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L^p-spectral independence of fractional Laplacians perturbed by potentials

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Abstract. Let $H_{\alpha,p,V}$ denote the fractional Laplacian $(-\Delta)^\alpha$ $\big(\alpha \in (0,1]\big)$ acting in $L^p(\mathbb{R}^N)$ $\big(p \in [1,\infty)\big)$ perturbed by a potential V. We prove spectral inclusion $\sigma(H_{\alpha,p,V}) \subset \sigma(H_{\alpha,q,V})$ $(1 \le q \le p \le 2$ or $2 \le p \le q < \infty)$ for a large class of potentials, and L^p-spectral independence $\sigma(H_{\alpha,p,V}) = \sigma(H_{\alpha,2,V})$ $\big(p \in [1,\infty)\big)$ under a certain condition. In addition, we prove that the spectrum of a perturbed fractional Dirichlet Laplacian acting in $L^p(O)$ is independent of $p \in [1,\infty)$ under a weak condition for potentials, where O is a bounded open subset of $\mathbb{R}^N$.

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Contents

§1. Introduction

Let O be an open subset of $\mathbb{R}^N$, and suppose that a C_0-semigroup $T_p = \big(T_p(t)\big)_{t\geq 0}$ on $L^p(O)$ with generator A_p is given for each $p \in [1,\infty)$. Assume further that T_p's are consistent in the sense that $T_p(t) = T_q(t)$ on $L^p(O) \cap L^q(O)$ for all $t \geq 0$ and $p, q \in [1,\infty)$. Under these assumptions, it is natural to expect L^p-spectral independence of the generators holds, that is to say,

$$\sigma(A_p) = \sigma(A_2) \tag{1.1}$$

for all $p \in [1,\infty)$. However, W. Arendt [1, Section 3] showed that this equality is not necessarily true. Nonetheless, L^p-spectral independence (1.1) is proved in many important cases. In fact, R. Hempel and J. Voigt [6, Theorem] proved that, for a potential V belonging to a large class including a Kato class, the spectrum of the Schrödinger operator $-\Delta/2 + V$ acting in $L^p(\mathbb{R}^N)$ is independent of $p \in [1,\infty)$ (for other references, see below in this section). Therefore we have an interest in L^p-spectral independence in the case where we replace the Laplacian in $-\Delta/2 + V$ with a fractional Laplacian, a no less fundamental operator than the Laplacian. For example, this means that we replace the heat semigroup

$$(e^{t\Delta} f)(x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

with the Poisson semigroup

$$\big(e^{-t(-\Delta)^{\frac{1}{2}}} f\big)(x) = \frac{\Gamma\big(\frac{N+1}{2}\big) t}{\pi^{\frac{N+1}{2}}} \int_{\mathbb{R}^N} \frac{f(y)}{\big(t^2 + |x-y|^2\big)^{\frac{N+1}{2}}}\, dy$$

(*cf.* [12, Example 1.10, 1.8] and [7, (3.248)]).

Let us state the aims and the main results of this paper in more detail. For that purpose, we have to make the main objects clear. In what follows, Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$. For all $\alpha \in (0,1]$, the fractional Laplacian $(-\Delta)^\alpha$ is positive definite self-adjoint and $-(-\Delta)^\alpha$ generates a C_0-semigroup on $L^2(\mathbb{R}^N)$. It is well known that the domain of $(-\Delta)^\alpha$ is the fractional order Sobolev space $H^{2\alpha}(\mathbb{R}^N)$. As will be stated in Proposition 2.2, there exists a C_0-semigroup $U_{\alpha,p} = \big(U_{\alpha,p}(t)\big)_{t\geq 0}$ on $L^p(\mathbb{R}^N)$ for each $p \in [1,\infty)$ such that $U_{\alpha,p}(t)$ and $e^{-t(-\Delta)^\alpha}$ are consistent for all $t \geq 0$. By this consistency, $U_{\alpha,p}$ is unique for each $p \in [1,\infty)$. Let $-H_{\alpha,p}$ denote the generator of $U_{\alpha,p}$. This is the definition of what was called the fractional Laplacian acting in $L^p(\mathbb{R}^N)$ in the abstract. It is also possible that the fractional Laplacian acting in $L^p(\mathbb{R}^N)$ is defined by $H^\alpha_{1,p}$. However, both of the definitions coincide (see the statement below the proof of Theorem 3.20 in [14]).

Then, we treat the formal expression: $H_{\alpha,p,V} := H_{\alpha,p} + V$. First, we have to consider whether the formal operator $H_{\alpha,p} + V$ makes sense. To that purpose, we modify the generalized Kato class defined in [18, p. 183] to make it suitable for the fractional Laplacians (Definition 2.10) and prove that for the potentials belonging to the modified Kato class, the formal expression $H_{\alpha,p} + V$ can be given a realization as a C_0-semigroup generator via the perturbation theory of Voigt [18], [19] (Theorem 2.14).

The purpose of this paper is to consider L^p-spectral independence of the operator $H_{\alpha,p,V} = H_{\alpha,p} + V$, which is given a realization described above. To prove L^p-spectral independence of $H_{\alpha,p,V}$, we prove the Feynman-Kac formula for $e^{-tH_{\alpha,p,V}}$ (Proposition 3.3) and by using the Feynman-Kac formula, we show L^p-L^q estimates for $e^{-tH_{\alpha,2,V}}$ (Proposition 3.6).

By using these estimates, we obtain the following spectral inclusion of $H_{\alpha,p,V}$ (Theorem 4.1): The relation

$$\sigma(H_{\alpha,p,V}) \subset \sigma(H_{\alpha,q,V})$$

holds for any $1 \leq q \leq p \leq 2$ or any $2 \leq p \leq q \leq \infty$, where $H_{\alpha,\infty,V} := H'_{\alpha,1,V}$ (the conjugate of $H_{\alpha,1,V}$). Moreover, in the special case of $N = 1$ and $1/2 < \alpha < 1$, we show L^p-spectral independence

$$\sigma(H_{\alpha,p,V}) = \sigma(H_{\alpha,2,V}) \quad \big(p \in [1,\infty)\big)$$

under a condition on potentials V (Theorem 4.2). On the other hand, let Δ_D and $H^D_{\alpha,p,V}$ denote the Dirichlet Laplacian in $L^2(O)$, where O is a bounded open subset of $\mathbb{R}^N$, and the fractional power $(-\Delta_D)^\alpha$ acting in $L^p(O)$ perturbed by a potential V for an $\alpha \in (0,1]$ and $p \in [1,\infty)$, respectively. A similar condition on V as in the case of $H_{\alpha,p,V}$ guarantees that $H^D_{\alpha,p,V}$ is given a realization. Under a certain condition on V without any restriction on N and α, we prove that the spectrum $\sigma(H^D_{\alpha,p,V})$ is independent of $p \in [1,\infty)$ (Theorem 4.18). Especially, our result implies that p-independence of $\sigma(H^D_{\alpha,p,V})$ holds provided that the boundary ∂O is smooth, $V_+ \in L^1_{loc}(O)$ and V_- is small enough.

We would like to refer to other references concerning L^p-spectral independence that have a close relation to this paper. We do not assume that O is bounded unless explicitly stated otherwise in the rest of this introduction. Let us recall that Hempel and Voigt [6] treated only the Schrödinger operators by a subtle argument using Feynman-Kac formula. However, Arendt [1] found that their result is closely connected to a specific property of the semigroups generated by the Schrödinger operators, and he succeeded in generalizing the result of [6] in an abstract direction. In more detail, Arendt called a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $L^2(O)$ ($O \subset \mathbb{R}^N$) satisfies an upper Gaussian estimate if there exist constants $M \geq 1, \omega \in \mathbb{R}$ and $b > 0$ such that

$$|T(t)u| \leq Me^{\omega t}e^{bt\Delta}|u| \tag{1.2}$$

for all $t \geq 0$ and $u \in L^2(O)$. In the right-hand side of this inequality, we regard $u \in L^2(O)$ as an element of $L^2(\mathbb{R}^N)$ by considering the value of u on $\mathbb{R}^N \setminus O$ to be 0. Arendt proved that if a C_0-semigroup T on $L^2(O)$ satisfies an upper Gaussian estimate, then there exists a C_0-semigroup T_p on $L^2(O)$ for each $p \in [1,\infty)$ which is consistent with T and the spectrum of the generator of T_p is independent of $p \in [1,\infty)$ provided the generator of T is self-adjoint ([1, Corollary 4.3]). (For the result in the non-self-adjoint case, see [1, Theorem 4.2].) For a large class of potentials, the Schrödinger semigroups satisfy upper Gaussian estimates (*cf.* [17, Theorem B.6.7]). Hence Arendt's result [1, Corollary 4.3] is a generalization of a considerable part of the results in [6]. However, it is not known whether all of the Schrödinger semigroups in [6] satisfies an upper Gaussian estimate. Hence the result in [6] is of independent interest.

On the other hand, after the work of Arendt, generalizations of his result were achieved by [10], [11], [13] and [14]. In the generalization process, the notion of an upper Gaussian estimate has been generalized. For example, the notion of a Gaussian estimate of order α ($\alpha \in (0,1]$) was defined in [14, Definition 3.1]. This estimate corresponds to what is obtained by replacing Δ with $-(-\Delta)^\alpha$ in (1.2). In some cases including the one where O is bounded, Miyajima and the author proved that if a C_0-semigroup T on $L^2(O)$ satisfies a Gaussian estimate of order α for some $\alpha \in (0,1]$, the same conclusion as Arendt's result above holds. However, for a similar reason as in the case of $-\Delta + V$, the problem of L^p-spectral independence of $(-\Delta)^\alpha + V$ has its own significance. This is the reason why the fractional Laplacians perturbed by potentials are particularly investigated in this paper.

Besides the references above, for L^p-spectral independence of second order differential operators, see the examples of the references above and the references therein.

This paper consists of three parts. In the next Section 2, we define the perturbed operator $-H_{\alpha,p,V}$, which is only formal at present, as the generator of a C_0-semigroup on $L^p(\mathbb{R}^N)$ for an appropriate potential V. In Section 3, we show the Feynman-Kac formula for $e^{-tH_{\alpha,p,V}}$, and then we prove L^p-L^q estimates for $e^{-tH_{\alpha,2,V}}$. In Section 4, by using this estimates, we prove L^p-spectral independence and L^p-spectral inclusion in the form as stated above.

Below, we will list function spaces and operator spaces frequently used in this paper. In this list, $p \in [1, \infty)$ and O denotes an open subset of $\mathbb{R}^N$ and X and Y designate Banach spaces.

$L^p(O)$	the usual Lebesgue space on O,
$L^\infty(O)$	the Lebesgue space of essentially bounded functions on O,
$L^p_{loc}(O)$	the usual L^p_{loc} space on O,
$C_\infty(\mathbb{R}^N)$	the space of continuous functions on $\mathbb{R}^N$ vanishing at infinity,
$C_c(O)$	the space of continuous functions with compact support in O,
$C^\infty(O)$	the space of infinite times differentiable functions on O,
$C_c^\infty(O) := C^\infty(O) \cap C_c(O)$,	
$H^s(O)$	the usual Sobolev space on O of order $s \in \mathbb{R}$,
$H_0^1(O)$	the closure of $C_c^\infty(O)$ in $H^1(O)$,
$\mathcal{S}(\mathbb{R}^N)$	the Schwartz space of rapidly decreasing functions,
$\mathcal{L}(X,Y)$	the space of bounded linear operators from X into Y,
$\mathcal{L}(X) := \mathcal{L}(X,X)$.	

In the case of $O = \mathbb{R}^N$, we may drop "$(\mathbb{R}^N)$", for example, $L^p = L^p(\mathbb{R}^N)$. In addition, we also use the following notations. $\|\cdot\|_p$ denotes $L^p(\mathbb{R}^N)$-norm for all $p \in [1, \infty]$ and $B(x, r)$ denotes the ball in $\mathbb{R}^N$ with center x and radius r. Constants "C" and "M" may vary from place to place.

§2. Perturbation of fractional Laplacians by potentials

2.1. Preliminaries

We shall discuss some basic properties of the semigroups generated by fractional Laplacians. Those will be used throughout this paper.

Definition 2.1. (i) Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$. For each $\alpha \in (0, 1]$, $(-\Delta)^\alpha$ is a positive definite self-adjoint operator in $L^2(\mathbb{R}^N)$.

(ii) For each $\alpha \in (0,1]$, $U_\alpha := \big(U_\alpha(t)\big)_{t\geq 0}$ is the C_0-semigroup on $L^2(\mathbb{R}^N)$ generated by $-(-\Delta)^\alpha$.

(iii) For each $\alpha \in (0,1]$, the function K_α is defined by

$$K_\alpha(t,x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\xi}e^{-t|\xi|^{2\alpha}}\,d\xi \quad (t>0, x\in\mathbb{R}^N).$$

($K_\alpha(t,x-y)$ is the integral kernel of $U_\alpha(t)$ $(t>0)$. See the next proposition.) As is well known, U_1 is the heat semigroup and $K_1(t,x)$ is the Gauss kernel (i.e. $K_1(t,x) = (4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$ for all $t>0, x\in\mathbb{R}^N$).

In the next proposition, we collect some properties of U_α and K_α, and state a relation between U_α and K_α. Moreover, it is proved from these properties that K_α defines a C_0-semigroup on $L^p(\mathbb{R}^N)$ for each $p\in[1,\infty)$.

Proposition 2.2. *For each $\alpha\in(0,1]$, the following assertions hold.*

(i) (a) *For all $t\geq 0$, $U_\alpha(t)$ is positive, i.e., $U_\alpha(t)u\geq 0$ for all positive $u\in L^2(\mathbb{R}^N)$.*

(b) *For each $t>0$,*

$$U_\alpha(t)u = K_\alpha(t,\cdot)*u$$

for all $u\in L^2(\mathbb{R}^N)$, where $f*g$ is the convolution of f and g.

(ii) (a) *$K_\alpha(t,x) = t^{-\frac{N}{2\alpha}}K_\alpha(1, t^{-\frac{1}{2\alpha}}x)$ for all $t>0$ and $x\in\mathbb{R}^N$.*

(b) *The function $(t,x)\mapsto K_\alpha(t,x)$ is continuous on $(0,\infty)\times\mathbb{R}^N$.*

(c) *There exists a constant $C_\alpha>0$ such that*

$$0\leq K_\alpha(t,x)\leq C_\alpha\frac{t}{\big(t^{\frac{1}{\alpha}}+|x|^2\big)^{\frac{N}{2}+\alpha}}$$

for all $t>0$ and $x\in\mathbb{R}^N$ (see also the estimate in Proposition 2.3 *below).*

(d) *For all $t>0$, $K_\alpha(t,\cdot)\in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} K_\alpha(t,x)\,dx = 1.$$

(e) *For each $t>0$ and $u\in C_\infty(\mathbb{R}^N)$, $K_\alpha(t,\cdot)*u$ belongs to $C_\infty(\mathbb{R}^N)$, and also $K_\alpha(t,\cdot)*u$ converges to u as $t\downarrow 0$ in $C_\infty(\mathbb{R}^N)$, i.e.,*

$$\int_{\mathbb{R}^N} K_\alpha(t,x-y)u(y)\,dy - u(x)\to 0$$

as $t\downarrow 0$ uniformly in $x\in\mathbb{R}^N$.

(iii) *For each* $p \in [1,\infty)$ *and* $t > 0$, *a bounded linear operator* $U_{\alpha,p}(t)$ *is defined on* $L^p(\mathbb{R}^N)$ *by the following formula:*

$$\big(U_{\alpha,p}(t)u\big)(x) := \big(K_\alpha(t,\cdot) * u\big)(x) \quad (u \in L^p(\mathbb{R}^N), x \in \mathbb{R}^N).$$

Then, $U_{\alpha,p} := \big(U_{\alpha,p}(t)\big)_{t\geq 0}$ *is a positive* C_0*-semigroup of contractions on* $L^p(\mathbb{R}^N)$ *for all* $p \in [1,\infty)$. *In addition,* $U_{\alpha,p}$ *and* $U_{\alpha,q}$ *are consistent for all* $p, q \in [1,\infty)$ *(i.e.* $U_{\alpha,p}(t) = U_{\alpha,q}(t)$ *on* $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *for all* $t \geq 0$ *and* $p, q \in [1,\infty)$*), and* $U_{\alpha,2}(t)u = U_\alpha(t)u$ *for all* $t > 0$ *and* $u \in L^2(\mathbb{R}^N)$.

$-H_{\alpha,p}$ will denote the generator of $U_{\alpha,p}$ for each $\alpha \in (0,1]$ and $p \in [1,\infty)$. By (iii) of this proposition, we may identify $\big(U_\alpha(t)u\big)(x)$ with $\big(K_\alpha(t,\cdot) * u\big)(x)$ for all $t > 0, x \in \mathbb{R}^N$ and $u \in L^2(\mathbb{R}^N)$. Under this convention, for all $u \in L^2(\mathbb{R}^N)$, the function $(t,x) \mapsto \big(U_\alpha(t)u\big)(x)$ is measurable on $(0,\infty) \times \mathbb{R}^N$.

(ii)-(e) of this proposition shows that the kernel $K_\alpha(t,x)$ generates a so-called Feller semigroup on $C_\infty(\mathbb{R}^N)$, and this fact will play an important role in Section 3. So we give a direct proof by using (ii)-(d) although (ii)-(e) is proved in [8, Example 4.1.3].

Proof. (i) The assertions (a) and (b) are proved in [14, Proposition 3.3].

(ii) (a) is verified by using the change of variables $t^{\frac{1}{2\alpha}}\xi = \xi'$ in the definition of $K_\alpha(t,x)$.

(b) is an easy consequence of Lebesgue's convergence theorem.

(c) and (d) are proved in [14] as Corollary 3.4 and Proposition 3.3 (see (3.3)), respectively.

(e) Let u be an arbitrary function in $C_\infty(\mathbb{R}^N)$. From the estimate

$$|K_\alpha(t,y)u(x-y)| \leq K_\alpha(t,y)\|u\|_\infty \quad (t > 0, x, y \in \mathbb{R}^N)$$

and Lebesgue's convergence theorem, it follows that for each $t > 0$ and $x_0 \in \mathbb{R}^N$,

$$\begin{aligned}\big(K_\alpha(t,\cdot) * u\big)(x) &= \int_{\mathbb{R}^N} K_\alpha(t,y)u(x-y)\,dy \\ &\to \int_{\mathbb{R}^N} K_\alpha(t,y)u(x_0-y)\,dy = \big(K_\alpha(t,\cdot) * u\big)(x_0)\end{aligned}$$

as $x \to x_0$. Hence $K_\alpha(t,\cdot) * u$ is a continuous function on $\mathbb{R}^N$ for each $t > 0$.

