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On k-Dependent Domination in Graphs

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Abstract. A subset D of vertices in a graph G is *k-dependent* if the maximum degree of a vertex in the subgraph $\langle D\rangle$ induced by D is at most k. The k-dependent domination number $\gamma^k(G)$ of a graph G is the minimum cardinality of a k-dependent dominating set of G. Any k-dependent dominating set D of a graph G with $|D| = \gamma^k(G)$ is called a γ^k-set of G. A vertex x of a graph G is called: (i) γ^k-good if x belongs to some γ^k-set, (ii) γ^k-fixed if x belongs to every γ^k-set, (iii) γ^k-free if x belongs to some γ^k-set but not to all γ^k-sets, (iv) γ^k-bad if x belongs to no γ^k-set. In this paper we deal with γ^k-good/bad/fixed/free vertices and present results on changing and unchanging of the k-dependent domination number when a graph is modified by adding an edge or deleting a vertex.

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§1. INTRODUCTION

We consider finite, simple graphs. For notation and graph theory terminology not presented here, we follow Haynes, et al. [5]. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S, G\rangle$. For a vertex x of G, $N(x, G)$ denote the set of all neighbors of x in G and $N[x, G] = N(x, G) \cup \{x\}$. The *maximum degree* of the graph G is denoted by $\Delta(G)$. For a graph G, let $x \in X \subseteq V(G)$. The *private neighbor set of x with respect to X* is $pn[x, X] = \{y \in V(G) : N[y, G] \cap X = \{x\}\}$.

Let G be a graph and $S \subseteq V(G)$. A set S is called *k-dependent* if $\Delta(\langle S, G\rangle) \leq k$. If $\Delta(\langle S, G\rangle) = 0$ then S is called *independent*. We let $i(G)$ denote the minimum cardinality of a maximal independent set of vertices in G. A *k-dependent dominating set* D in a graph G is a vertex subset which is both k-dependent and dominating. The minimum cardinality of an k-dependent dominating set of G is called the *k-dependent domination number* and is denoted by $\gamma^k(G)$. The concept of k-dependent domination was introduced by Favaron, Hedetniemi, Hedetniemi and Rall [2]. Note that $\gamma^{\Delta(G)}(G) = \gamma(G)$ - the ordinary domination number of a graph and $\gamma^0(G) = i(G)$.

Much has been written about the effects on domination related parameters when a graph is modified by deleting a vertex, adding an edge or deleting an edge. For surveys see [5, Chapter 5], [6, Chapter 16]. In this paper we present results on changing and unchanging of the k-dependent domination number when an edge is added or a vertex is deleted.

§2. VERTEX DELETION AND EDGE ADDITION

Let $\mu(G)$ be a numerical invariant of a graph G defined in such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ with a given property P. A set with property P and with $\mu(G)$ vertices in G is called a μ-set of G. A graph G is *vertex-μ-critical* if $\gamma(G-v) \neq \gamma(G)$ for all v in $V(G)$. A vertex v of a graph G is defined to be

(a) [4] *μ-good*, if v belongs to some μ-set of G;
(b) [4] *μ-bad*, if v belongs to no μ - set of G;
(c) [8] *μ-fixed* if v belongs to every μ-set;
(d) [8] *μ-free* if v belongs to some μ-set but not to all μ-sets.

For a graph G we define:
$\mathbf{G}^k(G) = \{x \in V(G) : x \text{ is } \gamma^k\text{-good }\}$;
$\mathbf{B}^k(G) = \{x \in V(G) : x \text{ is } \gamma^k\text{-bad }\}$;
$\mathbf{Fi}^k(G) = \{x \in V(G) : x \text{ is } \gamma^k\text{-fixed }\}$;
$\mathbf{Fr}^k(G) = \{x \in V(G) : x \text{ is } \gamma^k\text{-free }\}$;
$\mathbf{V}^k_0(G) = \{x \in V(G) : \gamma^k(G-x) = \gamma^k(G)\}$;
$\mathbf{V}^k_-(G) = \{x \in V(G) : \gamma^k(G-x) < \gamma^k(G)\}$;
$\mathbf{V}^k_+(G) = \{x \in V(G) : \gamma^k(G-x) > \gamma^k(G)\}$.

