(t-\tau)}{\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi(t-\tau)}\right)^{\frac{3}{2}} e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}} + \frac{\partial^2\Phi}{\partial y_j \partial y_i}. \end{aligned} \tag{I.16}$$

**Remarks**. All of the components on the fourth column in $[U^*_{IL}]$ are zero. This implies that we must find another fundamental solutions, independent on the first three column vectors in (I.15) to determine the pressure. Such fundamental solutions are discussed in Oseen [10, p. 48]. We also remark that $U^*_{ij,i} = 0$.

Let $x$ be an internal point of $\Omega$. $U^*_{ij}$ are singular for $y = x$, $\tau = t$, but they are regular elsewhere. For the application of $U^*_{ij}$ to (I.5), we must exclude the point of singularity. This can be done by replacing the interval $[0, t]$ of the integrations by $[0, t - \varepsilon]$ with a small positive number $\varepsilon$. Then we have

$$\int_0^{t-\varepsilon} \int_\Omega (L_{IJ} U_J) \, U^*_{Ij} \, dV d\tau = \int_0^{t-\varepsilon} \int_\Omega B_i U^*_{ij} \, dV d\tau, \tag{I.17}$$

and

$$\int_0^t \int_\Omega U_I \left( L^*_{IJ} U^*_{JL} \right) \, dV d\tau = 0. \tag{I.18}$$

After this replacement, we consider the limiting process when $\varepsilon \to 0$. According to the discussion in Oseen [10, Section 5], we can see that

$$\lim_{\varepsilon \to 0} \mathrm{Re} \int_\Omega \left( u_i U^*_{ij} \right)_{\tau = t - \varepsilon} dV = u_j(x, t) + \frac{1}{4\pi} \int_\Gamma u_i n_i \frac{\partial}{\partial y_j} \left( \frac{1}{r} \right) dS, \tag{I.19}$$

and

$$\lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_\Omega B_i U^*_{ij} \, dV d\tau = \int_0^t \int_\Omega B_i U^*_{ij} \, dV d\tau. \tag{I.20}$$

The functions involved in the integrations on the surface $\Gamma$ are not singular, because $r = |y - x| > 0$ for arbitrary but fixed $x$.

**Derivation of the Green formula (I.5)**

Dropping the index $L$, we see by integration by parts that the following holds.

$$\begin{aligned}
&\int_0^t \int_\Omega (L_{IJ} U_J) \, U^*_I \, dV d\tau \\
&\qquad = \int_0^t \int_\Omega \left\{ \left( -\mathrm{Re} \dot{u}_i + u_{i,jj} + u_{j,ij} + U_{4,j} \delta_{ij} \right) U^*_i + u_{j,j} U^*_4 \right\} dV d\tau \\
&\qquad = -\mathrm{Re} \int_0^t \int_\Omega \left\{ \frac{\partial}{\partial t} \left( u_i U^*_i \right) - u_i \dot{U}^*_i \right\} dV d\tau \\
&\qquad\quad + \int_0^t \left\{ \int_\Gamma \left( u_{i,j} n_j U^*_i + u_{j,i} n_j U^*_i - u_i n_j U^*_{i,j} - u_j n_i U^*_{i,j} + U_4 n_j \delta_{ij} U^*_i \right) dS \right\} d\tau \\
&\qquad\quad + \int_0^t \int_\Omega \left( u_i U^*_{i,jj} + u_j U^*_{i,ji} - U_4 \delta_{ij} U^*_{i,j} \right) dV d\tau \\
&\qquad\quad + \int_0^t \int_\Gamma u_j n_j U^*_4 \, dS d\tau - \int_0^t \int_\Omega u_j U^*_{4,j} \, dV d\tau.
\end{aligned}$$