Next, for any $t > 0, \varepsilon > 0$ and $u \in C_\infty(\mathbb{R}^N)$, we can take an $R > 0$ such that

$$0 \leq \int_{|x|>R} K_\alpha(t,x)\,dx < \varepsilon$$

by (ii)-(d) and $|u(x)| < \varepsilon$ for $|x| \geq R$. Hence, if $|x| \geq 2R$, we have

$$\begin{aligned}
\big|\big(K_\alpha(t,\cdot) * u\big)(x)\big| &\leq \int_{x\in\mathbb{R}^N} K_\alpha(t,x-y)|u(y)|\,dy \\
&= \Big(\int_{|y|\geq R} + \int_{|y|<R}\Big) K_\alpha(t,x-y)|u(y)|\,dy \\
&< \varepsilon \int_{|y|\geq R} K_\alpha(t,x-y)\,dy + \|u\|_\infty \int_{|x-y|>R} K_\alpha(t,x-y)\,dy \\
&< \varepsilon \int_{\mathbb{R}^N} K_\alpha(t,x-y)\,dy + \|u\|_\infty \int_{|y|>R} K_\alpha(t,y)\,dy \\
&< \varepsilon + \|u\|_\infty \varepsilon = (1+\|u\|_\infty)\varepsilon.
\end{aligned}$$

Thus, $K_\alpha(t,\cdot) * u \in C_\infty(\mathbb{R}^N)$ for all $t > 0$ and $u \in C_\infty(\mathbb{R}^N)$.

Let $\varepsilon > 0$ and $u \in C_\infty(\mathbb{R}^N)$. Then, there exists a $\delta > 0$ such that if $|x-y| < \delta$, then $|u(x) - u(y)| < \varepsilon$. Hence, for any $t > 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
\big|\big(K_\alpha(t,\cdot) * u\big)(x) - u(x)\big| &= \Big|\int_{\mathbb{R}^N} K_\alpha(t,x-y)u(y)\,dy - u(x)\Big| \\
&= \Big|\int_{\mathbb{R}^N} K_\alpha(t,x-y)\big(u(y)-u(x)\big)\,dy\Big| \quad \big(\text{by (ii)-(d)}\big) \\
&\leq \Big(\int_{|x-y|<\delta} + \int_{|x-y|\geq\delta}\Big) K_\alpha(t,x-y)|u(y)-u(x)|\,dy \\
&< \varepsilon \int_{|x-y|<\delta} K_\alpha(t,x-y)\,dy \\
&\qquad + \int_{|x-y|\geq\delta} K_\alpha(t,x-y)\,dy \cdot 2\|u\|_\infty \\
&\leq \varepsilon + 2\|u\|_\infty \int_{|y|\geq t^{-\frac{1}{2\alpha}}\delta} K_\alpha(1,y)\,dy.
\end{aligned}$$

For the last inequality, we used (ii)-(a) and an elementary change of variables. By this inequality and $K_\alpha(1,\cdot) \in L^1(\mathbb{R}^N)$,

$$\limsup_{t\downarrow 0} \|K_\alpha(t,\cdot) * u - u\|_\infty \leq \varepsilon.$$

Thus, for all $u \in C_\infty(\mathbb{R}^N)$, $K_\alpha(t,\cdot) * u$ converges to u as $t \downarrow 0$ in $C_\infty(\mathbb{R}^N)$.

(iii) is proved in [14, Proposition 3.3]. $\square$

We can prove also a lower estimate for the kernel K_α. This estimate will be used in the proof of Lemma 2.17.

Proposition 2.3. *Let $\alpha \in (0,1)$. There exist constants $C_\alpha, C'_\alpha > 0$ such that*

$$C'_\alpha \frac{t}{\left(t^{\frac{1}{\alpha}} + |x|^2\right)^{\frac{N}{2}+\alpha}} \le K_\alpha(t,x) \le C_\alpha \frac{t}{\left(t^{\frac{1}{\alpha}} + |x|^2\right)^{\frac{N}{2}+\alpha}} \tag{2.1}$$

for all $t > 0$ and $x \in \mathbb{R}^N$.

For the proof of this proposition, we introduce the function $f_{t,\alpha}$ defined in [20, Chapter IX, Section 11]: For each $\alpha \in (0,1)$ and $t > 0$,

$$f_{t,\alpha}(\lambda) := \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\sigma - i\infty}^{\sigma + i\infty} e^{z\lambda - tz^\alpha}\, dz & (\lambda \ge 0, \sigma > 0), \\ 0 & (\lambda < 0), \end{cases}$$

where the branch of z^α is so taken that $\operatorname{Re} z^\alpha > 0$ for $\operatorname{Re} z > 0$. ($f_{t,\alpha}$ is independent of $\sigma > 0$.) Lemma 2.4 below shows that $f_{t,\alpha}$ is a density function on $\mathbb{R}$ for all $\alpha \in (0,1)$ and $t > 0$, and also Proposition 3 in [20, Chapter IX, Section 11] implies that $f_{t,\alpha}$ defines a convolution semigroup on $\mathbb{R}$. According to [20, Chapter IX, Section 11 Theorem 2], for each $\alpha \in (0,1)$ and $t > 0$, $U_\alpha(t)$ is represented by $f_{t,\alpha}$ and U_1 as follows:

$$U_\alpha(t) = \int_0^\infty f_{t,\alpha}(s) U_1(s)\, ds, \tag{2.2}$$

i.e., U_α is subordinate to U_1. This representation yields that of $K_\alpha(t,x)$ by $K_1(t,x)$ for all $t > 0$ and $x \in \mathbb{R}^N$:

$$K_\alpha(t,x) = \int_0^\infty f_{t,\alpha}(s) K_1(s,x)\, ds,$$

which will be given a detailed proof and used in the proof of Proposition 2.3. To verify this representation, we prove the next lemma concerning properties of the function $f_{t,\alpha}$. Although these properties must be known, we state it here with a proof, since we could not find an appropriate literature.

Lemma 2.4. *Let $0 < \alpha < 1$ and $t > 0$. The function $f_{t,\alpha}$ above satisfies the following:*

(i) $f_{t,\alpha} \ge 0$,
(ii) $f_{t,\alpha} \in C^\infty(\mathbb{R})$,
(iii) $f_{t,\alpha} \in L^1(\mathbb{R})$ *and* $\|f_{t,\alpha}\|_{L^1(\mathbb{R})} = 1$,
(iv) *For all* $j \in \mathbb{N} \cup \{0\}$, $f_{t,\alpha}(\lambda) = o(\lambda^j)$ *as* $\lambda \to 0$.

Proof. (i) is proved in [20, Chapter IX, Section 11 Proposition 2].

(ii) It is clear that $f_{t,\alpha} \in C^\infty\big((-\infty,0)\big)$. To prove that $f_{t,\alpha} \in C^\infty\big((0,\infty)\big)$, take any $\lambda_0 > 0$ and note the following estimate: For all $z = \sigma + i\eta$ ($\sigma > 0, \eta \in \mathbb{R}$),

$$\begin{aligned} \text{(2.3)} \qquad \Big|\frac{d^j}{d\lambda^j} e^{z\lambda - tz^\alpha}\Big| &= |z^j e^{z\lambda - tz^\alpha}| \\ &\le e^{\sigma(\lambda_0+1)}(\sigma^2+\eta^2)^{\frac{j}{2}} e^{-t|\eta|^\alpha \cos\frac{\pi\alpha}{2}} \end{aligned}$$

for all $\lambda \in (0, \lambda_0+1)$ and $j \in \mathbb{N}\cup\{0\}$. The rightmost function of (2.3) with respect to η is independent of $\lambda \in (0,\lambda_0+1)$ and is integrable on $\mathbb{R}$. Hence, by Lebesgue's convergence theorem, $\int_{\sigma-i\infty}^{\sigma+i\infty} e^{z\lambda - tz^\alpha}\,dz$ is infinitely differentiable with respect to λ under the integral sign, with

$$f_{t,\alpha}^{(j)}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} z^j e^{z\lambda - tz^\alpha}\,dz$$

for all $\lambda > 0$ and $j \in \mathbb{N}\cup\{0\}$.

Next we prove that

$$\text{(2.4)} \qquad \lim_{\lambda\downarrow 0} f_{t,\alpha}^{(j)}(\lambda) = 0$$

for all $j \in \mathbb{N}\cup\{0\}$ and accordingly $f_{t,\alpha}$ is continuous at the origin. By the estimate (2.3) and Lebesgue's convergence theorem again,

$$\text{(2.5)} \qquad \lim_{\lambda\downarrow 0} f_{t,\alpha}^{(j)}(\lambda) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} z^j e^{-tz^\alpha}\,dz$$

for all $j \in \mathbb{N}\cup\{0\}$. Now, we can prove that the right-hand side of this equality is 0 for all $j \in \mathbb{N}\cup\{0\}$. To that purpose, we fix an arbitrary $\sigma > 0$ and define $C_R := \big\{z \in \mathbb{C}\big| |z| = R, -\theta_R \le \arg z \le \theta_R\big\}$ for all $R > \sigma$, where $\theta_R := \tan^{-1}\frac{\sqrt{R^2-\sigma^2}}{\sigma}$. Since the function $z \mapsto z^j e^{-tz^\alpha}$ is holomorphic on $\{z \in \mathbb{C} | \operatorname{Re} z > 0\}$,

$$\begin{aligned} \Big|\int_{\sigma-i\sqrt{R^2-\sigma^2}}^{\sigma+i\sqrt{R^2-\sigma^2}} z^j e^{-tz^\alpha}\,dz\Big| &= \Big|\int_{C_R} z^j e^{-tz^\alpha}\,dz\Big| \\ &\le \int_{-\theta_R}^{\theta_R} R^j e^{-tR^\alpha\cos\frac{\pi\alpha}{2}} \cdot R\,d\phi \\ &\le \pi R^{j+1} e^{-tR^\alpha \cos\frac{\pi\alpha}{2}} \end{aligned}$$

for all $R > \sigma$. Hence

$$\begin{aligned} \text{(2.6)} \qquad \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} z^j e^{-tz^\alpha}\,dz &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-i\sqrt{R^2-\sigma^2}}^{\sigma+i\sqrt{R^2-\sigma^2}} z^j e^{-tz^\alpha}\,dz \\ &= 0 \end{aligned}$$

for all $j \in \mathbb{N} \cup \{0\}$. By this equality and (2.5), we have (2.4). The equality (2.6) for $j = 0$ means $f_{t,\alpha}(0) = 0$. Since $f_{t,\alpha}$ is right continuous at the origin by (2.4) for $j = 0$ and is left continuous there by definition, $f_{t,\alpha}$ is continuous at the origin.

Now, it follows from (2.5) and (2.6) that $f_{t,\alpha}^{(j)}$ is continuous on $\mathbb{R}$ for all $t > 0$ and $j \in \mathbb{N} \cup \{0\}$ since $f_{t,\alpha} = 0$ on $(-\infty, 0]$.

(iii) is proved in [20, Chapter IX, Section 11 Proposition 3].

(iv) is an easy consequence of the fact that $f_{t,\alpha} \in C^\infty(\mathbb{R})$ and $f_{t,\alpha}^{(j)}(0) = 0$ for all $j \in \mathbb{N} \cup \{0\}$ and the mean value theorem. □

Proof of Proposition 2.3. **Step 1**. We prove the following representation of K_α by K_1 for all $t > 0$ and $x \in \mathbb{R}^N$:

$$K_\alpha(t,x) = \int_0^\infty f_{t,\alpha}(s) K_1(s,x)\,ds. \tag{2.7}$$

For this purpose, we need verify that for all $t > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$, the function $(s,x,y) \mapsto f_{t,\alpha}(s)K_1(s,x-y)u(y)$ is integrable on $(0,\infty) \times E \times \mathbb{R}^N$, where E is an arbitrary bounded measurable subset of $\mathbb{R}^N$. This integrability follows from the estimate

$$|f_{t,\alpha}(s)K_1(s,x-y)u(y)| \le \frac{1}{(4\pi s)^{\frac{N}{2}}} f_{t,\alpha}(s)|u(y)|$$

for all $(s,x,y) \in (0,\infty) \times E \times \mathbb{R}^N$ and the fact that the function $s \mapsto s^{-\frac{N}{2}} f_{t,\alpha}(s)$ is integrable on $(0,\infty)$ by Lemma 2.4 (iii) and (iv). Hence, for a.e. $(s,x) \in (0,\infty) \times E$, the function $y \mapsto f_{t,\alpha}(s)K_1(s,x-y)u(y)$ is integrable on $\mathbb{R}^N$, and the function $(s,x) \mapsto \int_{\mathbb{R}^N} f_{t,\alpha}(s)K_1(s,x-y)u(y)\,dy = f_{t,\alpha}(s)\big(U_1(s)u\big)(x)$ is integrable on $(0,\infty) \times E$. In addition, for a.e. $s \in (0,\infty)$, the function $x \mapsto f_{t,\alpha}(s)\big(U_1(s)u\big)(x)$ is integrable on E, and the function $s \mapsto \int_E f_{t,\alpha}(s)\big(U_1(s)u\big)(x)\,dx$ is integrable on $(0,\infty)$. Hence, for all $t > 0, u \in C_c^\infty(\mathbb{R}^N)$ and bounded measurable $E \subset \mathbb{R}^N$,

$$\begin{aligned}
&\int_E \Big(\int_{\mathbb{R}^N} K_\alpha(t,x-y)u(y)\,dy\Big)\,dx \\
&\quad = \int_E \big(U_\alpha(t)u\big)(x)\,dx \\
&\quad = \int_E \Big(\int_0^\infty f_{t,\alpha}(s)U_1(s)u\,ds\Big)(x)\,dx \quad \text{(by (2.2))} \\
&\quad = \int_E \Big(\int_0^\infty f_{t,\alpha}(s)\big(U_1(s)u\big)(x)\,ds\Big)\,dx
\end{aligned}$$

$$= \int_E \Big(\int_0^\infty f_{t,\alpha}(s) \Big(\int_{\mathbb{R}^N} K_1(s, x-y) u(y)\, dy \Big)\, ds \Big)\, dx$$
$$= \int_E \Big(\int_{\mathbb{R}^N} \Big(\int_0^\infty f_{t,\alpha}(s) K_1(s, x-y)\, ds \Big) u(y)\, dy \Big)\, dx$$

by Fubini's theorem. Since E is arbitrary, we have for all $t > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} K_\alpha(t, x-y) u(y)\, dy = \int_{\mathbb{R}^N} \Big(\int_0^\infty f_{t,\alpha}(s) K_1(s, x-y)\, ds \Big) u(y)\, dy \tag{2.8}$$

for a.e. $x \in \mathbb{R}^N$. Since both sides are continuous with respect to x on $\mathbb{R}^N$, (2.8) holds for all $x \in \mathbb{R}^N$. In fact, the continuity of the left-hand side is proved by using the estimate

$$|K_\alpha(t, x-y) u(y)| \le t^{-\frac{N}{2\alpha}} \|K_\alpha(1, \cdot)\|_\infty |u(y)|$$

for all $t > 0$ and $x, y \in \mathbb{R}^N$ and Lebesgue's convergence theorem. Next we prove that the right-hand side of (2.8) is continuous with respect to x on $\mathbb{R}^N$. Since the estimate

$$|f_{t,\alpha}(s) K_1(s, x-y)| \le \frac{1}{(4\pi s)^{\frac{N}{2}}} f_{t,\alpha}(s)$$

holds for all $s > 0$ and $x, y \in \mathbb{R}^N$, the function $(x, y) \mapsto \int_0^\infty f_{t,\alpha}(s) K_1(s, x-y)\, ds$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ by Lemma 2.4 (iii) and (iv) and Lebesgue's convergence theorem, with

$$\begin{aligned} 0 &\le \int_0^\infty f_{t,\alpha}(s) K_1(s, x-y)\, ds \\ &\le \frac{1}{(4\pi)^{\frac{N}{2}}} \int_0^\infty s^{-\frac{N}{2}} f_{t,\alpha}(s)\, ds < \infty \end{aligned} \tag{2.9}$$

for all $x, y \in \mathbb{R}^N$. By this continuity and the estimate (2.9) and Lebesgue's convergence theorem, the desired continuity is proved. The fundamental lemma of calculus of variations yields

$$K_\alpha(t, x-y) = \int_0^\infty f_{t,\alpha}(s) K_1(s, x-y)\, ds$$

for all $t > 0$, $x \in \mathbb{R}^N$ and a.e. $y \in \mathbb{R}^N$. Since both of these functions are continuous with respect to y on $\mathbb{R}^N$, this equality holds for all $y \in \mathbb{R}^N$. Thus,

$$K_\alpha(t, x) = \int_0^\infty f_{t,\alpha}(s) K_1(s, x)\, ds$$

for all $t > 0$ and $x \in \mathbb{R}^N$.

Step 2. Since $K_1(s,x) \geq K_1(s,y)$ for $s > 0$ and $|x| \leq |y|$, by the representation above,

$$K_\alpha(t,x) \geq K_\alpha(t,y) \tag{2.10}$$

for $t > 0$ and $|x| \leq |y|$. The asymptotic expansion formula [14, Proposition 2.1] implies that there exist constants $R, C_{\alpha,1}, C_{\alpha,2} > 0$ such that

$$C_{\alpha,1}\frac{1}{|x|^{N+2\alpha}} \leq K_\alpha(1,x) \leq C_{\alpha,2}\frac{1}{|x|^{N+2\alpha}} \tag{2.11}$$

for all $|x| \geq R$. For this $R > 0$, it follows from (2.10) and this estimate that

$$K_\alpha(1,0) \geq K_\alpha(1,x) \geq K_\alpha(1,R\boldsymbol{e}) > 0$$

for all $|x| \leq R$, where $\boldsymbol{e} \in \mathbb{R}^N$ and $|\boldsymbol{e}| = 1$ (note that $K_\alpha(1,R\boldsymbol{e})$ is independent of such $\boldsymbol{e}$'s). By this inequality and (2.11), there exist constants $C'_{\alpha,1}, C'_{\alpha,2} > 0$ such that

$$C'_{\alpha,1}\frac{1}{(1+|x|^2)^{\frac{N}{2}+\alpha}} \leq K_\alpha(1,x) \leq C'_{\alpha,2}\frac{1}{(1+|x|^2)^{\frac{N}{2}+\alpha}}$$

for all $x \in \mathbb{R}^N$. Since $K_\alpha(t,x) = t^{-\frac{N}{2\alpha}}K_\alpha(1, t^{-\frac{1}{2\alpha}}x)$ for all $t > 0$ and $x \in \mathbb{R}^N$,

$$C'_{\alpha,1}\frac{t}{(t^{\frac{1}{\alpha}}+|x|^2)^{\frac{N}{2}+\alpha}} \leq K_\alpha(t,x) \leq C'_{\alpha,2}\frac{t}{(t^{\frac{1}{\alpha}}+|x|^2)^{\frac{N}{2}+\alpha}}$$

for all $t > 0$ and $x \in \mathbb{R}^N$. $\square$

2.2. Perturbation of fractional Laplacians by potentials

Before proving the main theorems on L^p-spectral independence and L^p-spectral inclusion, we have to consider whether the formal operator $H_{\alpha,p} + V$ makes sense for each $\alpha \in (0,1]$ and $p \in [1,\infty)$, where $V\colon \mathbb{R}^N \to \mathbb{R}$ is a measurable function. We use the same symbol for the function V and also for the associated maximal multiplication operator in L^p defined by V. For convenience, we will consider such a problem in a more general situation. Let O be an open subset of $\mathbb{R}^N$ and $U = \big(U(t)\big)_{t\geq 0}$ a positive C_0-semigroup on $L^p(O)$ with generator T for a $p \in [1,\infty)$. If $V\colon O \to \mathbb{R}$ is a bounded measurable function, then the operator sum $T - V$ generates a C_0-semigroup on $L^p(O)$. In the case where V is unbounded, as was stated in [18], it is reasonable to consider only the potential V satisfying the following conditions: The strong limit

$$U_V(t) := \text{s-}\lim_{n\to\infty} \exp\big(t(T - V^{(n)})\big) \quad (t \geq 0) \tag{2.12}$$

exists and $U_V := \big(U_V(t)\big)_{t\ge 0}$ is a (positive) C_0-semigroup on $L^p(O)$, where $V^{(n)} := (\operatorname{sign} V)\big(|V| \wedge n\big)$ $(n \in \mathbb{N})$. Now, following [18, Definition 2.2, 2.5] with [19, Theorem 2.6] taken into account, and define the notion of "semigroup admissibility" as follows:

Definition 2.5. Let O be an open subset of $\mathbb{R}^N$ and $V\colon O \to \mathbb{R}$ a measurable function, and suppose that $U = \big(U(t)\big)_{t\ge 0}$ is a positive C_0-semigroup on $L^p(O)$ with generator T for a $p \in [1, \infty)$.