Clearly, $\{\mathbf{V}^k_-(G), \mathbf{V}^k_0(G), \mathbf{V}^k_+(G)\}$ and $\{\mathbf{G}^k(G), \mathbf{B}^k(G)\}$ are partitions of $V(G)$, and $\{\mathbf{Fi}^k(G), \mathbf{Fr}^k(G)\}$ is a partition of $\mathbf{G}^k(G)$.

Proposition 2.1. *Let G be a graph and $v \in \mathbf{V}^k_-(G)$. Then:*

(1) $\gamma^k(G-v) = \gamma^k(G)-1$*; for any γ^k-set M of $G-v$ the set $M_v = M \cup \{v\}$ is a γ^k-set of G and any neighbor of v is a γ^k-bad vertex in $G-v$;*
(2) $\mathbf{G}^k(G-v) \subseteq \mathbf{G}^k(G)$, $\mathbf{Fi}^k(G-v) \supseteq \mathbf{Fi}^k(G) - \{v\}$ *and* $\mathbf{B}^k(G-v) \supseteq \mathbf{B}^k(G)$*;*
(3) *if u is a γ^k-fixed vertex of G and $u \neq v$ then $uv \notin E(G)$.*

Proof. (1) Let M be an arbitrary γ^k-set of $G - v$. If $u \in M$ then $u \notin N(v, G)$ - otherwise M will be a k-dependent dominating set of G, which is a contradiction with $\gamma^k(G-v) < \gamma^k(G)$. Then M_v is a k-dependent dominating set of G and $\gamma^k(G) \leq |M_v| = \gamma^k(G - v) + 1 \leq \gamma^k(G)$.

(2) Immediately follows by (1).

(3) By (2), $u \in \mathbf{Fi}^k(G - v)$ and by (1), $uv \notin E(G)$. □

Proposition 2.2. *Let G be a graph and $v \in V(G)$.*

(1) ([1] when $k = \Delta(G)$) *Let $v \in \mathbf{V}^k_+(G)$. Then v is a γ^k-fixed vertex of G;*
(2) *If v is a γ^k-bad vertex of G then $\gamma^k(G - v) = \gamma^k(G)$.*

Proof. (1) Let M be a γ^k-set of G. Assume $v \notin M$. Then M is a k-dependent dominating set of $G - v$ which implies $\gamma^k(G) < \gamma^k(G - v) \leq |M| = \gamma^k(G)$ - a contradiction.

(2) By (1), $\gamma^k(G - v) \leq \gamma^k(G)$ and by Proposition 2.1(1), $\gamma^k(G - v) \geq \gamma^k(G)$. □

Since for every $v \in V(G)$, $\gamma^k(G-v) \leq |V(G)|-1$ and because of Proposition 2.1 we have $\gamma^k(G - v) = \gamma^k(G) + p$, where $p \in \{-1, 0, .., |V(G)| - 2\}$. This motivated us to define for a graph G:

$\mathbf{Fr}^k_-(G) = \{x \in \mathbf{Fr}^k(G) : \gamma^k(G - x) = \gamma^k(G) - 1\}$;
$\mathbf{Fr}^k_0(G) = \{x \in \mathbf{Fr}^k(G) : \gamma^k(G - x) = \gamma^k(G)\}$;
$\mathbf{Fi}^k_p(G) = \{x \in \mathbf{Fi}^k(G) : \gamma^k(G-x) = \gamma^k(G)+p\}$, $p \in \{-1, 0, .., |V(G)|-2\}$.

Let G be a graph of order n. By Propositions 2.1 and 2.2 we have:

(e) $\{\mathbf{Fr}^k_-(G), \mathbf{Fr}^k_0(G)\}$ is a partition of $\mathbf{Fr}^k(G)$;
(f) $\{\mathbf{Fi}^k_{-1}(G), \mathbf{Fi}^k_0(G), \ldots, \mathbf{Fi}^k_{n-2}(G)\}$ is a partition of $\mathbf{Fi}^k(G)$;
(g) $\{\mathbf{Fi}^k_{-1}(G), \mathbf{Fr}^k_-(G)\}$ is a partition of $\mathbf{V}^k_-(G)$;
(h) $\{\mathbf{Fi}^k_0(G), \mathbf{Fr}^k_0(G), \mathbf{B}^k(G)\}$ is a partition of $\mathbf{V}^k_0(G)$;
(i) $\{\mathbf{Fi}^k_1(G), \mathbf{Fi}^k_2(G), \ldots, \mathbf{Fi}^k_{n-2}(G)\}$ is a partition of $\mathbf{V}^k_+(G)$.