Note that $\delta_{ij}U^*_{i,j} = U^*_{j,j}$, $u_j n_j = u_i\delta_{ij}n_j$, and $u_jU^*_{4,j} = u_iU^*_{4,j}\delta_{ij}$. Using (I.4) we have

$$
\begin{aligned}
&\int_0^t \int_\Omega (L_{IJ}U_J)\, U_I^*\, dV d\tau \\
&\qquad = -\mathrm{Re}\left[\int_\Omega u_iU_i^*\, dV\right]_{\tau=0}^t \\
&\qquad\quad + \int_0^t \int_\Gamma \left\{ (u_{i,j} + u_{j,i} + U_4\delta_{ij})\, n_jU_i^* - u_i\left(U^*_{i,j} + U^*_{j,i} - U_4^*\delta_{ij}\right) n_j \right\} dS d\tau \\
&\qquad\quad + \int_0^t \int_\Omega \left\{ u_i\left(\mathrm{Re}\dot{U}_i^* + U^*_{i,jj} + U^*_{j,ij} - U^*_{4,j}\delta_{ij}\right) - U_4U^*_{j,j}\right\} dV d\tau \\
&\qquad = -\mathrm{Re}\left[\int_\Omega u_iU_i^*\, dV\right]_0^t \\
&\qquad\quad + \int_0^t \int_\Gamma (\sigma_iU_i^* - u_i\Sigma_i^*)\, dS d\tau + \int_0^t \int_\Omega U_I\,(L^*_{IJ}U_J^*)\, dV d\tau.
\end{aligned}
$$

**Derivation of $\Phi$ in (I.14)**.

We put $\Psi = \Delta\Phi$ in (I.13), then $\Psi$ satisfies

$$(\mathrm{Re}D_\tau + \Delta_y)\,\Psi = \delta(x)\delta(t).$$

The solution with spherical symmetry is given by

$$\Psi = -\frac{1}{\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi(t-\tau)}\right)^{\frac{3}{2}} e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}}. \tag{I.21}$$

Hence $\Phi$ is given as a solution of the equation:

$$\Delta\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \Psi. \tag{I.22}$$

To find $\Phi$, we see that

$$
\begin{aligned}
\frac{\partial}{\partial r}(r\Phi) &= \frac{1}{\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi(t-\tau)}\right)^{\frac{3}{2}} \int_r^\infty \rho e^{-\frac{\mathrm{Re}\rho^2}{4(t-\tau)}}\, d\rho \\
&= \frac{1}{2\pi\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi(t-\tau)}\right)^{\frac{1}{2}} e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}} = E(r, t-\tau).
\end{aligned}
$$

Then we see that

$$\Phi = \frac{1}{r}\int_0^r E(\rho, t-\tau)\, d\rho.$$

Using the relation:

$$\int_0^r E(\rho, t-\tau)\, d\rho = \frac{1}{4\pi\mathrm{Re}}\mathrm{Erf}\left(\frac{r}{2}\sqrt{\frac{\mathrm{Re}}{t-\tau}}\right),$$

we have (I.14).

**Proof of (I.19)**.

The proof follows Oseen [10, p. 42]: From (3.5), (I.21), (I.22) we see for $\tau < t$ that

$$U^*_{ij} = \delta_{ij}\frac{1}{\mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi(t-\tau)}\right)^{\frac{3}{2}} e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}} + \frac{\partial^2\Phi}{\partial y_j \partial y_i}. \tag{I.23}$$

Using the incompressibility condition (2.2), we have

$$\begin{aligned}
&\mathrm{Re}\int_\Omega \left(u_i U^*_{ij}\right)_{\tau=t-\varepsilon} dV(y) \\
&= \int_\Omega u_j(y,t-\varepsilon)\left(\frac{\mathrm{Re}}{4\pi\varepsilon}\right)^{\frac{3}{2}} e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}\, dV + \mathrm{Re}\int_\Omega \left(u_i \frac{\partial^2\Phi}{\partial y_j\partial y_i}\right)_{\tau=t-\varepsilon} dV \\
&= \left(\frac{\mathrm{Re}}{4\pi}\right)^{\frac{3}{2}} \int_\Omega u_j(y,t-\varepsilon)\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV + \mathrm{Re}\int_\Gamma \left(u_i n_i \frac{\partial\Phi}{\partial y_j}\right)_{\tau=t-\varepsilon} dS(y).
\end{aligned}$$

Since the convergence of the limit:

$$\lim_{\varepsilon\to 0}\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}} = 0$$

is uniform for any $r \geq \delta$ with some small but fixed $\delta > 0$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega u_j(y,t-\varepsilon)\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV = \lim_{\varepsilon\to 0}\int_{r<\delta} u_j(y,t-\varepsilon)\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV.$$