(i) If V is bounded below, V will be called U-admissible if the U_V above is a (positive) C_0-semigroup on $L^p(O)$ (see also the following Remark (ii)).

(ii) If V is bounded above, V will be called U-admissible if $U_V(t)$ of (2.12) exists for all $t \ge 0$ and U_V is a (positive) C_0-semigroup on $L^p(O)$.

(iii) In general, V will be called U-admissible if both V_+ and $-V_-$ are U-admissible, where V_+ [resp. V_-] is the positive [resp. negative] part of V: $V_+ := V \vee 0$ [resp. $V_- := (-V) \vee 0$]. In this case, since the U_V above is a C_0-semigroup on $L^p(O)$ (see the following Remark (iii)), we may write the generator of U_V as T_V.

Remark 2.6. (i) If $V \in L^\infty(O)$, V is U-admissible and $T_V = T - V$.

(ii) In Definition (i), the dominated convergence theorem yields the existence of $U_V(t)$ for all $t \ge 0$ ([18, Remark 2.1 (c)]).

(iii) If V is U-admissible, then the $U_V(t)$ above exists and V_+ [resp. $-V_-$] is U_{-V_-}-admissible [resp. U_{V_+}-admissible]. Moreover, the equality

$$U_V(t) = \big(U_{-V_-}\big)_{V_+}(t) = \big(U_{V_+}\big)_{-V_-}(t)$$

holds for all $t \ge 0$. Hence, U_V is a positive C_0-semigroup on $L^p(O)$. For details, see [18, p. 174] and [19, Theorem 2.6].

In this paper, we have to consider the situation where a C_0-semigroup U_p on $L^p(O)$ is given for each $p \in [1, \infty)$ (e.g., $U_p(t) = e^{t\Delta}$ on $L^p(\mathbb{R}^N)$). Therefore, we introduce the following definition to firmly set the starting point.

Definition 2.7. Let O be an open subset of $\mathbb{R}^N$ and suppose that a positive C_0-semigroup $U_p = \big(U_p(t)\big)_{t\ge 0}$ on $L^p(O)$ is given for each $p \in [1, \infty)$. We say that the family $\{U_p; p \in [1, \infty)\}$ is self-adjoint and consistent if the following conditions are satisfied:

(i) $U_2(t)$ is self-adjoint for all $t \ge 0$,

(ii) $U_p(t)$ and $U_q(t)$ are consistent for all $p, q \in [1, \infty)$, i.e., $U_p(t) = U_q(t)$ on $L^p \cap L^q$.

Remark 2.8. If $\{U_p; p \in [1, \infty)\}$ is self-adjoint and consistent, then it can be easily verified that $U_p(t)' = U_{p'}(t)$ for all $t \ge 0$ and $p \in (1, 2) \cup (2, \infty)$, where $U_p(t)'$ is the conjugate of $U_p(t)$ and p' is the conjugate exponent of p.

Under this definition, we can prove the following proposition (*cf.* [18, Proposition 3.2]).

Proposition 2.9. *Let U_p be a positive C_0-semigroup on $L^p(O)$ with generator T_p for each $p \in [1,\infty)$ and $\{U_p; p \in [1,\infty)\}$ self-adjoint and consistent. Moreover, let $V\colon O \to \mathbb{R}$ be a measurable function. Assume that $-V_-$ is U_1-admissible and V_+ is U_{p_0}-admissible for some $p_0 \in [1,\infty)$, then V is U_p-admissible for all $p \in [1,\infty)$. Moreover, $U_{p,V}$ is a positive C_0-semigroup on $L^p(O)$ for all $p \in [1,\infty)$ and $\{U_{p,V}; p \in [1,\infty)\}$ is self-adjoint and consistent.*

Proof. First, note that if $V \in L^\infty(O)$, then V is U_p-admissible for all $p \in [1,\infty)$ and the Trotter product formula implies the last assertion of this proposition.

Next, if $V \geq 0$ (i.e. $V = V_+$), then by the assumption for V_+ and [18, Remark 2.1 (c)], the operator $U_{p,V}(t)$ exists as a bounded operator on $L^p(O)$ for all $t \geq 0$ and $p \in [1,\infty)$, and $0 \leq U_{p,V}(t) \leq U_p(t)$ holds for all $t \geq 0$ and $p \in [1,\infty)$, i.e., $u \leq U_{p,V}(t)u \leq U_p(t)u$ for all positive $u \in L^p$ and $t \geq 0, p \in [1,\infty)$. Since $U_{p_0,V}$ is a C_0-semigroup on $L^{p_0}(O)$ and consistent with $U_{p,V}$ for all $p \in [1,\infty)$, by a similar argument as in [1, p. 1160], $U_{p,V}$ is proved to be a C_0-semigroup on $L^p(O)$ for all $p \in [1,\infty)$. Since $U_{p,V}(t)$ is the strong limit of $U_{p,V^{(n)}}(t)$ as $n \to \infty$ for all $t \geq 0$ and $p \in [1,\infty)$, the last assertion of this proposition is shown by the result in the case of $V \in L^\infty(O)$.

If $V \leq 0$ (i.e. $V = -V_-$), then by the assumption for V_- and [19, Proposition 2.2],

$$\sup\Big\{\|e^{t(T_1-V^{(n)})}\| \;\Big|\; 0 \leq t \leq 1, n \in \mathbb{N}\Big\} < \infty. \tag{2.13}$$

Since $U_{p,V^{(n)}}(t)$ is the interpolating operator between $U_{1,V^{(n)}}(t)$ and $U_{1,V^{(n)}}(t)'$ for all $t \geq 0, p \in [1,\infty)$ and $n \in \mathbb{N}$, (2.13) and its dual imply by Riesz-Thorin convexity theorem that

$$\sup\Big\{\|e^{t(T_p-V^{(n)})}\| \;\Big|\; 0 \leq t \leq 1, n \in \mathbb{N}\Big\} < \infty$$

for all $p \in [1,\infty)$. By [19, Proposition 2.2] again, V is U_p-admissible for all $p \in [1,\infty)$. The remainder of the proof is as in the case of $V \geq 0$.

For an arbitrary V, the results above state that both V_+ and $-V_-$ are U_p-admissible for all $p \in [1,\infty)$. Hence V is U_p-admissible for all $p \in [1,\infty)$. To conclude the proof of this proposition, note that U_{p,V_+} is a positive C_0-semigroup on $L^p(O)$ for all $p \in [1,\infty)$ and the family $\{U_{p,V_+}; p \in [1,\infty)\}$ is self-adjoint and consistent. By applying the result in the case of $V \in L^\infty(O)$ to $(U_{p,V_+})_{-V_-^{(n)}}(t)$ and using the fact stated in Remark 2.6 (iii): $U_{p,V}(t) = (U_{p,V_+})_{-V_-}(t) = \text{s-}\lim_{n\to\infty}(U_{p,V_+})_{-V_-^{(n)}}(t)$ for all $t \geq 0$ and $p \in [1,\infty)$, the last assertion of this proposition is proved. $\square$

From now on, we will treat the C_0-semigroup $U_\alpha(t) = e^{-t(-\Delta)^\alpha}$ and give a sufficient condition for a potential V to be $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$. Since the case of $\alpha = 1$ is treated in [18, Section 5, 6], we assume $\alpha \in (0,1)$ in what follows. To state the condition for V in the case of $\alpha \in (0,1)$, we modify the generalized Kato class defined in [18, p. 183] to be suitable for the fractional Laplacians.

Definition 2.10. Let $\alpha \in (0,1)$.

(i) The function $g_{N,\alpha} \colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is defined as follows:

$$g_{N,\alpha}(x) := \begin{cases} |x| & \left(\frac{N}{2} < \alpha \text{ i.e. } N = 1, \frac{1}{2} < \alpha < 1\right), \\ \dfrac{1}{\pi} \log|x| & \left(\frac{N}{2} = \alpha \text{ i.e. } N = 1, \alpha = \frac{1}{2}\right), \\ \dfrac{1}{4^\alpha \pi^{\frac{N}{2}}} \cdot \dfrac{\Gamma\left(\frac{N}{2} - \alpha\right)}{\Gamma(\alpha)} |x|^{-N+2\alpha} & \left(\frac{N}{2} > \alpha\right). \end{cases}$$

(ii) The function space $\hat{K}_{N,\alpha}$ is defined as follows:

$$\hat{K}_{N,\alpha} := \left\{ V \in L^1_{loc}(\mathbb{R}^N) \;\middle|\; \|V\|_{\hat{K}_{N,\alpha}} < \infty \right\}, \text{where}$$
$$\|V\|_{\hat{K}_{N,\alpha}} := \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{|x-y|<1} |g_{N,\alpha}(x-y)||V(y)|\, dy.$$

(iii) For all $V \in \hat{K}_{N,\alpha}$, the quantity $c_{N,\alpha}(V)$ is defined by

$$c_{N,\alpha}(V) := \lim_{\rho \downarrow 0} \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{|x-y|<\rho} |g_{N,\alpha}(x-y)||V(y)|\, dy.$$

Remark 2.11. (i) In the case of $\frac{N}{2} > \alpha$, the function $g_{N,\alpha}$ is the Riesz kernel of order 2α.

(ii) $\hat{K}_{N,\alpha}$ is a Banach space with norm $\|\cdot\|_{\hat{K}_{N,\alpha}}$.

(iii) In Definition (ii), we may replace the integral region by $|x - y| < \delta$ for any $\delta > 0$, since there exist constants $C_1, C_2 > 0$ such that

$$C_1 \big\| |g_{N,\alpha,\delta}| * |V| \big\|_\infty \le \big\| |g_{N,\alpha,1}| * |V| \big\|_\infty \le C_2 \big\| |g_{N,\alpha,\delta}| * |V| \big\|_\infty,$$

where $g_{N,\alpha,\eta} := g_{N,\alpha}\chi_{B(0,\eta)}$ for any $\eta > 0$.

(iv) For all $V \in \hat{K}_{N,\alpha}$, the quantity $c_{N,\alpha}(V)$ is clearly finite.

(v) $g_{N,\alpha}$ and $\hat{K}_{N,\alpha}$ have the following relation to g_N and $\hat{K}_N$ defined in [18, p. 183]: For all $\alpha \in (0,1)$, $|g_N(x)| \le |g_{N,\alpha}(x)|$ holds if $|x| > 0$ small enough, and hence $\hat{K}_{N,\alpha} \subset \hat{K}_N$ for all $\alpha \in (0,1)$.

We define function spaces which have a relation to $\hat{K}_{N,\alpha}$.

Definition 2.12. For each $p \in [1,\infty)$, the function space $L^p_{loc,unif}$ is defined as follows:

$$L^p_{loc,unif} = L^p_{loc,unif}(\mathbb{R}^N) := \big\{ V \in L^p_{loc}(\mathbb{R}^N) \bigm| \|V\|_{p,loc,unif} < \infty \big\}, \text{where}$$
$$\|V\|_{p,loc,unif} := \operatorname*{ess.sup}_{x\in\mathbb{R}^N} \Big(\int_{|x-y|<1} |V(y)|^p \, dy \Big)^{\frac{1}{p}}.$$

$L^p_{loc,unif}$ is a Banach space with norm $\|\cdot\|_{p,loc,unif}$ for all $p \in [1,\infty)$. It is clear that $L^\infty \hookrightarrow L^q_{loc,unif} \hookrightarrow L^p_{loc,unif}$ for all $1 \le p \le q < \infty$, where $\hookrightarrow$ means the continuous embedding. A relation to $\hat{K}_{N,\alpha}$ is stated in the next proposition.

Proposition 2.13. *Let* $\alpha \in (0,1)$.

(i) *In the case of* $\frac{N}{2} < \alpha$ *(i.e.* $N = 1, \frac{1}{2} < \alpha < 1$*),*

$$L^1_{loc,unif}(\mathbb{R}) = \hat{K}_{1,\alpha}$$

(as linear spaces, and the norms are equivalent).

(ii) *In the case of* $\frac{N}{2} \ge \alpha$,

$$L^p_{loc,unif}(\mathbb{R}^N) \hookrightarrow \hat{K}_{N,\alpha} \hookrightarrow L^1_{loc,unif}(\mathbb{R}^N)$$

for all $p \in \big(\frac{N}{2\alpha}, \infty\big)$.

In each case, if V *belongs to the leftmost space, then* $c_{N,\alpha}(V) = 0$.

Proof. We first prove assertion (i) and $c_{1,\alpha}(V) = 0$ for all $V \in L^1_{loc,unif}(\mathbb{R})$. It is easy to see that if $V \in L^1_{loc,unif}(\mathbb{R})$, then $V \in \hat{K}_{1,\alpha}$ and $\|V\|_{\hat{K}_{1,\alpha}} \le \|V\|_{1,loc,unif}$. Conversely, let $V \in \hat{K}_{1,\alpha}$ and we put $I_y := \big[y-1, y-\frac{1}{2}\big] \cup \big[y+\frac{1}{2}, y+1\big]$ for all $y \in \mathbb{R}$. Note that $[-1,1] \subset \bigcup_{j=-1}^{1} I_j$ and $g_{1,\alpha}(x) \ge \frac{1}{2}$ for all $x \in I_0$. For a.e. $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{|y|<1} |V(x-y)| \, dy &\le \sum_{j=-1}^{1} \int_{I_j} |V(x-y)| \, dy \\
&= \sum_{j=-1}^{1} \int_{I_0} |V(x-j-y)| \, dy \\
&\le 2 \sum_{j=-1}^{1} \int_{I_0} g_{1,\alpha}(y) |V(x-j-y)| \, dy \\
&\le 2 \sum_{j=-1}^{1} \big\| |V| * g_{1,\alpha} \big\|_\infty \\
&= 6 \big\| |V| * g_{1,\alpha} \big\|_\infty = 6 \|V\|_{\hat{K}_{1,\alpha}}.
\end{aligned}$$

Hence, $V \in L^1_{loc,unif}(\mathbb{R})$, $\|V\|_{1,loc,unif} \leq 6\|V\|_{\hat{K}_{1,\alpha}}$. In addition, if $V \in L^1_{loc,unif}(\mathbb{R})$, then for each $\frac{1}{2} < \alpha < 1$ and $0 < \rho \leq 1$,

$$\operatorname*{ess.sup}_{x\in\mathbb{R}} \int_{|x-y|<\rho} |g_{1,\alpha}(x-y)||V(y)|\,dy \leq \rho\|V\|_{1,loc,unif}.$$

By taking the limit as $\rho \downarrow 0$, we obtain $c_{1,\alpha}(V) = 0$.

Next, we prove $\hat{K}_{N,\alpha} \hookrightarrow L^1_{loc,unif}(\mathbb{R}^N)$ in the case of $\frac{N}{2} \geq \alpha$. Let $V \in \hat{K}_{N,\alpha}$. We can take finite points $x_1, \ldots, x_n \in \mathbb{R}^N$ such that

$$B(0,1) \subset \bigcup_{j=1}^{n} B\big(x_j, \tfrac{1}{2}\big).$$

Note that for all $x \in \mathbb{R}^N$, $B(x,1) \subset \bigcup_{j=1}^{n} B\big(x + x_j, \frac{1}{2}\big)$ and that by the definition of $\|\cdot\|_{\hat{K}_{N,\alpha}}$ and Fatou's lemma,

$$\int_{B(x+x_j,1)} |g_{N,\alpha}(x+x_j-y)||V(y)|\,dy \leq \|V\|_{\hat{K}_{N,\alpha}}$$

for all $x \in \mathbb{R}^N$ and $j = 1, \ldots, n$. Now we put $C_{N,\alpha} := \big|g_{N,\alpha}\big(\frac{1}{2}\boldsymbol{e}\big)\big|$, where $\boldsymbol{e} \in \mathbb{R}^N$ and $|\boldsymbol{e}| = 1$ (note that $C_{N,\alpha}$ is independent of such $\boldsymbol{e}$'s), then $|g_{N,\alpha}(x)| > C_{N,\alpha}$ for all $x \in B\big(0, \frac{1}{2}\big)$. Hence, for all $x \in \mathbb{R}^N$,

$$\begin{aligned}
\int_{B(x,1)} |V(y)|\,dy &\leq \sum_{j=1}^{n} \int_{B(x+x_j,\frac{1}{2})} |V(y)|\,dy \\
&\leq \frac{1}{C_{N,\alpha}} \sum_{j=1}^{n} \int_{B(x+x_j,\frac{1}{2})} |g_{N,\alpha}(x+x_j-y)||V(y)|\,dy \\
&\leq \frac{n}{C_{N,\alpha}} \|V\|_{\hat{K}_{N,\alpha}}.
\end{aligned}$$

Thus, $V \in L^1_{loc,unif}$ and $\|V\|_{1,loc,unif} \leq \frac{n}{C_{N,\alpha}} \|V\|_{\hat{K}_{N,\alpha}}$.