Theorem 2.3. *Let G be a graph of order $n \geq 2$. Then G is a vertex-γ^k-critical graph if and only if $\gamma^k(G - v) = \gamma^k(G) - 1$ for all $v \in V(G)$.*

Proof. Necessity is obvious.

Sufficiency: Let G be a γ^k-critical graph. For every isolated vertex $v \in V(G)$, $\gamma^k(G - v) = \gamma^k(G) - 1$. So, let G have a component of order at least two, say Q. By Propositions 2.1 and 2.2 it follows that either for all $v \in V(Q)$, $\gamma^k(Q - v) > \gamma^k(Q)$ or for all $v \in V(Q)$, $\gamma^k(Q - v) = \gamma^k(Q) - 1$. Suppose, for all $v \in V(Q)$, $\gamma^k(Q - v) > \gamma^k(Q)$. But then Proposition 2.2(1) implies that $V(Q)$ is a γ^k-set of Q. This is a contradiction with $\gamma^k(Q - v) > \gamma^k(Q)$. □

When $k \in \{0, \Delta(G)\}$ the theorem above due to Ao and MacGillivray (as is referred in [6]) and Carrington, Harary and Haynes [1] respectively.

Theorem 2.4. *Let x and y be two nonadjacent vertices in a graph G. If $\gamma^k(G+xy) < \gamma^k(G)$ then $\gamma^k(G+xy) = \gamma^k(G) - 1$. Moreover, $\gamma^k(G+xy) = \gamma^k(G) - 1$ if and only if at least one of the following holds:*

(i) *$x \in \mathbf{V}^k_-(G)$ and y is a γ^k-good vertex of $G - x$;*

(ii) *x is a γ^k-good vertex of $G - y$ and $y \in \mathbf{V}^k_-(G)$.*

Proof. Let $\gamma^k(G+xy) < \gamma^k(G)$ and M be a γ^k-set of $G + xy$. Then $|\{x, y\} \cap M| = 1$, otherwise M will be a k-dependent dominating set of G which is a contradiction. Let without loss of generalities $x \notin M$ and $y \in M$. Since M is no dominating set of G then $M \cap N(x, G) = \emptyset$. Hence $M_1 = M \cup \{x\}$ is a k-dependent dominating set of G with $|M_1| = \gamma^k(G + xy) + 1$ which implies $\gamma^k(G) = \gamma^k(G + xy) + 1$. Since M is a k-dependent dominating set of $G - x$, $\gamma^k(G - x) \le \gamma^k(G + xy)$. Hence $\gamma^k(G) \ge \gamma^k(G - x) + 1$ and by Proposition 2.1 follows $\gamma^k(G) = \gamma^k(G - x) + 1$. Thus x is in $\mathbf{V}^k_-(G)$ and M is a γ^k-set of $G - x$. Since $y \in M$ then y is a γ^k-good vertex of $G - x$.

For the converse let without loss of generalities (i) hold. Then there is a γ^k-set M of $G - x$ with $y \in M$. Certainly M is a k-dependent dominating set of $G + xy$ and then $\gamma^k(G+xy) \le |M| = \gamma^k(G-x) = \gamma^k(G)-1 \le \gamma^k(G+xy)$. □

Corollary 2.5. *Let x and y be two nonadjacent vertices in a graph G and $x \in \mathbf{V}^k_-(G)$. Then $\gamma^k(G) - 1 \le \gamma^k(G + xy) \le \gamma^k(G)$.*

Proof. Let M be a γ^k-set of $G - x$. If $y \in \mathbf{G}^k(G - x)$ then by Theorem 2.4 $\gamma^k(G)-1 = \gamma^k(G+xy)$. So that, let $y \in \mathbf{B}^k(G-x)$. By Proposition 2.1, $M_1 = M \cup \{x\}$ is a γ^k-set of G and $M_1 \cap N(x, G) = \emptyset$. Hence M_1 is a k-dependent dominating set of $G+xy$ and $\gamma^k(G+xy) \le |M_1| = \gamma^k(G-x)+1 = \gamma^k(G)$. □

We will refine the definitions of the γ^k-free vertex and the γ^k-fixed vertex as follows. Let x be a vertex of a graph G.