We write the integral in the form:

$$\begin{aligned}
\int_{r<\delta} u_j(y,t-\varepsilon)\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV &= u_j(x,t)\int_{r<\delta}\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV \\
&\quad + \int_{r<\delta}\left\{u_j(y,t-\varepsilon) - u_j(x,t)\right\}\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV.
\end{aligned}$$

From the relation $\displaystyle\int_0^\infty z^2 e^{-z^2}\, dz = \frac{\sqrt{\pi}}{4}$, the limit of the integral involved in the first term on the right hand side is calculated as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{r<\delta}\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}}\, dV \\
&= \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\delta}\frac{e^{-\frac{\mathrm{Re}r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}} r^2\sin\theta\, dr d\theta d\varphi \\
&= \frac{32\pi}{\mathrm{Re}^{\frac{3}{2}}}\lim_{\varepsilon\to 0}\int_0^{\delta\sqrt{\frac{\mathrm{Re}}{4\varepsilon}}}\alpha^2 e^{-\alpha^2}\, d\alpha = \left(\frac{4\pi}{\mathrm{Re}}\right)^{\frac{3}{2}}.
\end{aligned}$$

The second term is evaluated as follows:

$$\left| \int_{r<\delta} \{u_j(y, t-\varepsilon) - u_j(x,t)\} \frac{e^{-\frac{\mathrm{Re} r^2}{4\varepsilon}}}{\varepsilon^{\frac{3}{2}}} \, dV \right| \leq \max_{\substack{|x-y|\leq\delta \\ |t-\tau|\leq\varepsilon}} |u_j(y,\tau) - u_j(x,t)| \left(\frac{4\pi}{\mathrm{Re}}\right)^{\frac{3}{2}}.$$

Since $u_j(x,t)$ is continuous, we can make the last expression arbitrarily small by taking sufficiently small $\delta$ and $\varepsilon$. Therefore we have

$$\lim_{\varepsilon\to 0} \int_\Omega u_j(y, t-\varepsilon) \left(\frac{\mathrm{Re}}{4\pi\varepsilon}\right)^{\frac{3}{2}} e^{-\frac{\mathrm{Re} r^2}{4\varepsilon}} \, dV = u_j(x,t).$$

Next we consider the limit:

$$\lim_{\varepsilon\to 0} \mathrm{Re} \int_\Gamma \left(u_i n_i \frac{\partial \Phi}{\partial y_j}\right)_{\tau=t-\varepsilon} dS.$$

From (3.6a) we see that

$$\left(\frac{\partial \Phi}{\partial y_j}\right)_{\tau=t-\varepsilon} = \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) \int_0^r E(\rho,\varepsilon)\, d\rho + E(\rho,\varepsilon) \frac{\partial r}{\partial y_j}.$$

Using the relation $\displaystyle\int_0^\infty e^{-z^2}\, dz = \frac{\sqrt{\pi}}{2}$, we know that

$$\lim_{\varepsilon\to 0} \int_0^r E(\rho,\varepsilon)\, d\rho = \frac{1}{4\pi \mathrm{Re}},$$

and

$$\lim_{\varepsilon\to 0} E(\rho,\varepsilon) = 0.$$

The convergence of these two limits is uniform for $r \geq \delta$ with the positive $\delta = \max_{y\in\Gamma^{(\tau)}, |t-\tau|\leq\varepsilon} |y-x|$, we can see that

$$\lim_{\varepsilon\to 0} \mathrm{Re} \int_\Gamma \left(u_i n_i \frac{\partial \Phi}{\partial y_j}\right)_{\tau=t-\varepsilon} dS = \frac{1}{4\pi} \int_\Gamma u_i n_i \frac{\partial}{\partial y_j}\left(\frac{1}{r}\right) dS.$$

**Proof of (I.20)**.

The proof follows Oseen [10, p. 45]: We shall show that the integral

$$\int_0^t \int_\Omega B_i U_{ij}^* \, dV d\tau$$

is absolutely convergent for continuous and bounded $B_i$. From this property, the relation (I.20) is clear. To this end, we show that the singularity of $U^*_{ij}$ with respect to the variable $y$ at $y = x$ and the variable $\tau$ at $\tau = t$ can be separated, and that the singularity is weak.