Finally, in the case of $\frac{N}{2} \geq \alpha$, we prove $L^p_{loc,unif}(\mathbb{R}^N) \hookrightarrow \hat{K}_{N,\alpha}$ for all $p \in \big(\frac{N}{2\alpha}, \infty\big)$ and the last assertion of this proposition. If $V \in L^p_{loc,unif}$ $\big(p \in \big(\frac{N}{2\alpha}, \infty\big)\big)$, then for all $\rho \in (0,1]$, a.e. $x \in \mathbb{R}^N$ and the conjugate exponent p' of p, we obtain

$$\begin{aligned}
&\int_{|x-y|<\rho} |g_{N,\alpha}(x-y)||V(y)|\,dy \\
&\quad\leq \Big(\int_{|x-y|<\rho} |g_{N,\alpha}(x-y)|^{p'}\,dy\Big)^{\frac{1}{p'}} \Big(\int_{|x-y|<\rho} |V(y)|^{p}\,dy\Big)^{\frac{1}{p}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
&\quad\leq \|g_{N,\alpha}\|_{L^{p'}(B(0,\rho))} \|V\|_{p,loc,unif}.
\end{aligned}$$

Since $g_{N,\alpha} \in L^{p'}\big(B(0,1)\big)$ for all $p \in \big(\frac{N}{2\alpha}, \infty\big)$, we have $V \in \hat{K}_{N,\alpha}$, $\|V\|_{\hat{K}_{N,\alpha}} \leq \|g_{N,\alpha}\|_{L^{p'}(B(0,1))}\|V\|_{p,loc,unif}$ for all $p \in \big(\frac{N}{2\alpha}, \infty\big)$ and

$$c_{N,\alpha}(V) \leq \lim_{\rho\downarrow 0}\|g_{N,\alpha}\|_{L^{p'}(B(0,\rho))}\|V\|_{p,loc,unif} = 0$$

holds. □

Theorem 2.14. *Suppose that $V_- \in \hat{K}_{N,\alpha}$ and $c_{N,\alpha}(V_-) < 1$ and that V_+ is U_α-admissible. Then V is $U_{\alpha,p}$-admissible for all $p \in [1, \infty)$.*

Before proving this theorem, we give an example of a potential which is $U_{\alpha,p}$-admissible for all $p \in [1, \infty)$. Let $\lambda > 0$ and a potential V be defined by $V(x) := |x|^{-\lambda}$ ($x \in \mathbb{R}^N \setminus \{0\}$). It is easy to see that for any $p \in [1, \infty)$, the following (i) and (ii) are equivalent: (i) $V \in L^p_{loc,unif}$, (ii) $\int_{|y|<1} V(y)^p\, dy < \infty$. In the case where $\frac{N}{2} < \alpha$ (i.e. $N = 1, \frac{1}{2} < \alpha < 1$), $0 < \lambda < 1$ and $1 \leq p < \frac{1}{\lambda}$ or in the case where $\frac{N}{2} \geq \alpha, \lambda \in (0, 2\alpha)$ and $\frac{N}{2\alpha} < p < \frac{N}{\lambda}$, condition (ii) is satisfied. Hence, $V \in L^p_{loc,unif}$ in each of the cases and hence, by Proposition 2.13 and this theorem, V is $U_{\alpha,p}$-admissible for all $p \in [1, \infty)$. In particular, the Coulomb potential $V(x) := c|x|^{-1}$ in $\mathbb{R}^3$ (c is a constant) is $U_{\alpha,p}$-admissible for all $\alpha \in \big(\frac{1}{2}, 1\big)$ and $p \in [1, \infty)$.

For the proof of Theorem 2.14, we need the following lemmas and proposition (*cf.* [18, Lemma B.1, B.2, Proposition 5.1]). To state the lemmas and proposition and prove Theorem 2.14, we introduce the notion of "semigroup boundedness" defined in [18, Definition 1.2].

Definition 2.15. Let $U := \big(U(t)\big)_{t\geq 0}$ be a C_0-semigroup on a Banach space X, with generator T. An operator B in X will be called U-bounded, if B is T-bounded and there exist an $\eta \in (0, \infty]$ and $\gamma \geq 0$ such that

$$\int_0^\eta \|BU(t)x\|\, dt \leq \gamma\|x\| \tag{2.14}$$

holds for all $x \in D(T)$ (see also the remark below). The number

$$\inf\{\gamma \geq 0 \,|\, \text{there exists an } \eta > 0 \text{ such that (2.14) holds for all } x \in D(T)\}$$

is called the U-bound of B. If B is U-bounded with U-bound < 1, then B will be called U-small.

Remark 2.16. By T-boundedness of B, for all $x \in D(T)$, $U(t)x \in D(T) \subset D(B)$ for all $t \geq 0$ and the X-valued function $t \mapsto BU(t)x$ is continuous on $[0, \infty)$. Hence, this function is Bochner integrable on $[0, \eta]$ for all $\eta > 0$.

Lemma 2.17. *For any $c > 1$ and $\eta > 0$, there exists a $\delta_0 \in (0, 1]$ such that*

$$\big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty \le c \Big\| V \int_0^\eta U_{\alpha,1}(t)\, dt \Big\|$$

for all $\delta \in (0, \delta_0]$ and V which is $U_{\alpha,1}$-bounded, where $\|V \int_0^\eta U_{\alpha,1}(t)\, dt\|$ denotes the $\mathcal{L}(L^1)$-norm of the composition of V and $\int_0^\eta U_{\alpha,1}(t)\, dt$. (If V is $U_{\alpha,1}$-bounded, then for a sufficiently small $\eta > 0$, the norm is finite by the equality (2.15) *below and Definition* 2.15*.)*

Proof. It is easy to verify the following equality corresponding to (B.1) in [18]: For all $U_{\alpha,1}$-bounded V and $\eta > 0$,

$$\begin{aligned}
(2.15) \qquad \Big\| V \int_0^\eta U_{\alpha,1}(t)\, dt \Big\| &\\
&= \sup \Big\{ \int_0^\eta \|V U_{\alpha,1}(t) u\|_1\, dt \;\Big|\; u \in L^1, \|u\|_1 \le 1 \Big\} \\
&\qquad\qquad\qquad \text{(by [18, Proposition 4.7 (a)])} \\
&= \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |V(x-y)| \Big(\int_0^\eta K_\alpha(t, y)\, dt \Big)\, dy \\
&= \alpha \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |V(x-y)| |y|^{-N+2\alpha} \\
&\qquad\qquad \times \Big(\int_{\eta^{-\frac{1}{\alpha}} |y|^2}^\infty \tau^{\frac{N}{2} - \alpha - 1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau \Big)\, dy,
\end{aligned}$$

where $\boldsymbol{e} \in \mathbb{R}^N$ and $|\boldsymbol{e}| = 1$. For the last equality, we use the fact that $K_\alpha(t, y) = t^{-\frac{N}{2\alpha}} K_\alpha(1, t^{-\frac{1}{2\alpha}} y)$ for all $t > 0$ and $y \in \mathbb{R}^N$, and use the change of variables $t^{-\frac{1}{\alpha}} |y|^2 = \tau$. Therefore, it is sufficient to show that there exists a $\delta_0 \in (0, 1]$ such that

$$(2.16) \qquad |g_{N,\alpha,\delta}(y)| \le c\alpha |y|^{-N+2\alpha} \int_{\eta^{-\frac{1}{\alpha}} |y|^2}^\infty \tau^{\frac{N}{2} - \alpha - 1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau$$

for all $\delta \in (0, \delta_0]$ and $y \in B(0, \delta)$.

First case: $\frac{N}{2} < \alpha$ (i.e. $N = 1, \frac{1}{2} < \alpha < 1$). The right-hand side of (2.16) is estimated as follows:

$$\begin{aligned}
&\alpha |y|^{-1+2\alpha} \int_{\eta^{-\frac{1}{\alpha}} |y|^2}^\infty \tau^{-\frac{1}{2} - \alpha} K_\alpha(1, \tau^{\frac{1}{2}})\, d\tau \\
&\qquad \ge C'_\alpha \alpha |y|^{-1+2\alpha} \int_{\eta^{-\frac{1}{\alpha}} |y|^2}^\infty \tau^{-\frac{1}{2} - \alpha} \frac{1}{(1+\tau)^{\frac{1}{2} + \alpha}}\, d\tau \quad \big(\text{by (2.1)}\big)
\end{aligned}$$

$$
\begin{aligned}
&= C'_\alpha \alpha |y|^{-1+2\alpha}\Big(\frac{2}{2\alpha-1}\cdot\frac{\eta^{1-\frac{1}{2\alpha}}|y|^{1-2\alpha}}{\big(1+\eta^{-\frac{1}{\alpha}}|y|^2\big)^{\frac{1}{2}+\alpha}} \\
&\qquad\qquad - \frac{2\alpha+1}{2\alpha-1}\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^{\infty}\frac{\tau^{\frac{1}{2}-\alpha}}{(1+\tau)^{\frac{3}{2}+\alpha}}\,d\tau\Big) \\
&\qquad\qquad\qquad \text{(by the integration by parts)} \\
&\geq C'_\alpha\cdot\frac{\alpha}{2\alpha-1}\left(\frac{2\eta^{1-\frac{1}{2\alpha}}}{\big(1+\eta^{-\frac{1}{\alpha}}|y|^2\big)^{\frac{1}{2}+\alpha}}\right. \\
&\qquad\qquad \left. -(2\alpha+1)\int_0^\infty \frac{\tau^{\frac{1}{2}-\alpha}}{(1+\tau)^{\frac{3}{2}+\alpha}}\,d\tau\cdot|y|^{-1+2\alpha}\right).
\end{aligned}
$$

Since the limit as $|y| \to 0$ of the right-hand side of the last inequality is $C'_\alpha \cdot \frac{2\alpha}{2\alpha-1}\eta^{1-\frac{1}{2\alpha}} > 0$,

$$
\alpha|y|^{-1+2\alpha}\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^{\infty}\tau^{-\frac{1}{2}-\alpha}K_\alpha(1,\tau^{\frac{1}{2}})\,d\tau \geq \text{const.} \geq |y| = g_{N,\alpha}(y)
$$

for all $|y| < \delta$ if δ is small enough.

Second case: $\frac{N}{2} = \alpha$ (i.e. $N = 1, \alpha = \frac{1}{2}$). In this case, $K_\alpha(t,x)$ is the Poisson kernel:

$$
K_\alpha(t,x) = \frac{1}{\pi}\cdot\frac{t}{t^2+x^2} \quad (t > 0, x \in \mathbb{R}).
$$

For $0 < \delta < \min\{1,\eta\}$ and $0 < |y| < \eta$, the right-hand side of (2.16) is estimated as follows:

$$
\begin{aligned}
\alpha\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^{\infty}&\tau^{-\frac{1}{2}-\alpha}K_\alpha(1,\tau^{\frac{1}{2}})\,d\tau \\
&= \frac{1}{2\pi}\int_{\eta^{-2}|y|^2}^{\infty}\frac{1}{\tau}\cdot\frac{1}{1+\tau}\,d\tau \\
&= \frac{1}{2\pi}\log\Big(1+\frac{\eta^2}{|y|^2}\Big) \\
&\geq \frac{1}{\pi}\log\frac{\eta}{|y|} \\
&= -\frac{1}{\pi}\log|y|\Big(1-\frac{\log\eta}{\log|y|}\Big).
\end{aligned}
$$

Since $1 - \frac{\log\eta}{\log|y|} \to 1$ as $|y| \to 0$, for c in the statement of this lemma,

$$
c\alpha\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^{\infty}\tau^{-1}K_\alpha(1,\tau^{\frac{1}{2}})\,d\tau \geq \frac{1}{\pi}\big|\log|y|\big| = |g_{N,\alpha}(y)|
$$

for $|y| < \delta$ if δ is small enough.

Third case: $\frac{N}{2} > \alpha$. We have only to prove that

$$\alpha |y|^{-N+2\alpha} \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau = g_{N,\alpha}(y)$$

for all $y \in \mathbb{R}^N$, where $\boldsymbol{e} \in \mathbb{R}^N$ and $|\boldsymbol{e}| = 1$. Recalling the representation (2.7) of K_α by K_1, we have

$$\begin{aligned}
&\int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau \\
&\quad = \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} \Big(\int_0^\infty f_{1,\alpha}(s) K_1(s, \tau^{\frac{1}{2}} \boldsymbol{e})\, ds \Big)\, d\tau \\
&\quad = \int_0^\infty f_{1,\alpha}(s) \Big(\int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_1(s, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau \Big)\, ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&\quad = \int_0^\infty f_{1,\alpha}(s) \Big(\frac{1}{(4\pi s)^{\frac{N}{2}}} \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} e^{-\frac{\tau}{4s}}\, d\tau \Big)\, ds \\
&\quad = \frac{1}{4^\alpha \pi^{\frac{N}{2}}} \int_0^\infty f_{1,\alpha}(s) s^{-\alpha} \Big(\int_0^\infty \tau^{\frac{N}{2}-\alpha-1} e^{-\tau}\, d\tau \Big)\, ds \\
&\quad = \frac{1}{4^\alpha \pi^{\frac{N}{2}}} \Gamma\big(\tfrac{N}{2} - \alpha\big) \int_0^\infty f_{1,\alpha}(s) s^{-\alpha}\, ds.
\end{aligned}$$

By using the equality (2.17) that is proved separately in Lemma 2.18, we obtain

$$\begin{aligned}
&\alpha |y|^{-N+2\alpha} \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\, d\tau \\
&\quad = \alpha |y|^{-N+2\alpha} \cdot \frac{1}{4^\alpha \pi^{\frac{N}{2}}} \Gamma\big(\tfrac{N}{2} - \alpha\big) \cdot \frac{1}{\alpha \Gamma(\alpha)} \\
&\quad = \frac{1}{4^\alpha \pi^{\frac{N}{2}}} \cdot \frac{\Gamma\big(\frac{N}{2} - \alpha\big)}{\Gamma(\alpha)} |y|^{-N+2\alpha} = g_{N,\alpha}(y).
\end{aligned}$$

Therefore, the proof is completed. $\square$

Lemma 2.18. *Let $\alpha \in (0,1)$. The equality*

$$\int_0^\infty f_{1,\alpha}(s) s^{-\alpha}\, ds = \frac{1}{\alpha \Gamma(\alpha)} \tag{2.17}$$

holds.

Proof. Let $0 < \sigma_0 < \varepsilon$. For all $\sigma \in (0, \sigma_0)$,

$$
\begin{aligned}
\int_0^\infty & e^{-\varepsilon s} f_{1,\alpha}(s) s^{-\alpha}\, ds \\
&= \frac{1}{2\pi i} \int_0^\infty \Big(\int_{\sigma - i\infty}^{\sigma + i\infty} e^{sz - z^\alpha}\, dz \Big) e^{-\varepsilon s} s^{-\alpha}\, ds \\
&= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Big(\int_0^\infty e^{-(\varepsilon - z)s} s^{-\alpha}\, ds \Big) e^{-z^\alpha}\, dz \\
&\qquad\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma(1-\alpha)}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha}\, dz.
\end{aligned} \tag{2.18}
$$

The reason why we could apply Fubini's theorem above is that the estimate

$$
|e^{sz - z^\alpha} e^{-\varepsilon s} s^{-\alpha}| \le e^{-(\varepsilon - \sigma)s} s^{-\alpha} e^{-|\eta|^\alpha \cos \frac{\pi\alpha}{2}}
$$

holds for all $z = \sigma + i\eta$ ($\eta \in \mathbb{R}$) and the right-hand side function is integrable with respect to (s, η) on $(0, \infty) \times \mathbb{R}$.

In addition, for $z = \sigma + i\eta$ ($0 < \sigma < \sigma_0, \eta \in \mathbb{R}$), the estimate

$$
\Big| \frac{1}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha} \Big| \le \frac{1}{(\varepsilon - \sigma_0)^{1-\alpha}} e^{-|\eta|^\alpha \cos \frac{\pi\alpha}{2}}
$$

holds and the right-hand side function is integrable with respect to η on $\mathbb{R}$. Since $f_{1,\alpha}$ is independent of $\sigma > 0$, by applying Lebesgue's convergence theorem to the rightmost side of (2.18), we have

$$
\begin{aligned}
\int_0^\infty & e^{-\varepsilon s} f_{1,\alpha}(s) s^{-\alpha}\, ds \\
&= \frac{\Gamma(1-\alpha)}{2\pi i} \lim_{\sigma \downarrow 0} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha}\, dz \\
&= \frac{\Gamma(1-\alpha)}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha}\, dz
\end{aligned} \tag{2.19}
$$

for all $\varepsilon > 0$. Since the function $s \mapsto f_{1,\alpha}(s) s^{-\alpha}$ is integrable on $(0, \infty)$ by Lemma 2.4 (iii) and (iv), by Lebesgue's convergence theorem,

$$
\lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon s} f_{1,\alpha}(s) s^{-\alpha}\, ds = \int_0^\infty f_{1,\alpha}(s) s^{-\alpha}\, ds. \tag{2.20}
$$

On the other hand, the integrand of the rightmost side of (2.19) satisfies the estimate

$$
\Big| \frac{1}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha} \Big| \le \frac{1}{|\eta|^{1-\alpha}} e^{-|\eta|^\alpha \cos \frac{\pi\alpha}{2}}
$$

for $z = i\eta$ ($\eta \in \mathbb{R}$) and the right-hand side function of this inequality is integrable with respect to η on $\mathbb{R}$. Hence, by Lebesgue's convergence theorem again,

$$\lim_{\varepsilon \downarrow 0} \int_{-i\infty}^{i\infty} \frac{1}{(\varepsilon - z)^{1-\alpha}} e^{-z^\alpha} \, dz = \int_{-i\infty}^{i\infty} \frac{1}{(-z)^{1-\alpha}} e^{-z^\alpha} \, dz. \tag{2.21}$$

By (2.19), (2.20) and (2.21), the equality

$$\begin{aligned}
&\int_0^\infty f_{1,\alpha}(s) s^{-\alpha} \, ds \\
&\quad = \frac{\Gamma(1-\alpha)}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{(-z)^{1-\alpha}} e^{-z^\alpha} \, dz \\
&\quad = \frac{\Gamma(1-\alpha)}{2\pi} \Big(\int_0^\infty \frac{1}{(-i\eta)^{1-\alpha}} e^{-i^\alpha \eta^\alpha} \, d\eta + \int_0^\infty \frac{1}{(i\eta)^{1-\alpha}} e^{-(-i)^\alpha \eta^\alpha} \, d\eta \Big) \\
&\quad = \frac{\Gamma(1-\alpha)}{2\pi} \Big(e^{i\frac{\pi}{2}(1-\alpha)} \cdot \frac{1}{\alpha} e^{-i\frac{\pi\alpha}{2}} + e^{-i\frac{\pi}{2}(1-\alpha)} \cdot \frac{1}{\alpha} e^{i\frac{\pi\alpha}{2}} \Big) \\
&\quad = \frac{2\cos(\frac{\pi}{2}(1-2\alpha))}{2\pi\alpha} \Gamma(1-\alpha) \\
&\quad = \frac{\sin \pi\alpha}{\pi\alpha} \Gamma(1-\alpha) = \frac{1}{\alpha \Gamma(\alpha)}
\end{aligned}$$

holds. ☐

Lemma 2.19. *For any $c > 1$ and $\delta \in (0,1]$, there exists an $\eta_0 > 0$ such that*

$$\Big\| V \int_0^\eta U_{\alpha,1}(t) \, dt \Big\| \leq c \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty$$

for all $\eta \in (0, \eta_0]$ and $V \in \hat{K}_{N,\alpha}$.

Proof. We first show that for all $c' \in (1, c)$ there exist a $\delta' > 0$ and $\eta_1 > 0$ such that

$$\operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{|y|<\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y) \, dt \Big) \, dy \leq c' \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty \tag{2.22}$$

for all $\eta \in (0, \eta_1]$, where c and δ are as in the statement of this lemma.