(j) x is called *γ^k_0-free* if $x \in \mathbf{Fr}^k_0(G)$;

(k) x is called *$\gamma^k_-(G)$-free* if $x \in \mathbf{Fr}^k_-(G)$ and

(l) x is called *$\gamma^k_q(G)$-fixed* if $x \in \mathbf{Fi}^k_q(G)$, where $q \in \{-1, 0, 1, .., |V(G)| - 2\}$.

We need the following useful lemma:

Lemma 2.6. *Let x be a γ^k_0-fixed vertex of a graph G. Then $N(x, G) \subseteq \mathbf{B}^k(G - x) \cap (\mathbf{V}^k_0(G) \cup \mathbf{Fi}^k_1(G))$.*

Proof. Let M be a γ^k-set of $G - x$ and $y \in N(x, G)$. If $y \in M$ then M will be a k-dependent dominating set of G of cardinality $|M| = \gamma^k(G - x) = \gamma^k(G)$ - a contradiction with $x \in \mathbf{Fi}^k(G)$. Thus $N(x, G) \subseteq \mathbf{B}^k(G - x)$. By Proposition 2.1(3) it follows $y \notin \mathbf{V}^k_-(G)$. Assume $y \in \mathbf{Fi}^k_p(G)$ for some $p \ge 2$. It follows by $M \cap N(x, G) = \emptyset$ that $M_2 = M \cup \{x\}$ is a k-dependent dominating set of G

with $|M_2| = \gamma^k(G-x)+1 = \gamma^k(G)+1$. But $y \notin M$ and then $|M_2| \geq \gamma^k(G)+p$. Thus we have a contradiction. □

It is well known fact that for any edge $e \in \overline{G}$, $\gamma(G+e) \leq \gamma(G)$ ([5]). In general, for γ^k this is not valid.

Theorem 2.7. *Let x and y be two nonadjacent vertices in a graph G. Then $\gamma^k(G+xy) > \gamma^k(G)$ if and only if every γ^k-set of G is no k-dependent set of $G+xy$ and one of the following holds:*

(1) *x is a γ^k_p-fixed vertex of G and y is a γ^k_q-fixed vertex of G for some $p, q \geq 1$;*
(2) *$x \in \mathbf{Fi}^k_0(G)$ and $y \in \mathbf{Fi}^k_1(G) \cap \mathbf{B}^k(G-x)$;*
(3) *$x \in \mathbf{Fi}^k_1(G) \cap \mathbf{B}^k(G-y)$ and $y \in \mathbf{Fi}^k_0(G)$;*
(4) *$x, y \in \mathbf{Fi}^k_0(G)$, $x \in \mathbf{B}^k(G-y)$ and $y \in \mathbf{B}^k(G-x)$.*

Proof. Let $\gamma^k(G+xy) > \gamma^k(G)$. By Corollary 2.5, $x, y \in \mathbf{V}^k_0(G) \cup \mathbf{V}^k_+(G)$. Assume to the contrary, that (without loss of generalities) $x \notin \mathbf{Fi}^k(G)$. Hence there is a γ^k-set M of G with $x \notin M$. But then M will be a k-dependent dominating set of $G+xy$ and $|M| = \gamma^k(G) < \gamma^k(G+xy)$ - a contradiction. Thus x and y are both γ^k-fixed vertices of G. This implies that each γ^k-set M of G is a dominating set of $G+xy$ and is no k-dependent set of $G+xy$.