In fact, we consider (I.23) with $t > \tau$ in the form:

$$U^*_{ij} = \delta_{ij} \frac{\mathrm{Re}}{2} \frac{E(r, t-\tau)}{t-\tau} + \frac{\partial^2 \Phi}{\partial y_j \partial y_i}. \tag{I.24}$$

For $r = \sqrt{(y_k - x_k)^2}$ we know that

$$\frac{\partial r}{\partial y_i} = \frac{r_i}{r}, \tag{I.25a}$$

$$\frac{\partial^2 r}{\partial y_j \partial y_i} = \frac{1}{r}\delta_{ij} - \frac{r_i r_j}{r^3}, \tag{I.25b}$$

$$\frac{\partial}{\partial y_i}\left(\frac{1}{r}\right) = \frac{r_i}{r^3}, \tag{I.25c}$$

$$\frac{\partial^2}{\partial y_j \partial y_i}\left(\frac{1}{r}\right) = -\frac{\delta_{ij}}{r^3} + \frac{3 r_i r_j}{r^5}. \tag{I.25d}$$

Using these relations we can see that

$$\frac{\partial \Phi}{\partial y_i} = \frac{\partial \Phi}{\partial r}\frac{r_i}{r}, \tag{I.26}$$

$$\frac{\partial^2 \Phi}{\partial y_j \partial y_i} = \delta_{ij}\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{r_i r_j}{r^2}\left(\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r}\right). \tag{I.27}$$

From (3.6a) we can see furthermore that

$$\frac{\partial \Phi}{\partial r} = -\frac{1}{r}\left\{\Phi - E(r, t-\tau)\right\}, \tag{I.28}$$

$$\frac{\partial^2 \Phi}{\partial r^2} = \frac{2}{r^2}\left\{\Phi - E(r, t-\tau)\right\} - \frac{\mathrm{Re}}{2}\frac{E(r, t-\tau)}{t-\tau}. \tag{I.29}$$

Therefore we have

$$\begin{aligned} U^*_{ij} &= \delta_{ij}\tfrac{\mathrm{Re}}{2}\tfrac{E}{t-\tau} - \tfrac{\delta_{ij}}{r^2}(\Phi - E) \\ &\quad + \tfrac{r_i r_j}{r^2}\left\{\tfrac{2}{r^2}(\Phi - E) - \tfrac{\mathrm{Re}}{2}\tfrac{E}{t-\tau} + \tfrac{1}{r^2}(\Phi - E)\right\} \\ &= \tfrac{\mathrm{Re}}{2}\left\{\delta_{ij} - \tfrac{r_i r_j}{r^2}\right\}\tfrac{E}{t-\tau} - \tfrac{1}{r^2}\left\{\delta_{ij} - 3\tfrac{r_i r_j}{r^2}\right\}(\Phi - E). \end{aligned} \tag{I.30}$$

We notice for $r > 0$ that $\left|\dfrac{r_i r_j}{r^2}\right| < 1$ and that

$$0 < E(r, t-\tau) < \frac{1}{4\pi\sqrt{\mathrm{Re}\pi(t-\tau)}}, \quad 0 < \Phi < \frac{1}{4\pi\sqrt{\mathrm{Re}\pi(t-\tau)}}.$$

From the last two inequalities we have

$$|\Phi - E| < \frac{1}{4\pi\sqrt{\mathrm{Re}\pi(t-\tau)}}.$$

Using the inequality $se^{-s} \leq e^{-1}$ for $s > 0$, we have

$$\begin{aligned} \frac{\mathrm{Re}}{2}\frac{E}{t-\tau} &= \frac{1}{2\pi r^2\sqrt{\mathrm{Re}\pi(t-\tau)}}\left[\frac{\mathrm{Re}r^2}{4(t-\tau)}\right]e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}} \\ &\leq \frac{1}{2e\pi r^2\sqrt{\mathrm{Re}\pi(t-\tau)}} < \frac{1}{4\pi r^2\sqrt{\mathrm{Re}\pi(t-\tau)}}. \end{aligned}$$

Therefore we obtain the inequality:

$$|U^*_{ij}| < \frac{6}{4\pi r^2\sqrt{\mathrm{Re}\pi(t-\tau)}}.$$

This implies that $|U^*_{ij}|$ are summable.