First case: $\frac{N}{2} < \alpha$ (i.e. $N = 1, \frac{1}{2} < \alpha < 1$). Since $0 \leq K_\alpha(t,y) \leq C_\alpha t (t^{\frac{1}{\alpha}} + |y|^2)^{-\frac{1}{2}-\alpha} \leq C_\alpha t^{-\frac{1}{2\alpha}}$ for all $t > 0$ and $y \in \mathbb{R}^N$ by (2.1),

$$\int_0^\eta K_\alpha(t,y) \, dt \leq C_\alpha \int_0^\eta t^{-\frac{1}{2\alpha}} \, dt = C\eta^{1-\frac{1}{2\alpha}}$$

for all $y \in \mathbb{R}^N$ and $\eta > 0$. Hence, taking $\delta' = 1$, we can estimate the left-hand side of (2.22) as follows:

$$\begin{aligned}
\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|<\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt\Big)\,dy \\
&\le C\eta^{1-\frac{1}{2\alpha}} \|V\|_{1,loc,unif} \\
&\le C\eta^{1-\frac{1}{2\alpha}} \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty \\
&\le c' \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty
\end{aligned}$$

for a sufficiently small $\eta > 0$ (note that the exponent $1 - \frac{1}{2\alpha}$ of η is positive).

Second case: $\frac{N}{2} = \alpha$ (i.e. $N = 1, \alpha = \frac{1}{2}$). Let $\eta \in (0, 1]$. As in the second case of the proof of Lemma 2.17, we have

$$\begin{aligned}
0 \le \int_0^\eta K_\alpha(t,y)\,dt &\le \int_0^1 K_\alpha(t,y)\,dt \\
&= \alpha \int_{|y|^2}^\infty \frac{1}{\tau} \cdot \frac{1}{1+\tau}\,d\tau \\
&= \frac{1}{2\pi} \log\Big(1 + \frac{1}{|y|^2}\Big) \\
&\le \frac{c'}{\pi} \big|\log|y|\big|
\end{aligned}$$

for $|y| < \delta'$ if δ' is small enough. Hence, let $\delta' > 0$ be such small and smaller than δ, then we can estimate the left-hand side of (2.22) as follows:

$$\begin{aligned}
\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|<\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt\Big)\,dy &\le c' \big\| |V| * |g_{N,\alpha,\delta'}| \big\|_\infty \\
&\le c' \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty.
\end{aligned}$$

Third case: $\frac{N}{2} > \alpha$. By the result in the third case of the proof of Lemma 2.17, we have

$$\begin{aligned}
0 &\le \int_0^\eta K_\alpha(t,y)\,dt \\
&\le \alpha |y|^{-N+2\alpha} \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\,d\tau \\
&= g_{N,\alpha}(y)
\end{aligned}$$

for all $\eta > 0$ and $y \in \mathbb{R}^N$. Hence, let $\delta' = \delta$, then the left-hand side of (2.22) is estimated as follows:

$$\begin{aligned}
\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|<\delta} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt\Big)\,dy &\le \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty \\
&\le c' \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty.
\end{aligned}$$

Hence, it is sufficient to prove the assertion of this lemma that for any $\varepsilon > 0$ and $\delta' > 0$, there exists an $\eta_2 > 0$ such that

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|>\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt \Big)\,dy \le \varepsilon \big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty \tag{2.23}$$

for all $\eta \in (0,\eta_2]$ and $V \in \hat{K}_{N,\alpha}$. Since $0 \le K_\alpha(t,y) \le C_\alpha t|y|^{-N-2\alpha}$ for all $t > 0$ and $y \ne 0$ by (2.1),

$$\begin{aligned}
&\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|>\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt \Big)\,dy \\
&\qquad\le \tfrac{1}{2} C_\alpha \eta^2 \operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|>\delta'} |V(x-y)||y|^{-N-2\alpha}\,dy
\end{aligned}$$

for all $\delta' > 0$ and $\eta > 0$. We will show the estimate

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|>\delta'} |V(x-y)||y|^{-N-2\alpha}\,dy \le C_{\alpha,\delta'} \|V\|_{1,loc,unif} \tag{2.24}$$

for all $\delta' > 0$. On one hand, for all $x \in \mathbb{R}^N$ and $\eta > 0$,

$$\begin{aligned}
&\int_{|y|>\frac{2}{3}\delta'} \Big(\int_{|z|<\frac{\delta'}{3}} |V(x-y+z)||y-z|^{-N-2\alpha}\,dz \Big)\,dy \\
&\qquad= \int_{|z|<\frac{\delta'}{3}} \Big(\int_{|y|>\frac{2}{3}\delta'} |V(x-y+z)||y-z|^{-N-2\alpha}\,dy \Big)\,dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Fubini's theorem)} \\
&\qquad\ge \int_{|z|<\frac{\delta'}{3}} \Big(\int_{|y-z|>\delta'} |V(x-y+z)||y-z|^{-N-2\alpha}\,dy \Big)\,dz \\
&\qquad= \int_{|z|<\frac{\delta'}{3}} \Big(\int_{|y|>\delta'} |V(x-y)||y|^{-N-2\alpha}\,dy \Big)\,dz \\
&\qquad= \big|B\big(0,\tfrac{\delta'}{3}\big)\big| \int_{|y|>\delta'} |V(x-y)||y|^{-N-2\alpha}\,dy.
\end{aligned}$$

On the other hand, if $|y| > \frac{2}{3}\delta'$ and $|z| < \frac{\delta'}{3}$, then $|y-z| > \frac{|y|}{2}$. Hence, for a.e. $x \in \mathbb{R}^N$ and all $\eta > 0$,

$$\begin{aligned}
&\int_{|y|>\frac{2}{3}\delta'} \Big(\int_{|z|<\frac{\delta'}{3}} |V(x-y+z)||y-z|^{-N-2\alpha}\,dz \Big)\,dy \\
&\qquad\le 2^{N+2\alpha} \int_{|y|>\frac{2}{3}\delta'} \Big(\int_{|z|<\frac{\delta'}{3}} |V(x-y+z)|\,dz \Big) |y|^{-N-2\alpha}\,dy \\
&\qquad\le 2^{N+2\alpha} \int_{|y|>\frac{2}{3}\delta'} |y|^{-N-2\alpha}\,dy \cdot \|V\|_{1,loc,unif}.
\end{aligned}$$

By Proposition 2.13, there exists a constant $C > 0$ such that $\|V\|_{1,loc,unif} \leq \|V\|_{\hat{K}_{N,\alpha}}$ for all $V \in \hat{K}_{N,\alpha}$. By Remark 2.11 (iii), there exists a constant $C' > 0$ such that $\|V\|_{\hat{K}_{N,\alpha}} \leq C'\||V| * |g_{N,\alpha,\delta}|\|_\infty$ for all $V \in \hat{K}_{N,\alpha}$. Thus, we obtain (2.24) and hence

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{|y|>\delta'} |V(x-y)| \Big(\int_0^\eta K_\alpha(t,y)\,dt \Big)\,dy \leq C_{\alpha,\delta',\delta}\,\eta^2 \big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty$$

for all $\eta > 0$ and $V \in \hat{K}_{N,\alpha}$. Hence, for all $\varepsilon > 0$ and $\delta' > 0$, there exists an $\eta_2 > 0$ such that (2.23) holds for all $\eta \in (0, \eta_2]$ and $V \in \hat{K}_{N,\alpha}$. Thus, the proof of this lemma is completed. $\square$

Proposition 2.20. *Let $V\colon \mathbb{R}^N \to \mathbb{R}$ be a measurable function. Then the following assertions hold.*

(i) *The following conditions are equivalent:*

  (a) *$V \in \hat{K}_{N,\alpha}$,*

  (b) *V is $U_{\alpha,1}$-bounded,*

  (c) *V is $H_{\alpha,1}$-bounded,*

  (d) *The following quantity $c'_{N,\alpha}(V)$ is finite.*

$$c'_{N,\alpha}(V) := \lim_{\eta\downarrow 0} \Big\| V \int_0^\eta U_{\alpha,1}(t)\,dt \Big\|. \tag{2.25}$$

(ii) *There exist constants $c_1, c_2, c_3 > 0$ such that*

$$\|V\|_{\hat{K}_{N,\alpha}} \leq c_1 \Big\| V \int_0^1 U_{\alpha,1}(t)\,dt \Big\| \leq c_2 \|V(H_{\alpha,1}+1)^{-1}\| \leq c_3 \|V\|_{\hat{K}_{N,\alpha}}$$

for all $V \in \hat{K}_{N,\alpha}$.

(iii) *If $V \in \hat{K}_{N,\alpha}$, then $c_{N,\alpha}(V) = c'_{N,\alpha}(V) = \lim_{\lambda\to\infty} \|V(H_{\alpha,1}+\lambda)^{-1}\|$ is the $U_{\alpha,1}$-bound of V. Therefore, we have $c_{N,\alpha}(V) = 0$ if and only if the $H_{\alpha,1}$-bound of V is 0.*

Remark 2.21. By (2.15), we can write $c'_{N,\alpha}(V)$ also as

$$c'_{N,\alpha}(V) = \alpha \lim_{\eta\downarrow 0} \operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} |V(x-y)||y|^{-N+2\alpha} \times \Big(\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^\infty t^{\frac{N}{2}-\alpha-1} K_\alpha\big(1, t^{\frac{1}{2}}\boldsymbol{e}\big)\,dt \Big)\,dy,$$

where $\boldsymbol{e} \in \mathbb{R}^N$ and $|\boldsymbol{e}| = 1$. Note that $K_\alpha(1, s\boldsymbol{e})$ is independent of such $\boldsymbol{e}$'s for all $s \geq 0$.

Proof. (i): (b) $\Leftrightarrow$ (c) holds by [18, Proposition 4.7 (a)].

(b) $\Rightarrow$ (a). If V is $U_{\alpha,1}$-bounded, then there exist an $\eta > 0$ and $\delta > 0$ such that

$$\big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty \le 2\Big\|V\int_0^\eta U_{\alpha,1}(t)\,dt\Big\| < \infty$$

by Lemma 2.17. Hence, $V \in \hat{K}_{N,\alpha}$.

(a) $\Rightarrow$ (d). If $V \in \hat{K}_{N,\alpha}$, then there exists an $\eta_0 > 0$ such that

$$\Big\|V\int_0^\eta U_{\alpha,1}(t)\,dt\Big\| \le 2\big\||V| * |g_{N,\alpha,1}|\big\|_\infty = 2\|V\|_{\hat{K}_{N,\alpha}} < \infty$$

for all $\eta \in (0, \eta_0]$ by Lemma 2.19. By the definition of $c'_{N,\alpha}(V)$, we have $c'_{N,\alpha}(V) < \infty$.

(d) $\Rightarrow$ (b). By the assumption (d), there exists an $\eta > 0$ such that the following c_η is finite:

$$c_\eta := \alpha \operatorname*{ess.sup}_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} |V(x-y)||y|^{-N+2\alpha}\Big(\int_{\eta^{-\frac{1}{\alpha}}|y|^2}^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}}\boldsymbol{e})\,d\tau\Big)\,dy.$$

For this $\eta > 0$ and all $u \in L^1$,

$$\int_0^\eta\Big(\int_{\mathbb{R}^N} |V(x)|\big|\big(U_{\alpha,1}(t)u\big)(x)\big|\,dx\Big)\,dt \le c_\eta\|u\|_1$$

by the fact that $U_{\alpha,1}(t)u = K_\alpha(t,\cdot) * u$ for all $t > 0$ and $u \in L^1$, and by a straightforward calculation. This inequality implies that for all $u \in L^1$, $U_{\alpha,1}(t)u \in D(V)$ in L^1 for a.e. $t \in [0, \eta)$ and the L^1-valued function $t \mapsto VU_{\alpha,1}(t)u$ is Bochner integrable on $[0, \eta)$, and in addition the estimate

$$\Big\|V\int_0^\eta U_{\alpha,1}(t)u\,dt\Big\| \le c_\eta\|u\|_1 \tag{2.26}$$

for all $u \in L^1$. Hence V is $H_{\alpha,1}$-bounded by [18, Proposition 1.3]. By this $H_{\alpha,1}$-boundedness and the estimate (2.26), V is $U_{\alpha,1}$-bounded.

(ii) Let an arbitrary $c > 1$ be fixed in the proof of this assertion.

First inequality: By Lemma 2.17, there exists a $\delta \in (0, 1]$ such that

$$\big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty \le c\Big\|V\int_0^1 U_{\alpha,1}(t)\,dt\Big\|$$

for all $V \in \hat{K}_{N,\alpha}$. As stated in Remark 2.11 (iii), there exists a constant $C_\delta > 0$ such that

$$\|V\|_{\hat{K}_{N,\alpha}} \le C_\delta\big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty$$

for all $V \in \hat{K}_{N,\alpha}$. Thus, the first inequality in the assertion (ii) holds.

Second and third inequality: By [18, Proposition 4.7 (b)], there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\Big\|V \int_0^1 U_{\alpha,1}(t)\,dt\Big\| \le \kappa_1 \big\|V(H_{\alpha,1}+1)^{-1}\big\| \le \kappa_2 \Big\|V \int_0^1 U_{\alpha,1}(t)\,dt\Big\|$$

for all $V \in \hat{K}_{N,\alpha}$. Hence, the second inequality in (ii) holds. In addition, by Lemma 2.19, there exists an $\eta > 0$ such that

$$\Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\| \le c\big\||V| * |g_{N,\alpha,1}|\big\|_\infty = c\|V\|_{\hat{K}_{N,\alpha}}$$

for all $V \in \hat{K}_{N,\alpha}$. For this $\eta > 0$ and $\frac{1}{m} < \eta$ ($m \in \mathbb{N}$),

$$\begin{aligned} \Big\|V \int_0^1 U_{\alpha,1}(t)\,dt\Big\| &\le \sum_{j=0}^{m-1} \Big\|V \int_{\frac{j}{m}}^{\frac{j+1}{m}} U_{\alpha,1}(t)\,dt\Big\| \\ &\le \sum_{j=0}^{m-1} \Big\|V \int_0^{\frac{1}{m}} U_{\alpha,1}(t)\,dt\Big\| \big\|U_{\alpha,1}\big(\tfrac{j}{m}\big)\big\| \\ &\le m\Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\|. \end{aligned}$$

For the last inequality, we used the contractivity of $U_{\alpha,1}$ by Proposition 2.2 (iii). Thus, the third inequality in (ii) holds.

(iii) Let $V \in \hat{K}_{N,\alpha}$. By assertion (i), V is $U_{\alpha,1}$-bounded. Let $c > 1$. Then, for all $\eta > 0$

$$\limsup_{\delta\downarrow 0} \big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty \le c\Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\|$$

holds by Lemma 2.17, hence

$$\limsup_{\delta\downarrow 0} \big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty \le c \liminf_{\eta\downarrow 0} \Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\|. \tag{2.27}$$

On the other hand, Lemma 2.19 implies

$$\limsup_{\eta\downarrow 0} \Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\| \le c\big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty$$

for every $\delta \in (0, 1]$, hence

$$\limsup_{\eta\downarrow 0} \Big\|V \int_0^\eta U_{\alpha,1}(t)\,dt\Big\| \le c \liminf_{\delta\downarrow 0} \big\||V| * |g_{N,\alpha,\delta}|\big\|_\infty. \tag{2.28}$$

Note that (2.27) and (2.28) holds for $c = 1$. Hence, by the definition of $c_{N,\alpha}(V)$ and (2.27), (2.28) for $c = 1$ and (2.25), the following limits exist and

$$c_{N,\alpha}(V) = \lim_{\delta\downarrow 0}\big\| |V| * |g_{N,\alpha,\delta}| \big\|_\infty = \lim_{\eta\downarrow 0}\Big\| V \int_0^\eta U_{\alpha,1}(t)\,dt \Big\| = c'_{N,\alpha}(V)$$

for all $V \in \hat{K}_{N,\alpha}$. By [18, Proposition 4.7 (c)], $c_{N,\alpha}(V)$ is $U_{\alpha,1}$-bound of V and the last statement of (iii) holds. □

Proof of Theorem 2.14. By the assumption for V_- and Proposition 2.20, $-V_-$ is $U_{\alpha,1}$-small. Hence, $-V_-$ is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ by [18, Remark 2.1 (b)] and Proposition 2.9. On the other hand, V_+ is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ by Proposition 2.9. Thus, V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$. □

Proposition 2.13 gives a sufficient condition that guarantees that a potential V satisfies $V \in \hat{K}_{N,\alpha}$ and $c_{N,\alpha}(V) < 1$, which are assumed for V_- in Theorem 2.14. In the following proposition, we give a necessary and sufficient condition for a positive potential V to be $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$.

Proposition 2.22 (*cf.* [18, Proposition 5.8 (a)]). *Let $V : \mathbb{R}^N \to \mathbb{R}$ be a non-negative measurable function. Then the following assertions are equivalent:*

(i) *V is $U_{\alpha,p}$-admissible for some (all) $p \in [1,\infty)$,*

(ii) *$H^\alpha \cap Q(V)$ is dense in L^2, where $Q(V)$ is the form domain of V.*

Hence, if $V \in L^1_{loc}(\mathbb{R}^N)$, then V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$.

Proof. We can prove that (i) and (ii) are equivalent in a similar way as in [18, Proposition 5.8 (a)]. If $V \in L^1_{loc}(\mathbb{R}^N)$, then $H^\alpha \cap Q(V)$ is dense in L^2 since $H^\alpha \cap Q(V)$ includes $C_c^\infty(\mathbb{R}^N)$. Hence, V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$. □

§3. The Feynman-Kac formula and L^p-L^q estimates for $e^{-tH_{\alpha,2,V}}$

In the proofs of the main theorems, L^p-L^q estimates for the semigroup $e^{-tH_{\alpha,p,V}}$ play an important role. These estimates follow from the Feynman-Kac formula and a corollary to the so-called Khas'minskii's lemma.

3.1. The Feynman-Kac formula for $e^{-tH_{\alpha,p,V}}$

Voigt established the Feynman-Kac formula for a Schrödinger semigroup ($= e^{-tH_{1,p,V}}$) for a rather general potential in [18, Proposition 6.1]. On the other hand, M. Demuth and J.A. van Casteren [4, Theorem 2.5] showed the

Feynman-Kac formulas for Feller semigroups perturbed by certain potentials. While their result can be applied to many Feller semigroups, in the case of the heat semigroup $e^{t\Delta}$, the coverage for potentials is restricted as compared with [18, Proposition 6.1]. In view of the difference between these results, there is a possibility that, for a larger class of potentials than the one in [4, Theorem 2.5], $e^{-tH_{\alpha,p,V}}$ is represented by the Feynman-Kac formula. In fact, we can prove the Feynman-Kac formula for $e^{-tH_{\alpha,p,V}}$ for such a larger class of potentials, by using some preliminaries concerning a Hunt process (*cf.* [2, p. 45] for the definition). First, we fix some notations. For any topological space X, the symbol $\mathscr{B}(X)$ denotes the family of the Borel sets of X. In particular, for $X = \mathbb{R}^N$, we define $\mathscr{E} := \mathscr{B}(\mathbb{R}^N)$. E_∞ and $\mathscr{E}_\infty$ denotes the one point compactification of $\mathbb{R}^N$ by a point x_∞ ($\notin \mathbb{R}^N$) and $\mathscr{B}(E_\infty)$, respectively.