Let x be γ^k_p-fixed, y be γ^k_q-fixed and without loss of generalities, $q \geq p \geq 0$. Assume (1) does not hold. Hence $p = 0$. Let M_1 be a γ^k-set of $G-x$. Then $|M_1| = \gamma^k(G-x) = \gamma^k(G) < \gamma^k(G+xy)$ and we have that y is a γ^k-bad vertex of $G-x$. By Lemma 2.6, $N(x,G) \cap M_1 = \emptyset$. Then $M_1 \cup \{x\}$ is a k-dependent dominating set of $G+xy$ which implies $\gamma^k(G+xy) = \gamma^k(G)+1$. Since $y \notin M_1 \cup \{x\}$ then $M_1 \cup \{x\}$ is a k-dependent dominating set of $G-y$ and then $\gamma^k(G)+1 = |M_1 \cup \{x\}| \geq \gamma^k(G-y) = \gamma^k(G)+q$. So, $q \in \{0,1\}$. If $q = 1$ then (2) holds. If $q = 0$ then by symmetry, it follows that x is a γ^k-bad vertex of $G-y$ and hence (4) holds.

For the converse, let every γ^k-set of G be no k-dependent set of $G+xy$ and let one of the conditions (1), (2), (3) and (4) holds. Assume to the contrary, that $\gamma^k(G+xy) \leq \gamma^k(G)$. By Theorem 2.4, $\gamma^k(G+xy) = \gamma^k(G)$. Let M_2 be a γ^k-set of $G+xy$. Hence $|M_2 \cap \{x,y\}| = 1$ - otherwise M_2 will be a γ^k-set of G. Let without loss of generalities $x \notin M_2$. Then M_2 is a k-dependent dominating set of $G-x$ which implies $\gamma^k(G-x) \leq |M_2| = \gamma^k(G+xy) = \gamma^k(G)$. Since $x \in V^k_0(G) \cup V^k_+(G)$, we have $\gamma^k(G-x) = \gamma^k(G+xy) = \gamma^k(G)$ and then M_2 is a γ^k-set of $G-x$. Hence x is a γ^k_0-fixed vertex of G and y is a γ^k-good vertex of $G-x$, which is a contradiction with each of (1) — (4). □

By Theorem 2.4 and Theorem 2.7 we immediately have:

Theorem 2.8. *Let x and y be two nonadjacent vertices in a graph G. Then $\gamma^k(G+xy) = \gamma^k(G)$ if and only if at least one of the following holds:*

(1) *$x \in \mathbf{V}^k_-(G) \cap \mathbf{B}^k(G-y)$ and $y \in \mathbf{V}^k_-(G) \cap \mathbf{B}^k(G-x)$;*
(2) *$x \in \mathbf{V}^k_-(G)$ and $y \in \mathbf{B}^k(G-x) - \mathbf{V}^k_-(G)$;*
(3) *$x \in \mathbf{B}^k(G-y) - \mathbf{V}^k_-(G)$ and $y \in \mathbf{V}^k_-(G)$;*
(4) *$x, y \notin \mathbf{V}^k_-(G)$ and $|\{x,y\} \cap \mathbf{Fi}^k(G)| \leq 1$;*
(5) *$x \in \mathbf{Fi}^k_0(G)$ and $y \in \mathbf{Fi}^k_s(G) \cap \mathbf{G}^k(G-x)$ for some $s \in \{0,1\}$;*
(6) *$x \in \mathbf{Fi}^k_s(G) \cap \mathbf{G}^k(G-y)$ and $y \in \mathbf{Fi}^k_0(G)$ for some $s \in \{0,1\}$;*
(7) *$x \in \mathbf{Fi}^k_0(G)$ and $y \in \mathbf{Fi}^k_q(G)$ for some $q \geq 2$;*
(8) *$x \in \mathbf{Fi}^k_q(G)$ and $y \in \mathbf{Fi}^k_0(G)$ for some $q \geq 2$;*
(9) *there is a γ^k-set of G which is a k-dependent set of $G+xy$ and one of the (1), (2), (3) and (4) of Theorem 2.7 holds.*

Corollary 2.9. *Let x and y be two nonadjacent vertices in a graph G. If $x \in \mathbf{B}^k(G)$ then $\gamma^k(G+xy) = \gamma^k(G)$.*

Proof. If $y \notin \mathbf{V}^k_-(G)$ then the result follows by Theorem 2.8(4). If $y \in \mathbf{V}^k_-(G)$ then by Proposition 2.1, $x \in \mathbf{B}^k(G-y)$ and the result now follows by Theorem 2.8(3). □