## APPENDIX II

In this appendix, we shall prove Lemma 3.1. The idea of the inequality estimates is due to Pogorzelski [11, p. 353]: From (I.30) each component $U^*_{ij}$ can be estimated in the following way:

$$|U^*_{ij}| \leq \mathrm{Re}\frac{E(r, t-\tau)}{t-\tau} + \frac{4}{r^2}|\Phi - E(r, t-\tau)|. \tag{II.1}$$

Using the inequality:

$$\xi^\alpha e^{-\xi} \leq \alpha^\alpha e^{-\alpha} \tag{II.2}$$

with any $\alpha > 0$, we can see from (3.7a) that

$$\begin{aligned} \mathrm{Re}\frac{E(r, t-\tau)}{t-\tau} &= \frac{1}{2^{2\mu-1}\pi^{\frac{3}{2}}\mathrm{Re}^{1-\mu}}\frac{1}{(t-\tau)^\mu}\frac{1}{r^{3-2\mu}}\left[\frac{\mathrm{Re}r^2}{4(t-\tau)}\right]^{\frac{3}{2}-\mu}\exp[-\frac{\mathrm{Re}r^2}{4(t-\tau)}] \\ &\leq \frac{G_1}{(t-\tau)^\mu}\frac{1}{r^{3-2\mu}} \end{aligned}$$

with $\alpha_1 = \frac{3}{2} - \mu > 0$. Here we put $G_1(\mu) = \dfrac{\alpha_1^{\alpha_1} e^{-\alpha_1}}{2^{2\mu-1}\pi^{\frac{3}{2}}\mathrm{Re}^{1-\mu}}$. We restrict $\mu$ as to satisfy $\mu > 1$ and $3 - 2\mu < 2$. This implies $\dfrac{1}{2} < \mu < 1$. In this case we see that

$$\begin{aligned}\int_0^t \int_{S_m} \mathrm{Re}\frac{E(r, t-\tau)}{t-\tau)}\, dS d\tau \quad &\leq \quad \int_0^t \frac{G_1}{(t-\tau)^\mu} \int_{S_m} \frac{dS}{r^{3-2\mu}}\, d\tau \\ &\leq \quad \int_0^t \frac{G_1}{(t-\tau)^\mu} \sup_{\substack{0<\tau<t\\ m\in N}} \int_{S_m} \frac{dS}{r^{3-2\mu}}. \qquad \text{(II.3)}\end{aligned}$$

Since each $S_m$ is a closed Lyapunov surface tending to $\Gamma$ as $m \to \infty$, the supremum is bounded by some constant. Owing to Oseen [10, p. 69] we know the relation:

$$\Phi - E(r, t-\tau) = \frac{\mathrm{Re}}{4} r^2 \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^2}\, ds. \qquad \text{(II.4)}$$

Thus, the second term in (II.1) is evaluated at follows:

$$\begin{aligned}\frac{4}{r^2}\left|\Phi - E(r, t-\tau)\right| &= \mathrm{Re}\int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^2}\, ds \\ &= \int_{-\infty}^{\tau} \frac{\mathrm{Re}^{\nu-2}}{2^{2\nu-3}\pi^{\frac{3}{2}}} \frac{1}{(t-s)^\nu} \frac{1}{r^{5-2\nu}} \left[\frac{\mathrm{Re}r^2}{4(t-s)}\right]^{\frac{5}{2}-\nu} \exp\left[-\frac{\mathrm{Re}r^2}{4(t-s)}\right] ds \\ &\leq \frac{\mathrm{Re}^{\nu-2}}{2^{2\nu-3}\pi^{\frac{3}{2}}} \frac{\nu-1}{(t-\tau)^{\nu-1}} \frac{1}{r^{5-2\nu}} \alpha_2^{\alpha_2} e^{-\alpha_2}\end{aligned}$$

with $\alpha_2 = \dfrac{5}{2} - \nu > 0$ for any $\nu > 1$. We restrict further as to satisfy $\nu - 1 < 1$ and $5 - 2\nu < 2$. This implies $\dfrac{3}{2} < \nu < 2$. We put $G_2(\nu) = \dfrac{(\nu-1)\alpha_2^{\alpha_2} e^{-\alpha_2}}{2^{2\nu-3}\pi^{\frac{3}{2}}\mathrm{Re}^{2-\nu}}$. In this case, we see that

$$\int_0^t \int_{S_m} \frac{4}{r^2}|\Phi - E(r, t-\tau)|\, dS(y) d\tau \leq \int_0^t \frac{G_2}{(t-\tau)^{\nu-1}}\, d\tau \sup_{\substack{0<\tau<t\\ m\in N}} \int_{S_m} \frac{dS(y)}{r^{5-2\nu}}.$$

The supremum is bounded by some constant.