Let $\alpha \in (0, 1]$. As is easily proved by using Proposition 2.2, we can define the Markov transition function $K_\alpha(t, x, A)$ by

$$K_\alpha(t,x,A) := \begin{cases} \displaystyle\int_A K_\alpha(t, x-y)\,dy & (t > 0, x \in \mathbb{R}^N, A \in \mathscr{E}), \\ \delta_x(A) & (t = 0, x \in \mathbb{R}^N, A \in \mathscr{E}), \end{cases}$$

where δ_x is the unit mass at x. It is immediately verified from Proposition 2.2 that this Markov transition function induces a Feller semigroup on $C_\infty(\mathbb{R}^N)$ (for the definition, see [4, B.11]) by defining the semigroup by

$$\int_{\mathbb{R}^N} K_\alpha(t, x, dy) u(y) = \big(K_\alpha(t, \cdot) * u\big)(x)$$

for all $t > 0, u \in L^\infty(\mathbb{R}^N)$ and a.e. $x \in \mathbb{R}^N$. Hence, the Markov transition function satisfies the assumptions in [2, Theorem 9.4] since the assumptions (1) and (2) is the same as the condition (i) and (v′) in [4, B.11], respectively.

Hence, there exists a Hunt process $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, W_x^\alpha)$ with state space $(\mathbb{R}^N, \mathscr{E})$ and transition function $K_\alpha(t, x, A)$ by Theorem 9.4 in [2]. This Hunt process consists of the following objects:

(i) $\Omega \subset E_\infty^{\boldsymbol{T}}$, where $\boldsymbol{T} := [0, \infty]$ and Ω consists of all the elements of $E_\infty^{\boldsymbol{T}}$ satisfying the following conditions:

  (a) the function $t \mapsto \omega(t)$ is right continuous on $[0, \infty)$ and has the left limit on $(0, \infty)$,

  (b) $\omega(\infty) = x_\infty$,

  (c) If $\omega(t) = x_\infty$ for some $t \geq 0$, then $\omega(s) = x_\infty$ for all $s \geq t$.

  Now, we define the special element $\omega_\infty \in \Omega$ by $\omega_\infty(t) = x_\infty$ for all $t \in \boldsymbol{T}$.

(ii) For all $t \in \boldsymbol{T}$, we define the function $X_t \colon \Omega \to E_\infty$ by $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. Note that $X_\infty(\omega) = \omega(\infty) = x_\infty$ for all $\omega \in \Omega$.

(iii) $(\Omega, \mathscr{F}_t, W_x^\alpha)$ is a probability space for all $x \in E_\infty$ constructed from the following probability space $(\Omega, \mathscr{F}_t^0, W_x^\alpha)$ as follows. $\mathscr{F}_t^0$ denotes the σ-algebra generated by $\{X_s; 0 \le s \le t\}$ in Ω for all $t \in \boldsymbol{T}$. W_x^α is a unique probability measure on $(\Omega, \mathscr{F}_\infty^0)$ satisfying the following conditions:

(a) In the case of $x \in \mathbb{R}^N$, for $0 \le t_1 < \cdots < t_n < \infty, B_k \in \mathscr{E}_\infty$ $(k = 1, \ldots, n)$,

$$\begin{aligned} &W_x^\alpha\big(\{\omega \in \Omega \big| (\omega(t_1), \ldots, \omega(t_n)) \in B_1 \times \cdots \times B_n\}\big) \\ &= \int_{(B_1 \setminus \{x_\infty\}) \times \cdots \times (B_n \setminus \{x_\infty\})} K_\alpha(t_1, x_1 - x) K_\alpha(t_2 - t_1, x_2 - x_1) \times \cdots \\ &\qquad\qquad \times K_\alpha(t_n - t_{n-1}, x_n - x_{n-1})\, dx_1 dx_2 \cdots dx_n, \end{aligned}$$

(b) In the case of $x = x_\infty$, $W_x^\alpha = \delta_{\omega_\infty}$, where δ_{ω_∞} is the probability measure on $(\Omega, \mathscr{F}_\infty^0)$ defined by $\delta_{\omega_\infty}(\{\omega_\infty\}) = 1$. (Note that $\{\omega_\infty\} = X_0^{-1}(x_\infty) \in \mathscr{F}_0^0 \subset \mathscr{F}_\infty^0$.)
$\mathscr{F}$ denotes the completion of $\mathscr{F}_\infty^0$ with respect to $\{W_x^\alpha; x \in \mathbb{R}^N\}$. $\mathscr{F}_t$ denotes the completion of $\mathscr{F}_t^0$ in $\mathscr{F}$ with respect to $\{W_x^\alpha; x \in \mathbb{R}^N\}$ for all $t \in \boldsymbol{T}$. For the definitions of these completions, see [2, (5.2) and Definition 5.3] and see also the remark below. We use the same symbol W_x^α for its extension to $\mathscr{F}$.

(iv) θ_t is the translation operator for all $t \in \boldsymbol{T}$, i.e., for all $t \in \boldsymbol{T}$, the function $\theta_t \colon \Omega \to \Omega$ satisfies $X_s \circ \theta_t = X_{s+t}$ for all $s \in \boldsymbol{T}$.

Remark 3.1. According to the statement above Definition 5.7 and (5.15) in [2], we should use the family of measures $\{W_x^\alpha; x \in E_\infty\}$ to define the σ-algebras $\mathscr{F}$ and $\mathscr{F}_t$ $(t \in \boldsymbol{T})$. However, it is easily verified that the completion of $\mathscr{F}_\infty^0$ with respect to $W_{x_\infty}^\alpha$ is the power set 2^Ω and that the completion of $\mathscr{F}_t^0$ in $\mathscr{F}$ with respect to $W_{x_\infty}^\alpha$ is $\mathscr{F}$. Hence, such completions with respect to $\{W_x^\alpha; x \in \mathbb{R}^N\}$ are the same as the ones with respect to $\{W_x^\alpha; x \in E_\infty\}$.

From now on, $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, W_x^\alpha)$ denotes the Hunt process associated with the transition function $K_\alpha(t, x, A)$. Then, for all $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$ and has the left limit on $(0, \infty)$. Hence, for all $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is continuous on $[0, \infty)$ except at most countable points (*cf.* [8, Lemma 3.3.5]). In the case of $\alpha = 1$, it is well known that for all $\nu \in \left(0, \frac{1}{2}\right), x \in E_\infty$ and W_x^α-a.e. $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is Hölder continuous of order ν. In the case of $\alpha \in (0, 1)$, such a regularity could not be proved. However, the continuity above suffices for us to prove the Feynman-Kac formula below.

Proposition 3.2. *Let $\alpha \in (0,1)$ and $p \in [1,\infty)$. If $V \in L^\infty(\mathbb{R}^N)$, then $U_{\alpha,p,V}(t)$ can be expressed by the Feynman-Kac formula: For all $t > 0$ and $u \in L^p$,*

$$(U_{\alpha,p,V}(t)u)(x) = \int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big) u\big(\omega(t)\big)\,dW_x^\alpha(\omega) \tag{3.1}$$

for a.e. $x \in \mathbb{R}^N$. We regard both values of u and V at x_∞ as 0.

Before proving this proposition, we state the Feynman-Kac formula for more general potentials in the next proposition, which corresponds to [18, Proposition 6.1] for the case of $\alpha = 1$.

Proposition 3.3. *Let $\alpha \in (0,1), p \in [1,\infty)$ and let $V\colon \mathbb{R}^N \to \mathbb{R}$ be Lebesgue measurable. Then the following assertions hold:*

(i) *Suppose that V is bounded above and $U_{\alpha,p}$-admissible. Then for all $t > 0$, $\int_0^t V\big(\omega(s)\big)\,ds > -\infty$ for a.e. $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$, and* (3.1) *holds.*

(ii) *Suppose that V is bounded below. Then the following conditions are equivalent:*

(a) *V is $U_{\alpha,p}$-admissible,*

(b) *for any sequence $\{t_n\}_n$ in $[0,\infty)$ satisfying $t_n \downarrow 0$, it is obtained that $\int_0^{t_n} V\big(\omega(s)\big)\,ds \to 0$ holds as $n \to \infty$ for a.e. $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$.*

Each of the conditions above implies (3.1).

(iii) *Suppose that V is $U_{\alpha,p}$-admissible. Then for all $t > 0$, $\int_0^t V\big(\omega(s)\big)\,ds \in (-\infty,\infty]$ holds for a.e. $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$, and* (3.1) *is valid.*

Proof. We can proceed as in the proof of [18, Proposition 6.1]. □

Although Proposition 3.2 is proved by a similar manner as in the proof of [16, Theorem X.68], for the reader's convenience, we give a detailed proof with a verification of measurability of functions appearing in the Feynman-Kac formula. To prove this proposition, we need the following lemma (*cf.* [16, p. 279 Lemma]).

Lemma 3.4. *Let $t > 0$ and S be a Borel null set in $\mathbb{R}^N$. Then the following assertions hold.*

(i) *$X_t^{-1}(S) \in \mathscr{F}_t^0$ and $W_x^\alpha\big(X_t^{-1}(S)\big) = 0$ for all $x \in \mathbb{R}^N$.*

(ii) *Let* $\Omega_{t,S}$ *be defined by*

$$\Omega_{t,S} := \{\omega \in \Omega \mid X_s(\omega) \notin S \text{ for } \mu\text{-a.e. } s \in (0,t)\},$$

where μ *is the Lebesgue measure on* $\mathbb{R}$. *Then,* $\Omega_{t,S} \in \mathscr{F}_t$ *and* $W_x^\alpha(\Omega_{t,S}) = 1$ *for all* $x \in \mathbb{R}^N$.

(iii) *Let* $m \in \mathbb{N}, 0 < t_1 < \cdots < t_m = t$ *and* S *be a Borel null set in* $\mathbb{R}^{mN}$, *where we identify* $\mathbb{R}^{mN}$ *with* $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$ *(m factors). Then, there exists a set* $\mathscr{N} \in \mathscr{F}_t^0$ *such that*

$$\big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \dots, \omega(t_m)\big) \in S\big\} \subset \mathscr{N}$$

and $W_x^\alpha(\mathscr{N}) = 0$ *for all* $x \in \mathbb{R}^N$.

Proof. (i) By the definition of $\mathscr{F}_t^0$, it is clear that $X_t^{-1}(S) \in \mathscr{F}_t^0$. For all $x \in \mathbb{R}^N$, by description (iii)-(a) concerning the Hunt process,

$$W_x^\alpha\big(X_t^{-1}(S)\big) = \int_S K_\alpha(t, x_1 - x)\, dx_1 = 0.$$

(ii) We first define $A_{t,S}$ by

$$A_{t,S} := \{(s,\omega) \in (0,t) \times \Omega \mid X_s(\omega) \in S\}$$

and prove that $A_{t,S} \in \mathscr{B}\big((0,t)\big) \otimes \mathscr{F}_t^0$ and $(\mu \otimes W_x^\alpha)(A_{t,S}) = 0$ for all $x \in \mathbb{R}^N$. To prove this, note that for all $s > 0$, X_s is $\mathscr{F}_s^0/\mathscr{E}_\infty$-measurable (i.e. for any $A \in \mathscr{E}_\infty$, the set $X_s^{-1}(A) \in \mathscr{F}_s^0$) and for all $\omega \in \Omega$, the function $s \mapsto X_s(\omega)$ is right continuous on $[0,\infty)$ by description (i)-(a) concerning the Hunt process. Hence, it is easily proved that the function $(s,\omega) \mapsto X_s(\omega)$ is $\big(\mathscr{B}\big((0,t)\big) \otimes \mathscr{F}_t^0\big)/\mathscr{E}_\infty$-measurable on $(0,t) \times \Omega$, and hence $A_{t,S} \in \mathscr{B}\big((0,t)\big) \otimes \mathscr{F}_t^0$. In addition, for all $x \in \mathbb{R}^N$, by Fubini's theorem and assertion (i),

$$(\mu \otimes W_x^\alpha)(A_{t,S}) = \int_0^t W_x^\alpha\big(X_s^{-1}(S)\big)\, d\mu(s) = 0.$$

By this equality and Fubini's theorem,

$$\int_\Omega \mu\big(\{s \in (0,t) \mid X_s(\omega) \in S\}\big)\, dW_x^\alpha(\omega) = (\mu \otimes W_x^\alpha)(A_{t,S}) = 0$$

for all $x \in \mathbb{R}^N$. Hence, for all $x \in \mathbb{R}^N$, there exists a W_x^α-null set $\mathscr{N}_x \in \mathscr{F}_t^0$ such that

$$\mu\big(\{s \in (0,t) \mid X_s(\omega) \in S\}\big) = 0$$

for all $\omega \in \Omega \setminus \mathscr{N}_x$. Hence, $\Omega \setminus \mathscr{N}_x \subset \Omega_{t,S} \subset \Omega$ for all $x \in \mathbb{R}^N$. Thus, $\Omega_{t,S}$ belongs to the completion of $\mathscr{F}_t^0$ with respect to W_x^α for each $x \in \mathbb{R}^N$. Since the intersection of these completion of $\mathscr{F}_t^0$ over $\{W_x^\alpha; x \in \mathbb{R}^N\}$ is included in $\mathscr{F}_t$ (see [2, (5.2), Definition 5.3, (5.5)-(ii)]), we have $\Omega_{t,S} \in \mathscr{F}_t$ and $W_x^\alpha(\Omega_{t,S}) = 1$ for all $x \in \mathbb{R}^N$.

(iii) For all $k \in \mathbb{N}$, we can take an open subset O_k of $\mathbb{R}^{mN}$ such that $S \subset O_k$ and $\mu_{mN}(O_k \setminus S) < \frac{1}{k}$, where μ_{mN} is the Lebesgue measure on $\mathbb{R}^{mN}$. For all $k \in \mathbb{N}$, there exist right-half-open intervals $J_{j,n}^{(k)}$ in $\mathbb{R}$ $(j = 1, \ldots, mN, n \in \mathbb{N})$ such that

$$\Big(\prod_{j=1}^m I_{j,n}^{(k)}\Big) \cap \Big(\prod_{j=1}^m I_{j,n'}^{(k)}\Big) = \emptyset$$

for $n \neq n'$, where $I_{j,n}^{(k)} := \prod_{l=(j-1)N+1}^{jN} J_{l,n}^{(k)}$, and

$$O_k = \bigcup_{n=1}^\infty \prod_{j=1}^m I_{j,n}^{(k)}.$$

Note that for all $k \in \mathbb{N}$, the set $\big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in O_k\big\}$ is the disjoint sum of the sets $\big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in \prod_{j=1}^m I_{j,n}^{(k)}\big\}$ over $n \in \mathbb{N}$. Hence, for all $x \in \mathbb{R}^N$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} &W_x^\alpha\big(\big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in O_k\big\}\big) \\ &\quad = \sum_{n=1}^\infty W_x^\alpha\Big(\Big\{\omega \in \Omega \;\Big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in \prod_{j=1}^m I_{j,n}^{(k)}\Big\}\Big) \\ &\quad = \sum_{n=1}^\infty \int_{\prod_{j=1}^m I_{j,n}^{(k)}} K_\alpha(t_1, x_1 - x) \cdots K_\alpha(t_m - t_{m-1}, x_m - x_{m-1})\, dx_1 \cdots dx_m \\ &\quad \leq C_m \sum_{n=1}^\infty \mu_{mN}\Big(\prod_{j=1}^m I_{j,n}^{(k)}\Big) \quad \text{(see the following statement)} \\ &\quad = C_m \mu_{mN}(O_k) < \frac{C_m}{k}. \end{aligned}$$

We can take the constant C_m as $C_m = \prod_{j=1}^m \|K_\alpha(t_j - t_{j-1}, \cdot)\|_\infty$ by Proposition 2.2 (ii)-(c), where $t_0 = 0$, and hence C_m is independent of $k \in \mathbb{N}$. By the argument above, the set $\mathscr{N} := \bigcap_{k=1}^\infty \big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in O_k\big\}$ belongs to $\mathscr{F}_t^0$ and is W_x^α-null set for all $x \in \mathbb{R}^N$, and includes the set $\big\{\omega \in \Omega \;\big|\; \big(\omega(t_1), \ldots, \omega(t_m)\big) \in S\big\}$. □

We will prove Proposition 3.2 in the next subsection.

3.2. Proof of Proposition 3.2

3.2.1. Measurability of functions appearing in the Feynman-Kac formula

Let $t > 0, V \in L^\infty$ and $u \in L^p$. We first show that (i): the function $\omega \mapsto u\bigl(\omega(t)\bigr)$ is $\mathscr{F}_t$-measurable, and that (ii): for all $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$, the function $s \mapsto V\bigl(\omega(s)\bigr)$ is Lebesgue measurable and integrable on $(0,t)$ and (iii): the function $\omega \mapsto \int_0^t V\bigl(\omega(s)\bigr)\,ds$ is $\mathscr{F}_t$-measurable on Ω and W_x^α-integrable on Ω for all $x \in \mathbb{R}^N$.

(i): We can take a sequence $\{u_n\}_n$ in $C_c(\mathbb{R}^N)$ and a Borel null set S in $\mathbb{R}^N$ such that $u_n(y) \to u(y)$ for all $y \in \mathbb{R}^N \setminus S$. By Lemma 3.4 (i), $X_t^{-1}(S) \in \mathscr{F}_t^0$ and $W_x^\alpha\bigl(X_t^{-1}(S)\bigr) = 0$ for all $x \in \mathbb{R}^N$. Hence, $u_n\bigl(\omega(t)\bigr) \to u\bigl(\omega(t)\bigr)$ as $n \to \infty$ for all $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$. Hence, the function $\omega \mapsto u\bigl(\omega(t)\bigr)$ is measurable relative to the completion of $\mathscr{F}_t^0$ with respect to W_x^α for each $x \in \mathbb{R}^N$. Thus, the function $\omega \mapsto u\bigl(\omega(t)\bigr)$ is measurable relative to $\mathscr{F}_t$.

(ii), (iii): There exist a sequence $\{V_n\}_n$ in $C_c(\mathbb{R}^N)$ and a Borel null set S in $\mathbb{R}^N$ such that $V_n(y) \to V(y)$ as $n \to \infty$ for all $y \in \mathbb{R}^N \setminus S$ and $\|V_n\|_\infty \le \|V\|_\infty$ for all $n \in \mathbb{N}$. Let $\Omega_{t,S}$ be as in Lemma 3.4 (ii). Then, by Lemma 3.4 (ii), for all $x \in \mathbb{R}^N$, we have that $W_x^\alpha(\Omega_{t,S}) = 1$ and that for all $\omega \in \Omega_{t,S}$,

$$V_n\bigl(\omega(s)\bigr) \to V\bigl(\omega(s)\bigr) \tag{3.2}$$

as $n \to \infty$ for a.e. $s \in (0,t)$ and

$$\bigl|V_n\bigl(\omega(s)\bigr)\bigr| \le \|V\|_\infty \tag{3.3}$$

for all $n \in \mathbb{N}$ and $s \in (0,t)$. By (3.2) and (3.3), for all $\omega \in \Omega_{t,S}$, the function $s \mapsto V\bigl(\omega(s)\bigr)$ is Lebesgue measurable and integrable on $(0,t)$ (we used the fact that the intersection of the completions of $\mathscr{F}_t^0$ with respect to W_x^α over $\{W_x^\alpha; x \in \mathbb{R}^N\}$ is included in $\mathscr{F}_t$).