Let $\mu \in \{\gamma, i\}$. A graph G is edge-μ-critical if $\mu(G+e) < \mu(G)$ for every edge e missing from G. These concepts were introduced by Sumner and Blitch [10] and Ao and MacGillivray [6, Chapter 16] respectively. Here we define a graph G to be *edge-γ^k-critical* if $\gamma^k(G+e) \neq \gamma^k(G)$ for every edge e of the complement of G. Relating edge addition to vertex removal, Sumner and Blitch [10] and Ao and MacGillivray showed that $\mathbf{V}^k_+(G)$ is empty for $k = \Delta(G)$ and $k = 0$ respectively. Furthermore Favaron, Sumner and Wojcicka [3] showed that if $\mathbf{V}^{\Delta(G)}_0(G) \neq \emptyset$ then $\left\langle \mathbf{V}^{\Delta(G)}_0(G), G \right\rangle$ is complete. In general, for edge-γ^k-critical graphs the following holds.

Theorem 2.10. *Let G be an edge-γ^k-critical graph. Then*

(1) *$V(G) = \mathbf{Fi}^k_{-1}(G) \cup \mathbf{Fr}^k(G)$ and if $\mathbf{Fr}^k_0(G) \neq \emptyset$ then $\left\langle \mathbf{Fr}^k_0(G), G \right\rangle$ is complete;*
(2) *$\gamma^k(G+e) < \gamma^k(G)$ for every edge e missing from G.*

Proof. (1) If $\gamma^k(G) = 1$ then obviously G is complete and the result is trivial. Assume $\gamma^k(G) \geq 2$. Let $x, y \in \mathbf{Fr}^k_0(G)$ and $xy \notin E(G)$. Then by Theorem 2.8(4) it follows $\gamma^k(G+xy) = \gamma^k(G)$ - a contradiction. By Corollary 2.9, $\mathbf{B}^k(G) = \emptyset$. Assume $x \in \mathbf{Fi}^k_q(G)$ for some $q \geq 0$. Let M be any γ^k-set of G. Hence there is $y \in pn[x, M] - \{x\}$ - otherwise $M - \{x\}$ becomes a γ^k-set of $G - x$, which implies $x \in \mathbf{V}^k_-(G)$. Since $pn[x, M] \cap \mathbf{V}^k_-(G) = \emptyset$ (by Proposition 2.1 when $q \geq 1$ and Lemma 2.6 when $q = 0$), $\mathbf{B}^k(G) = \emptyset$ and $y \notin M$, we have $y \in \mathbf{Fr}^k_0(G)$. Let M_1 be a γ^k-set of G and $y \in M_1$. Then there is $z \in (pn[x, M_1] - \{x\}) \cap \mathbf{Fr}^k_0(G)$. Hence $y, z \in \mathbf{Fr}^k_0(G)$ and $yz \notin E(G)$ - a contradiction. Thus $\mathbf{Fi}^k(G) = \mathbf{Fi}^k_{-1}(G)$ and the result follows.

(2) Immediately follows by (1) and Theorem 2.7. □

§3. OPEN PROBLEMS

• Characterize/study the following classes of graphs.
(We use acronyms as follows: C represents changing; U: unchanging; V: vertex; E: edge; R: removal; A: addition.)

$(CVR)^k$ $\quad \gamma^k(G-v) \neq \gamma^k(G)$ for all $v \in V(G)$;

$(CER)^k$ $\quad \gamma^k(G-e) \neq \gamma^k(G)$ for all $e \in E(G)$;

$(CEA)^k$ $\quad \gamma^k(G+e) \neq \gamma^k(G)$ for all $e \in E(\overline{G})$;

$(UVR)^k$ $\quad \gamma^k(G-v) = \gamma^k(G)$ for all $v \in V(G)$;

$(UER)^k$ $\quad \gamma^k(G-e) = \gamma^k(G)$ for all $e \in E(G)$;

$(CEA)^k$ $\quad \gamma^k(G+e) = \gamma^k(G)$ for all $e \in E(\overline{G})$.

Note that Chapter 5 [5] surveys the results of studies attempting to characterize the graphs G in the six classes above provided $k = \Delta(G)$. Additional facts on classes $(CEA)^{\Delta(G)}$ and $(CVR)^{\Delta(G)}$ can be found in [6, Chapter 16] and [9]. Some relationships among these six classes are established by Haynes [5, pp. 150–153] and Haynes and Henning [7].

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