**Derivation of (II.4)**.

We notice that $E(\rho, t-s)$ satisfies

$$\frac{\partial E}{\partial \rho} = -\frac{\mathrm{Re}\rho}{2(t-s)} E, \qquad \text{(II.5)}$$

and

$$\mathrm{Re}\frac{\partial E}{\partial s} + \frac{\partial^2 E}{\partial \rho^2} = 0.$$

From the last relation we can see for $\tau < t$ that

$$\begin{aligned} 0 &= \int_{-\infty}^{\tau}\int_0^r \left(\mathrm{Re}\frac{\partial E}{\partial s} + \frac{\partial^2 E}{\partial \rho^2}\right) d\rho ds \\ &= \mathrm{Re}\int_0^r E(\rho, t-\tau)\, d\rho + \int_{-\infty}^{\tau} \frac{\partial}{\partial r} E(r, t-s)\, ds. \end{aligned}$$

Therefore, the integration by parts yields:

$$\begin{aligned} &\int_0^r E(\rho, t-\tau)\, d\rho = -\frac{1}{\mathrm{Re}}\int_{-\infty}^{\tau} \frac{\partial}{\partial r} E(r, t-s)\, ds \\ &= r\int_{-\infty}^{\tau} \frac{1}{2\pi \mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi}\right)^{\frac{1}{2}} \frac{e^{-\frac{\mathrm{Re} r^2}{4(t-s)}}}{2(t-s)^{\frac{3}{2}}}\, ds \\ &= \frac{r}{2\pi \mathrm{Re}}\left(\frac{\mathrm{Re}}{4\pi}\right)^{\frac{1}{2}} \left\{ \left[\frac{e^{-\frac{\mathrm{Re} r^2}{4(t-s)}}}{\sqrt{t-s}}\right]_{s=-\infty}^{\tau} - \int_{-\infty}^{\tau} \frac{1}{\sqrt{t-s}}\frac{\partial}{\partial s} e^{-\frac{\mathrm{Re} r^2}{4(t-s)}}\, ds \right\} \\ &= rE(r, t-\tau) + \frac{\mathrm{Re}}{4} r^3 \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^2}\, ds. \end{aligned}$$

By dividing the first and last expressions by $r$ and from (3.6a), we obtain (II.4).

## APPENDIX III

In this appendix, we shall derive (3.11): From (I.24), (I.25a) and (II.5) we have

$$\begin{aligned} \frac{\partial U^*_{ij}}{\partial y_k} &= \delta_{ij}\frac{\mathrm{Re}}{2}\frac{1}{t-\tau}\frac{\partial E}{\partial r}\frac{r_k}{r} + \frac{\partial}{\partial y_k}\left(\frac{\partial^2 \Phi}{\partial y_j \partial y_i}\right) \\ &= -\frac{\delta_{ij} r_k}{r}\left(\frac{\mathrm{Re}}{2}\right)^2 \frac{rE}{(t-\tau)^2} + \frac{\partial}{\partial y_k}\left(\frac{\partial^2 \Phi}{\partial y_j \partial y_i}\right). \end{aligned} \tag{III.1}$$

The last term is calculated by using (I.27) as follows:

$$\begin{aligned} \frac{\partial}{\partial y_k}\left(\frac{\partial^2 \Phi}{\partial y_j \partial y_i}\right) &= \delta_{ij}\left(-\frac{1}{r^2}\right)\frac{r_k}{r}\frac{\partial \Phi}{\partial r} + \delta_{ij}\frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2}\frac{r_k}{r} \\ &\quad + \left(-\frac{2}{r^3}\right)\frac{r_k}{r} r_i r_j + \frac{1}{r^2}\left(\delta_{ik} r_j + \delta_{jk} r_i\right)\left(\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r}\right) \\ &\quad + \frac{r_i r_j}{r^2}\left(\frac{\partial^3 \Phi}{\partial r^3} + \frac{1}{r^2}\frac{\partial \Phi}{\partial r} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2}\right)\frac{r_k}{r} \\ &= \frac{\delta_{ij} r_k + \delta_{jk} r_i + \delta_{ki} r_j}{r}\left(\frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r}\right) \\ &\quad + \frac{r_i r_j r_k}{r}\left(\frac{\partial^3 \Phi}{\partial r^3} - \frac{3}{r}\frac{\partial^2 \Phi}{\partial r^2} + \frac{3}{r^3}\frac{\partial \Phi}{\partial r}\right). \end{aligned} \tag{III.2}$$