Moreover, for all $\omega \in \Omega_{t,S}$, by (3.2), (3.3) and Lebesgue's convergence theorem,

$$\int_0^t V_n\bigl(\omega(s)\bigr)\,ds \to \int_0^t V\bigl(\omega(s)\bigr)\,ds$$

as $n \to \infty$.

Since for all $\omega \in \Omega$ the function $s \mapsto \omega(s)$ is continuous on $[0,\infty)$ except at most countable points, the function $s \mapsto V_n\bigl(\omega(s)\bigr)$ is Riemann integrable on $(0,t)$ for each $n \in \mathbb{N}$ and $\omega \in \Omega$ and hence

$$\sum_{j=1}^m \frac{1}{m} V_n\bigl(\omega\bigl(\tfrac{j}{m}t\bigr)\bigr) \to \int_0^t V_n\bigl(\omega(s)\bigr)\,ds$$

as $m \to \infty$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Since for all $n \in \mathbb{N}$, this Riemann sum is measurable in ω relative to $\mathscr{F}_t^0$, the function $\omega \mapsto \int_0^t V_n\bigl(\omega(s)\bigr)\,ds$ $(n \in \mathbb{N})$ is measurable and hence the function $\omega \mapsto \int_0^t V\big(\omega(s)\big)\,ds$ is measurable relative to the completion of $\mathscr{F}_t^0$ with respect to W_x^α for each $x \in \mathbb{R}^N$. Thus, the function $\omega \mapsto \int_0^t V\big(\omega(s)\big)\,ds$ is measurable relative to $\mathscr{F}_t$.

3.2.2. The Feynman-Kac formula for $V \in C_c(\mathbb{R}^N)$

Next, in the case of $V \in C_c(\mathbb{R}^N)$, we prove the Feynman-Kac formula. It is easily verified that we can apply the Trotter product formula to the C_0-semigroup $e^{-tH_{\alpha,p,V}}$. Hence, for all $u \in L^p$,

$$\big(e^{-\frac{t}{m}H_{\alpha,p}}e^{-\frac{t}{m}V}\big)^m u \to e^{-tH_{\alpha,p,V}}u$$

as $m \to \infty$ in L^p. Hence, there exist a strictly monotone increasing sequence of natural numbers $\{m_k\}_k$ and a Borel null set S in $\mathbb{R}^N$ such that

$$\Big(\big(\exp\big(-\tfrac{t}{m_k}H_{\alpha,p}\big)\exp\big(-\tfrac{t}{m_k}V\big)\big)^{m_k}u\Big)(x) \to \big(e^{-tH_{\alpha,p,V}}u\big)(x)$$

as $k \to \infty$ for all $x \in \mathbb{R}^N \setminus S$. For all $k \in \mathbb{N}$ and $x \in \mathbb{R}^N \setminus S$, by Fubini's theorem and description (iii)-(a) of the Hunt process, we obtain that

$$\begin{aligned}
&\Big(\big(\exp\big(-\tfrac{t}{m_k}H_{\alpha,p}\big)\exp\big(-\tfrac{t}{m_k}V\big)\big)^{m_k}u\Big)(x) \\
&\quad= \int_{\mathbb{R}^{m_k N}} K_\alpha\big(\tfrac{t}{m_k}, x_1 - x\big)\cdots K_\alpha\big(\tfrac{t}{m_k}, x_{m_k} - x_{m_k-1}\big) \\
&\qquad\qquad \times \exp\Big(-\frac{t}{m_k}\sum_{j=1}^{m_k} V(x_j)\Big)u(x_{m_k})\,dx_1\cdots dx_{m_k} \\
&\quad= \int_\Omega \exp\Big(-\frac{t}{m_k}\sum_{j=1}^{m_k} V\big(\omega\big(\tfrac{j}{m_k}t\big)\big)\Big)u\big(\omega(t)\big)\,dW_x^\alpha(\omega).
\end{aligned} \tag{3.4}$$

We have to verify that (a): we could apply Fubini's theorem and (b): the last equality of (3.4) holds. Assertion (a) is ensured by the following estimates:

$$0 \le \exp\Big(-\frac{t}{m_k}\sum_{j=1}^{m_k} V(x_j)\Big) \le e^{t\|V\|_\infty} < \infty$$

for all $k \in \mathbb{N}$ and $x_1, \dots, x_{m_k} \in \mathbb{R}^N$, and

$$\begin{aligned}
0 \le \int_{\mathbb{R}^N} & K_\alpha\big(\tfrac{t}{m_k}, x_1 - x\big)\Big(\int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_2 - x_1\big)\cdots \\
&\Big(\int_{\mathbb{R}^N} K_\alpha(\tfrac{t}{m_k}, x_{m_k} - x_{m_k-1})|u(x_{m_k})|\,dx_{m_k}\Big)\cdots dx_2\Big)\,dx_1
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_1 - x\big) \Big(\int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_2 - x_1\big) \cdots \\
&\qquad \Big(\int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_{m_k} - x_{m_k-1}\big)^{p'} \, dx_{m_k} \Big)^{\frac{1}{p'}} \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}^N} |u(x_{m_k})|^p \, dx_{m_k} \Big)^{\frac{1}{p}} \cdots \, dx_2 \Big) \, dx_1 \\
&= \int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_1 - x\big) \Big(\int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_2 - x_1\big) \cdots \\
&\qquad \Big(\int_{\mathbb{R}^N} K_\alpha\big(\tfrac{t}{m_k}, x_{m_k-1} - x_{m_k-2}\big) \, dx_{m_k-1} \Big) \cdots \, dx_2 \Big) \, dx_1 \\
&\qquad \times \big\| K_\alpha\big(\tfrac{t}{m_k}, \cdot\big) \big\|_{p'} \|u\|_p \\
&= \big\| K_\alpha\big(\tfrac{t}{m_k}, \cdot\big) \big\|_1^{m_k - 1} \big\| K_\alpha\big(\tfrac{t}{m_k}, \cdot\big) \big\|_{p'} \|u\|_p < \infty
\end{aligned}
$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^N$, where p' is the conjugate exponent of p. (Note that $K_\alpha(t, \cdot) \in L^1 \cap L^\infty$ by Proposition 2.2 (ii)-(c) and (d), and hence $K_\alpha(t, \cdot) \in L^q$ for all $q \in [1, \infty]$.)

To verify assertion (b), let $F\colon \mathbb{R}^{mN} \to \mathbb{R}$ be a Lebesgue measurable function. We have only to prove that for all $m \in \mathbb{N}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^{mN}} K_\alpha\big(\tfrac{t}{m}, x_1 - x\big) \cdots K_\alpha\big(\tfrac{t}{m}, x_m - x_{m-1}\big) F(x_1, \ldots, x_m) \, dx_1 \cdots dx_m \\
&\qquad = \int_\Omega F\big(\omega\big(\tfrac{t}{m}\big), \ldots, \omega(t)\big) \, dW_x^\alpha(\omega).
\end{aligned}
$$

Without loss of generality, we may assume that $F \ge 0$. For this F, there exist a sequence of Borel measurable and simple functions $\{F_n\}_n$ and a Borel null set S in $\mathbb{R}^{mN}$ such that

$$0 \le F_n(x_1, \ldots, x_m) \nearrow F(x_1, \ldots, x_m)$$

as $n \to \infty$ for all $(x_1, \ldots, x_m) \in \mathbb{R}^{mN} \setminus S$. We can write F_n as

$$F_n = \sum_{k=1}^{m_n} a_k^{(n)} \chi_{E_k^{(n)}}$$

for each $n \in \mathbb{N}$, where $E_k^{(n)}$ is a Borel measurable subset of $\mathbb{R}^{mN}$ and $E_k^{(n)} \cap E_{k'}^{(n)} = \emptyset$ if $k \ne k'$. Hence, we have

$$
\begin{aligned}
(3.5)\quad &\int_{\mathbb{R}^{mN}} K_\alpha\big(\tfrac{t}{m}, x_1 - x\big) \cdots K_\alpha\big(\tfrac{t}{m}, x_m - x_{m-1}\big) \\
&\qquad\qquad \times F_n(x_1, \ldots, x_m) \, dx_1 \cdots dx_m \\
&= \sum_{k=1}^{m_n} a_k^{(n)} \int_{E_k^{(n)}} K_\alpha\big(\tfrac{t}{m}, x_1 - x\big) \cdots K_\alpha\big(\tfrac{t}{m}, x_m - x_{m-1}\big) \, dx_1 \cdots dx_m
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{m_n} a_k^{(n)} W_x^\alpha\Big(\big\{\omega \in \Omega \;\big|\; \big(\omega\big(\tfrac{t}{m}\big), \omega\big(\tfrac{2t}{m}\big), \dots, \omega(t)\big) \in E_k^{(n)}\big\}\Big) \\
&= \int_\Omega \sum_{k=1}^{m_n} a_k^{(n)} \chi_{E_k^{(n)}}\big(\omega\big(\tfrac{t}{m}\big), \omega\big(\tfrac{2t}{m}\big), \dots, \omega(t)\big)\, dW_x^\alpha(\omega) \\
&= \int_\Omega F_n\big(\omega\big(\tfrac{t}{m}\big), \omega\big(\tfrac{2t}{m}\big), \dots, \omega(t)\big)\, dW_x^\alpha(\omega)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. By Lemma 3.4 (iii), for all $m \in \mathbb{N}$ and $x \in \mathbb{R}^N$,

$$
F_n\big(\omega\big(\tfrac{t}{m}\big), \dots, \omega(t)\big) \nearrow F\big(\omega\big(\tfrac{t}{m}\big), \dots, \omega(t)\big)
$$

as $n \to \infty$ for W_x^α-a.e. $\omega \in \Omega$. Hence, by applying the monotone convergence theorem to both sides of (3.5), the equality

$$
\begin{aligned}
&\int_{\mathbb{R}^{mN}} K_\alpha\big(\tfrac{t}{m}, x_1 - x\big) \cdots K_\alpha\big(\tfrac{t}{m}, x_m - x_{m-1}\big) F(x_1, \dots, x_m)\, dx_1 \cdots dx_m \\
&\qquad = \int_\Omega F\big(\omega\big(\tfrac{t}{m}\big), \dots, \omega(t)\big)\, dW_x^\alpha(\omega)
\end{aligned}
$$

holds for all $m \in \mathbb{N}$ and $x \in \mathbb{R}^N$.

Now, we can apply Lebesgue's convergence theorem to the rightmost side of (3.4). In fact, since the function $s \mapsto \omega(s)$ is continuous on $[0, \infty)$ except at most countable points for all $\omega \in \Omega$, the function $s \mapsto V\big(\omega(t)\big)$ is Riemann integrable on $(0, t)$. Hence,

$$
\exp\Big(-\frac{t}{m_k} \sum_{j=1}^{m_k} V\big(\omega\big(\tfrac{j}{m_k} t\big)\big)\Big) u\big(\omega(t)\big) \to \exp\Big(-\int_0^t V\big(\omega(s)\big)\, ds\Big) u\big(\omega(t)\big)
$$

as $k \to \infty$ for all $\omega \in \Omega$. Moreover, the estimate

$$
\Big|\exp\Big(-\frac{t}{m_k} \sum_{j=1}^{m_k} V\big(\omega\big(\tfrac{j}{m_k} t\big)\big)\Big) u\big(\omega(t)\big)\Big| \le e^{t\|V\|_\infty} \big|u\big(\omega(t)\big)\big|
$$

holds for all $k \in \mathbb{N}$ and $\omega \in \Omega$, and it is verified that the function $\omega \mapsto u\big(\omega(t)\big)$ is W_x^α-integrable on Ω for all $x \in \mathbb{R}^N$ by the following estimate.

$$
\begin{aligned}
\int_\Omega \big|u\big(\omega(t)\big)\big|\, dW_x^\alpha(\omega) &\le \Big(\int_\Omega \big|u\big(\omega(t)\big)\big|^p\, dW_x^\alpha(\omega)\Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathbb{R}^N} K_\alpha(t, x - y) |u(y)|^p\, dy\Big)^{\frac{1}{p}} \\
&\le \|K_\alpha(t, \cdot)\|_1^{\frac{1}{p}} \|u\|_p < \infty
\end{aligned}
$$

by Young's inequality. Hence, we can apply Lebesgue's convergence theorem to the rightmost side of (3.4) and conclude that

$$\big(e^{-tH_{\alpha,p,V}}u\big)(x) = \int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big)u\big(\omega(t)\big)\,dW_x^\alpha(\omega)$$

for a.e. $x \in \mathbb{R}^N$.

3.2.3. The Feynman-Kac formula for $V \in L^\infty(\mathbb{R}^N)$

Finally, in the case of $V \in L^\infty$, we prove the Feynman-Kac formula. We can take a sequence $\{V_n\}_n$ in $C_c(\mathbb{R}^N)$ and a Borel null set S in $\mathbb{R}^N$ such that $\|V_n\|_\infty \le \|V\|_\infty$ for all $n \in \mathbb{N}$ and $V_n(y) \to V(y)$ as $n \to \infty$ for all $y \in \mathbb{R}^N \setminus S$. It is easily verified that for all $t > 0$ and $u \in L^p$, we can use Trotter-Kato approximation theorem. Hence,

$$e^{-tH_{\alpha,p,V_n}}u \to e^{-tH_{\alpha,p,V}}u$$

as $n \to \infty$ in L^p. There exist a Borel null set S' including S and a strictly monotone increasing sequence of natural numbers $\{n_k\}_k$ such that

$$\big(\exp(-tH_{\alpha,p,V_{n_k}})u\big)(x) \to \big(\exp(-tH_{\alpha,p,V})u\big)(x)$$

as $k \to \infty$ for all $x \in \mathbb{R}^N \setminus S'$. By the result in the case of $V \in C_c(\mathbb{R}^N)$, for all $k \in \mathbb{N}$, we have

$$\big(\exp(-tH_{\alpha,p,V_{n_k}})u\big)(x) = \int_\Omega \exp\Big(-\int_0^t V_{n_k}\big(\omega(s)\big)\,ds\Big)u\big(\omega(s)\big)\,dW_x^\alpha(\omega) \tag{3.6}$$

for a.e. $x \in \mathbb{R}^N$. It is clear that we may assume that this equality holds for all $x \in \mathbb{R}^N \setminus S'$, where S' is the set above.

Now, we will prove that we can apply Lebesgue's convergence theorem to the right-hand side of (3.6). In fact, for all $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$,

$$V_{n_k}\big(\omega(s)\big) \to V\big(\omega(s)\big)$$

holds as $k \to \infty$ for a.e. $s > 0$ by Lemma 3.4 (ii), and the estimate

$$\big|V_{n_k}\big(\omega(s)\big)\big| \le \|V\|_\infty$$

is clear for all $k \in \mathbb{N}, \omega \in \Omega$ and $s > 0$ by the way of taking of V_{n_k}. Hence, by Lebesgue's convergence theorem, for all $x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$, we have

$$\int_0^t V_{n_k}\big(\omega(s)\big)\,ds \to \int_0^t V\big(\omega(s)\big)\,ds$$

as $k \to \infty$. Furthermore, the estimate

$$\left|\exp\Big(-\int_0^t V_{n_k}\big(\omega(s)\big)\,ds\Big)u\big(\omega(t)\big)\right| \le e^{t\|V\|_\infty}\big|u\big(\omega(t)\big)\big|$$

holds for all $k \in \mathbb{N}, x \in \mathbb{R}^N$ and W_x^α-a.e. $\omega \in \Omega$, and the function $\omega \mapsto u\big(\omega(t)\big)$ is W_x^α-integrable on Ω for all $x \in \mathbb{R}^N$. Hence, applying Lebesgue's convergence theorem to the right-hand side of (3.6), it converges to

$$\int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big)u\big(\omega(t)\big)\,dW_x^\alpha(\omega)$$

for all $x \in \mathbb{R}^N$. Thus, we obtain

$$\big(e^{-tH_{\alpha,p,V}}u\big)(x) = \int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big)u\big(\omega(t)\big)\,dW_x^\alpha(\omega)$$

for a.e. $x \in \mathbb{R}^N$. The proof of this proposition is completed.

3.3. L^p-L^q estimates for $e^{-tH_{\alpha,2,V}}$

The next lemma is used to prove L^p-L^q estimates for $e^{-tH_{\alpha,2,V}}$, which is essentially Khas'minskii's lemma (*cf.* [4, Corollary 3.6]).

Lemma 3.5. *Let $\alpha \in (0,1), V \ge 0, V \in \hat{K}_{N,\alpha}$ and $c_{N,\alpha}(V) < 1$. Then, for any $\mu \in \big(c_{N,\alpha}(V), 1\big)$, there exists a $t_{\mu,V} > 0$ such that*

$$\left\|V\int_0^{t_{\mu,V}} U_{\alpha,1}(t)\,dt\right\| < \mu.$$

Moreover, for $M_\mu := \frac{1}{1-\mu} > 1$ and $b_{\mu,V} := \frac{1}{t_{\mu,V}}\log M_\mu > 0$, the estimate

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} E_x^\alpha\Big(\exp\Big(\int_0^t V\circ X_s\,ds\Big)\Big) \le M_\mu e^{tb_{\mu,V}}$$

holds for all $t \ge 0$, where E_x^α denotes the expectation for the probability measure W_x^α.

Proof. By Proposition 2.20 (iii) and Remark 2.21, we can take a $t_{\mu,V}$ satisfying the condition in the lemma.

By replacing V with $V \wedge n$ $(n \in \mathbb{N})$ and taking the limit as $n \to \infty$, it is seen that we may assume that V is bounded above. First, by a similar argument as in the proof of [5, Lemma 1], we can prove that

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} E_x^\alpha\Big(\exp\Big(\int_0^t V\circ X_s\,ds\Big)\Big) < \frac{1}{1-\mu}$$

for all $t \in [0, t_{\mu,V}]$. Next, by a similar argument in the last paragraph of the proof of [4, Proposition 3.5], we obtain the estimate

$$\operatorname*{ess.sup}_{x\in\mathbb{R}^N} E_x^\alpha\Big(\exp\Big(\int_0^t V\circ X_s\,ds\Big)\Big) \le M_\mu e^{tb_{\mu,V}}$$

for all $t \ge 0$. □

The Feynman-Kac formula and Lemma 3.5 yield the following L^p-L^q estimates. To state these estimates, we use the symbol, L^p-L^q norm $\|\cdot\|_{p,q}$. In what follows, we use this symbol only for linear operators whose domains include the Schwartz space $\mathcal{S}$, and so we define in this paper,

$$\|T\|_{p,q} := \sup\Big\{\frac{\|Tu\|_q}{\|u\|_p}\ \Big|\ u\in\mathcal{S}, \|u\|_p \le 1\Big\}$$

for all $1 \le p \le q \le \infty$. If $\|T\|_{p,q} < \infty$ for some $1 \le p \le q < \infty$, then T can be extended to an operator belonging to $\mathcal{L}(L^p, L^q)$ since $\mathcal{S}$ is dense in L^p.