The last result is symmetric for indices $i, j, k$. $\dfrac{\partial \Phi}{\partial r}$ is given by (I.28). $\dfrac{\partial^2 \Phi}{\partial r^2}$ is given by (I.29). Therefore we know that

$$\begin{aligned}
&\frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r} \\
&\quad = \frac{1}{r}\left\{\frac{2}{r^2}(\Phi - E) - \frac{\mathrm{Re}}{2}\frac{E}{t-\tau}\right\} - \frac{1}{r^2}\left\{-\frac{1}{r}(\Phi - E)\right\} \\
&\quad = \frac{3}{r^2}(\Phi - E) - \frac{\mathrm{Re}}{2r}\frac{E}{t-\tau} \\
&\quad = \frac{3\mathrm{Re}}{4r}\int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^2}\, ds - \frac{\mathrm{Re}}{2r}\frac{E}{t-\tau}.
\end{aligned}$$

The last equality follows from (II.4). We notice the following equality:

$$\int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^2}\, ds = \frac{2}{3}\frac{E(t-\tau)}{t-\tau} + \frac{\mathrm{Re}}{6} r^2 \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^3}\, ds. \tag{III.3}$$

Hence we have

$$\frac{1}{r}\frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \Phi}{\partial r} = \frac{1}{2}\left(\frac{\mathrm{Re}}{2}\right)^2 r \int_{-\infty}^{\tau} \frac{E(r, t-s)}{(t-s)^3}\, ds. \tag{III.4}$$

Moreover, $\dfrac{\partial^3 \Phi}{\partial r^3}$ is given as follows:

$$\begin{aligned}
\frac{\partial^3 \Phi}{\partial r^3} &= -\frac{4}{r^3}(\Phi - E) + \frac{2}{r^2}\left(\frac{\partial \Phi}{\partial r} - \frac{\partial E}{\partial r}\right) - \frac{\mathrm{Re}}{2(t-\tau)}\frac{\partial E}{\partial r} \\
&= -\frac{4}{r^3}(\Phi - E) + \frac{2}{r^2}\left(-\frac{\Phi - E}{r} - \frac{\partial E}{\partial r}\right) - \frac{\mathrm{Re}}{2(t-\tau)}\frac{\partial E}{\partial r} \\
&= -\frac{6}{r^3}(\Phi - E) - \left\{\frac{2}{r^2} + \frac{\mathrm{Re}}{2(t-\tau)}\right\}\frac{\partial E}{\partial r} \\
&= -\frac{6}{r^3}(\Phi - E) + \left\{\frac{1}{r} + \frac{\mathrm{Re}}{4(t-\tau)} r\right\} \mathrm{Re}\frac{E}{t-\tau}.
\end{aligned}$$

The last equality follows from (II.5). Therefore we know that

$$\begin{aligned}
&\frac{\partial^3\Phi}{\partial r^3} - \frac{3}{r}\frac{\partial^2\Phi}{\partial r^2} + \frac{3}{r^2}\frac{\partial\Phi}{\partial r} \\
&\quad = -\frac{6}{r^3}(\Phi - E) + \left\{\frac{1}{r} + \frac{\mathrm{Re}}{4(t-\tau)}r\right\}\mathrm{Re}\frac{E}{t-\tau} \\
&\quad\quad -\frac{3}{r}\left\{\frac{2}{r^2}(\Phi - E) - \frac{\mathrm{Re}}{2}\frac{E}{t-\tau}\right\} + \frac{3}{r^2}\left\{-\frac{1}{r}(\Phi - E)\right\} \\
&\quad = -\frac{15}{r^3}(\Phi - E) + \frac{5\mathrm{Re}}{2r}\frac{E}{t-\tau} + \frac{\mathrm{Re}^2 r}{4}\frac{E}{(t-\tau)^2} \\
&\quad = -\frac{15\mathrm{Re}}{4r}\int_{-\infty}^{\tau}\frac{E}{(t-s)^2}\,ds + \frac{5\mathrm{Re}}{2r}\frac{E}{t-\tau} + \left(\frac{\mathrm{Re}}{2}\right)^2 r\frac{E}{(t-\tau)^2} \\
&\quad = -\frac{5}{2}\left(\frac{\mathrm{Re}}{2}\right)^2 r\int_{-\infty}^{\tau}\frac{E}{(t-s)^3}\,ds + \left(\frac{\mathrm{Re}}{2}\right)^2 r\frac{E}{(t-\tau)^2}.
\end{aligned}$$

The last equality follows from (III.3). We notice the following equality:

$$\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^3}\,ds = \frac{2}{5}\frac{E(r,t-s)}{(t-s)^2} + \frac{\mathrm{Re}}{10}r^2\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^4}\,ds. \qquad \text{(III.5)}$$

Hence we have

$$\frac{\partial^3\Phi}{\partial r^3} - \frac{3}{r}\frac{\partial^2\Phi}{\partial r^2} + \frac{3}{r^2}\frac{\partial\Phi}{\partial r} = -\frac{1}{2}\left(\frac{\mathrm{Re}}{2}\right)^3 r^3\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^4}\,ds. \qquad \text{(III.6)}$$

By substituting (III.4) and (III.6) into (III.2), we have

$$\begin{aligned}
\frac{\partial^3\Phi}{\partial y_k\partial y_j\partial y_i} &= \frac{\delta_{ij}r_k + \delta_{jk}r_i + \delta_{ki}r_j}{r}\frac{1}{2}\left(\frac{\mathrm{Re}}{2}\right)^2 r\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^3}\,ds \\
&\quad - \frac{r_i r_j r_k}{r^3}\frac{1}{2}\left(\frac{\mathrm{Re}}{2}\right)^3 r^3\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^4}\,ds.
\end{aligned}$$

Hence, (3.11) follows immediately from (III.1).

**Proof of (III.3).**

From (3.7a) we can see that

$$\begin{aligned}
&\int_{-\infty}^{\tau}\frac{E(r,t-s)}{(t-s)^2}\,ds = \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\int_{-\infty}^{\tau}\frac{e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{(t-s)^{\frac{5}{2}}}\,ds \\
&= \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\left\{\left[\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{3(t-s)^{\frac{3}{2}}}\right]_{s=-\infty}^{\tau} - \int_{-\infty}^{\tau}\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{3(t-s)^{\frac{3}{2}}}\left(-\frac{\mathrm{Re}r^2}{4(t-s)^2}\right)ds\right\} \\
&= \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\left\{\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}}}{3(t-\tau)^{\frac{3}{2}}} + \frac{\mathrm{Re}}{6}r^2\int_{-\infty}^{\tau}\frac{e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{(t-s)^{\frac{7}{2}}}\,ds\right\}.
\end{aligned}$$

**Proof of (III.5).**

$$\begin{aligned}
\int_{-\infty}^{\tau} \frac{E(r,t-s)}{(t-s)^3}\,ds &= \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\int_{-\infty}^{\tau}\frac{e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{(t-s)^{\frac{7}{2}}}\,ds \\
&= \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\left\{\left[\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{5(t-s)^{\frac{5}{2}}}\right]_{s=-\infty}^{\tau} - \int_{-\infty}^{\tau}\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{5(t-s)^{\frac{5}{2}}}\left(-\frac{\mathrm{Re}r^2}{4(t-s)^2}\right)ds\right\} \\
&= \frac{1}{2^2\pi^{\frac{3}{2}}\mathrm{Re}^{\frac{1}{2}}}\left\{\frac{2e^{-\frac{\mathrm{Re}r^2}{4(t-\tau)}}}{5(t-\tau)^{\frac{5}{2}}} + \frac{\mathrm{Re}}{10}r^2\int_{-\infty}^{\tau}\frac{e^{-\frac{\mathrm{Re}r^2}{4(t-s)}}}{(t-s)^{\frac{9}{2}}}\,ds\right\}.
\end{aligned}$$

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