Proposition 3.6. *Let $\alpha \in (0,1)$ and $1 \le p \le q \le \infty$. Assume that $V_- \in \hat{K}_{N,\alpha}, c_{N,\alpha}(V_-) < 1$ and V_+ is U_α-admissible, then V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ by Theorem 2.14. Moreover, there exist constants $M = M(V_-, p, q)$, $b = b(V_-, p, q)$ such that*

$$\big\|e^{-tH_{\alpha,2,V}}\big\|_{p,q} \le Mt^{-\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})}e^{tb}$$

for all $t > 0$.

Proof. Although this proposition is proved in a similar way as in [18, Proposition 6.3], we give a proof since this proposition is important to prove L^p-spectral inclusion and independence in the next section. We divide the proof into several cases. As is stated in the proof of [18, Proposition 6.3], the proof of [5, Lemma 2] is a reference in Case 1 and 2 for $\alpha = 1$, the way of which is of use also in this proof.

Case 1: $p = q = \infty$. In this case, by the Feynman-Kac formula and Lemma 3.5, it is clear that for all $t > 0$ and $u \in L^2 \cap L^\infty$,

$$\big|\big(e^{-tH_{\alpha,2,V}}u\big)(x)\big| \le M_\mu e^{tb_{\mu,V_-}}\|u\|_\infty$$

for a.e. $x \in \mathbb{R}^N$, where M_μ and b_{μ,V_-} is as in Lemma 3.5. Hence,

$$\|e^{-tH_{\alpha,2,V}}\|_{\infty,\infty} \le M_\mu e^{tb_{\mu,V_-}}$$

holds for all $t > 0$.

In the rest of the proof, we write simply c instead of $c_{N,\alpha}(V_-)$.

Case 2: $\frac{\mu}{\mu-c} < p < q = \infty$. If $u \in L^2 \cap L^p$, then we obtain that

$$\begin{aligned}
\big|\big(e^{-tH_{\alpha,2,V}}u\big)(x)\big| &\le \int_\Omega \exp\Big(\int_0^t V_-\big(\omega(s)\big)\,ds\Big)\big|u\big(\omega(t)\big)\big|\,dW_x^\alpha(\omega) \\
&\qquad\qquad\qquad \text{(by the Feynman-Kac formula)} \\
&\le \Big(\int_\Omega \exp\Big(p'\int_0^t V_-\big(\omega(s)\big)\,ds\Big)\,dW_x^\alpha(\omega)\Big)^{\frac{1}{p'}} \\
&\qquad\qquad \times \Big(\int_\Omega \big|u\big(\omega(t)\big)\big|^p\,dW_x^\alpha(\omega)\Big)^{\frac{1}{p}} \\
&\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
&= \Big(E_x^\alpha\Big[\exp\Big(\int_0^t (p'V_-)\circ X_s\,ds\Big)\Big]\Big)^{\frac{1}{p'}} \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}^N} K_\alpha(t,x-y)|u(y)|^p\,dy\Big)^{\frac{1}{p}} \\
&\le \Big(M_\mu \exp(tb_{\mu,p'V_-})\Big)^{\frac{1}{p'}} \cdot t^{-\frac{N}{2\alpha}\cdot\frac{1}{p}}\|K_\alpha(1,\cdot)\|_\infty^{\frac{1}{p}}\|u\|_p \\
&\qquad \text{(by Lemma 3.5 and Proposition 2.2 (ii)-(a))} \\
&= M't^{-\frac{N}{2\alpha}\cdot\frac{1}{p}}e^{tb'}\|u\|_p
\end{aligned}$$

for a.e. $x \in \mathbb{R}^N$ (the p' above is the conjugate exponent of p and the constants M' and b' are independent of $t > 0$). The reason why we could apply Lemma 3.5 above is that $c_{N,\alpha}(p'V_-) = p'c_{N,\alpha}(V_-) < \mu$ by $\frac{\mu}{\mu-c_{N,\alpha}(V_-)} < p$. Hence, we have

$$\|e^{-tH_{\alpha,2,V}}\|_{p,\infty} \le M't^{-\frac{N}{2\alpha}\cdot\frac{1}{p}}e^{tb'}$$

for all $t > 0$.

Case 3: $1 = p \le q < \frac{\mu}{c}$. The self-adjointness of $e^{-tH_{\alpha,2,V}}$ $(t > 0)$ shows that

$$\|e^{-tH_{\alpha,2,V}}\|_{1,q} = \|e^{-tH_{\alpha,2,V}}\|_{q',\infty},$$

where q' is the conjugate exponent of q. Since $\frac{\mu}{\mu-c} < q' \le \infty$ by $1 \le q < \frac{\mu}{c}$, we can apply the result in Case 1 if $q = 1$ or in Case 2 if $1 < q < \frac{\mu}{c}$ to the right-hand side of this equality. Hence, there exist constants M and b such that

$$\|e^{-tH_{\alpha,2,V}}\|_{q',\infty} \le Mt^{-\frac{N}{2\alpha}\cdot\frac{1}{q'}}e^{tb}$$

for all $t > 0$. Thus,

$$\|e^{-tH_{\alpha,2,V}}\|_{1,q} \le Mt^{-\frac{N}{2\alpha}(1-\frac{1}{q})}e^{tb}$$

holds for all $t > 0$.

Case 4: $1 \le p \le q \le \infty$ and $1 \le \frac{q}{p} < \frac{\mu}{c}$. Riesz-Thorin convexity theorem implies the estimate

$$\|e^{-tH_{\alpha,2,V}}\|_{p,q} \le \|e^{-tH_{\alpha,2,V}}\|_{1,\frac{q}{p}}^{\frac{1}{p}} \|e^{-tH_{\alpha,2,V}}\|_{\infty,\infty}^{1-\frac{1}{p}}$$

for all $t > 0$. By $1 \le \frac{q}{p} < \frac{\mu}{c}$, we can apply the result in Case 3 to $\|e^{-tH_{\alpha,2,V}}\|_{1,\frac{q}{p}}$. Applying the result in Case 3 and in Case 1 to $\left(1, \frac{q}{p}\right)$-norm and (∞, ∞)-norm, respectively, we have

$$\|e^{-tH_{\alpha,2,V}}\|_{p,q} \le Mt^{-\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})} e^{tb}$$

for some constants M and b which are independent of $t > 0$, and for all $t > 0$.

Case 5: $1 \le p \le q = \infty$. Let r be in $\left(1, \frac{\mu}{c}\right)$. Since the case of $p > \frac{\mu}{\mu - c}$ is treated in Case 1 and 2, we may assume that $1 \le p \le \frac{\mu}{\mu-c}$. Then, there exists an $n_0 \in \mathbb{N}$ such that $pr^{n_0} > \frac{\mu}{\mu-c}$. Hence, we have

$$\begin{aligned}
\|e^{-tH_{\alpha,2,V}}\|_{p,\infty} &= \left\| \left(\exp\left(-\tfrac{t}{n_0+1} H_{\alpha,2,V}\right)\right)^{n_0+1} \right\|_{p,\infty} \\
&\le \left\|\exp\left(-\tfrac{t}{n_0+1} H_{\alpha,2,V}\right)\right\|_{p_{n_0},\infty} \left\|\exp\left(-\tfrac{t}{n_0+1} H_{\alpha,2,V}\right)\right\|_{p_{n_0-1},p_{n_0}} \cdots \\
&\qquad \times \left\|\exp\left(-\tfrac{t}{n_0+1} H_{\alpha,2,V}\right)\right\|_{p_0,p_1}
\end{aligned}$$

for all $t > 0$, where $p_j := pr^j$ $(j = 0, 1, \ldots, n_0)$. Applying the result in Case 2 and in Case 4 to the (p_{n_0}, ∞)-norm and the other norms in the right-hand side of this inequality $\left(\text{by } \frac{p_j}{p_{j-1}} = r \in \left(1, \frac{\mu}{c}\right)\text{, we can use the result in Case 4}\right)$, respectively, we obtain the asserted estimate of this proposition.

Case 6: $1 \le p \le q \le \infty$. By Riesz-Thorin convexity theorem, the estimate

$$\|e^{-tH_{\alpha,2,V}}\|_{p,q} \le \|e^{-tH_{\alpha,2,V}}\|_{1,1}^{\frac{1}{q}} \|e^{-tH_{\alpha,2,V}}\|_{r,\infty}^{1-\frac{1}{q}}$$

holds for all $t > 0$, where $r := \left(1 - \frac{1}{q}\right) \Big/ \left(\frac{1}{p} - \frac{1}{q}\right)$. By using the results in Case 4 and 5, we obtain that there exist constants M and b such that

$$\|e^{-tH_{\alpha,2,V}}\|_{p,q} \le Mt^{-\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})} e^{tb}$$

for all $t > 0$. □

§4. L^p-spectral inclusion and independence

4.1. The case of $e^{-tH_{\alpha,p,V}}$ on $\mathbb{R}^N$

First, we prove L^p-spectral inclusion for $H_{\alpha,p,V}$ under the same assumption as in Theorem 2.14. As in Section 2, $U_\alpha(t) = e^{-t(-\Delta)^\alpha}$ for each $\alpha \in (0, 1]$ and $t \ge 0$.

Theorem 4.1 (*cf.* [6, Proposition 2.1]). *Suppose that $V_- \in \hat{K}_{N,\alpha}$ satisfies $c_{N,\alpha}(V_-) < 1$ and V_+ be U_α-admissible. Then V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ by Theorem 2.14, and so the operator $H_{\alpha,p,V}$ makes sense. Moreover, let $H_{\alpha,\infty,V}$ be defined by $H'_{\alpha,1,V}$. Then, for any $1 \le q \le p \le 2$ or $2 \le p \le q \le \infty$, we have $\rho(H_{\alpha,q,V}) \subset \rho(H_{\alpha,p,V})$ and the consistency*

$$(H_{\alpha,q,V} - \zeta)^{-1}|_{L^p \cap L^q} = (H_{\alpha,p,V} - \zeta)^{-1}|_{L^p \cap L^q}$$

holds for every $\zeta \in \rho(H_{\alpha,q,V})$.

Proof. Let $1 \le p < q \le \infty$ and $t > 0$. Since $\|e^{-tH_{\alpha,p,V}}\|_{p,q} < \infty$ by Proposition 3.6, it is proved that if $u \in D(H_{\alpha,p,V})$, then $e^{-tH_{\alpha,p,V}}u \in D(H_{\alpha,q,V})$ and $H_{\alpha,q,V}e^{-tH_{\alpha,p,V}}u = e^{-tH_{\alpha,p,V}}H_{\alpha,p,V}u$. Hence, in the same way as in the proof of [6, Proposition 2.1], the consistency

$$(H_{\alpha,q,V} - \zeta)^{-1}\big|_{L^p \cap L^q} = (H_{\alpha,p,V} - \zeta)^{-1}\big|_{L^p \cap L^q}$$

is shown for all $\zeta \in \rho(H_{\alpha,p,V}) \cap \rho(H_{\alpha,q,V})$.

Next let $1 \le q \le p \le 2$ and $\zeta \in \rho(H_{\alpha,q,V})\big(= \rho(H_{\alpha,q',V})\big)$. We have only to prove the assertion of this theorem in this case by duality. By the result above, $(H_{\alpha,q,V} - \zeta)^{-1}|_{L^q \cap L^{q'}} = (H_{\alpha,q',V} - \zeta)^{-1}|_{L^q \cap L^{q'}}$. Riesz-Thorin convexity theorem implies $\|(H_{\alpha,q,V} - \zeta)^{-1}\|_{p,p} < \infty$. By [1, Proposition 2.3], we obtain $\zeta \in \rho(H_{\alpha,p,V})$ and $(H_{\alpha,q,V} - \zeta)^{-1}|_{L^p \cap L^q} = (H_{\alpha,p,V} - \zeta)^{-1}|_{L^p \cap L^q}$. □

Next we prove L^p-spectral independence. Since the kernel $K_\alpha(t,x)$ does not decay exponentially as $|x| \to \infty$ for any $\alpha \in (0,1)$ (*cf.* [14, Proposition 2.1]), it is hard to prove L^p-spectral independence without a strict condition on space dimension N, α and potentials V.

Theorem 4.2. *Let $N = 1$ and $\frac{1}{2} < \alpha < 1$. Assume the following three conditions:*

(i) *$V_- \in \hat{K}_{1,\alpha}$,*

(ii) *V_+ is U_α-admissible,*

(iii) *V is $(-\Delta)^\alpha$-bounded with relative bound < 1.*

Then, V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ and $\sigma(H_{\alpha,p,V})$ is independent of $p \in [1,\infty)$.

We prepare lemmas and propositions to prove this theorem. Most of them correspond to the ones in [6] for the case of $\alpha = 1$. Hempel and Voigt used the weight function $e^{-\varepsilon \cdot x}$ ($\varepsilon, x \in \mathbb{R}^N$), however we have to define another weight function because of the polynomial decay of $K_\alpha(t,x)$.

Let c be in $(\frac{1}{2}, \alpha)$ and fixed. For all $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$, we define the weight function $w_{\varepsilon,z}$ by

$$w_{\varepsilon,z}(x) := (1 + \varepsilon|x - z|^2)^c \quad (x \in \mathbb{R}^N).$$

We use the same symbol $w_{\varepsilon,z}$ for the function $w_{\varepsilon,z}\colon \mathbb{R}^N \to \mathbb{R}$ and also for the associated maximal multiplication operator in L^2 defined by $w_{\varepsilon,z}$. The same convention is valid for $w_{\varepsilon,z}^{-1}$. Note that the domain of the operator $w_{\varepsilon,z}$ includes the Schwartz space $\mathcal{S}$ for all $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$.

In this subsection, all the lemmas and propositions do not require $N = 1$. More precisely, for an arbitrary N, assume the following conditions:

$$\begin{cases} \text{(i) } \alpha \in \left(\frac{1}{2}, 1\right) \text{ and } c \in \left(\frac{1}{2}, \alpha\right), \\ \text{(ii) } V_- \in \hat{K}_{N,\alpha},\ c_{N,\alpha}(V_-) = 0, \\ \text{(iii) } V_+ \text{ is } U_\alpha\text{-admissible}, \\ \text{(iv) } V \text{ is } (-\Delta)^\alpha\text{-bounded with relative bound} < 1, \end{cases} \tag{4.1}$$

then all of them is proved. This assumption is more general than the one in the theorem above since for any $\frac{1}{2} < \alpha < 1$ and $V \in \hat{K}_{1,\alpha}$, we have already proved $c_{1,\alpha}(V) = 0$ in Proposition 2.13 (i).

Lemma 4.3. *For all $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$, the weight function $w_{\varepsilon,z}$ satisfies the following estimates:*
(i) $0 \le w_{\varepsilon,z}(x)^{-1} w_{\varepsilon,z}(y) \le 2^c (1 + \varepsilon|x-y|^2)^c \quad (x, y \in \mathbb{R}^N)$,
(ii) $|w_{\varepsilon,z}(x)^{-1} w_{\varepsilon,z}(y) - 1| \le 2^{1+2c} c\sqrt{\varepsilon}(1 + |x-y|^2)^c \quad (x, y \in \mathbb{R}^N)$.

The inequality of (i) is Peetre's inequality (2.6) in [8] for the case of $s = c$.

Proof. (i) Put $x' := x - z, y' := y - z$, then $x' - y' = x - y$. The estimate is verified by the following straightforward calculation:

$$\begin{aligned} 0 \le w_{\varepsilon,z}(x)^{-\frac{1}{c}} w_{\varepsilon,z}(y)^{\frac{1}{c}} &= \frac{1 + \varepsilon|y'|^2}{1 + \varepsilon|x'|^2} \\ &\le \frac{1 + \varepsilon(|x' - y'| + |x'|)^2}{1 + \varepsilon|x'|^2} \\ &= \frac{1 + \varepsilon(|x-y|^2 + 2|x-y||x'| + |x'|^2)}{1 + \varepsilon|x'|^2} \\ &\le \frac{1 + 2\varepsilon(|x-y|^2 + |x'|^2)}{1 + \varepsilon|x'|^2} \\ &\le \frac{1 + 2\varepsilon|x'|^2}{1 + \varepsilon|x'|^2} + 2\varepsilon\frac{|x-y|^2}{1 + \varepsilon|x'|^2} \\ &\le 2 + 2\varepsilon|x-y|^2. \end{aligned}$$

Thus, $0 \le w_{\varepsilon,z}(x)^{-1} w_{\varepsilon,z}(y) \le 2^c(1 + \varepsilon|x-y|^2)^c$ for all $x, y \in \mathbb{R}^N$.

(ii) We define the function f by $f(u) := (1 + \varepsilon u^2)^c$ for $u \ge 0$. Taking any $u, v \ge 0$, by the mean value theorem, for some ξ $(u \le \xi \le v$ or $v \le \xi \le u)$, we have

$$\begin{aligned}|f(u)-f(v)| &= 2c\varepsilon(1+\varepsilon\xi^2)^{c-1}\xi|u-v| \\ &\le 2c\sqrt{\varepsilon}(1+\varepsilon\xi^2)^{c-\frac{1}{2}}|u-v|.\end{aligned}$$

If $|y-z| \le 2|x-z|$, then for some ξ ($|y-z| \le \xi \le |x-z|$ or $|x-z| \le \xi \le |y-z|$),

$$\begin{aligned}|w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)-1| &= \frac{1}{(1+\varepsilon|x-z|^2)^c}|f(|y-z|)-f(|x-z|)| \\ &\le \frac{1}{(1+\varepsilon|x-z|^2)^c}\cdot 2c\sqrt{\varepsilon}(1+\varepsilon\xi^2)^{c-\frac{1}{2}}\big||y-z|-|x-z|\big| \\ &\le \frac{\big(1+\varepsilon(2|x-z|)^2\big)^c}{(1+\varepsilon|x-z|^2)^c}\cdot 2c\sqrt{\varepsilon}|x-y| \\ &\le 2^{1+2c}c\sqrt{\varepsilon}|x-y| \\ &< 2^{1+2c}c\sqrt{\varepsilon}(1+|x-y|^2)^c.\end{aligned}$$

For the last inequality, we used $c > \frac{1}{2}$.

If $|y-z| > 2|x-z|$, then $|x-y| \ge |y-z|-|x-z| > \frac{1}{2}|y-z|$. Hence, for some ξ ($|x-z| \le \xi \le |y-z|$),

$$\begin{aligned}|w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)-1| &\le \frac{1}{(1+\varepsilon|x-z|^2)^c}\cdot 2c\sqrt{\varepsilon}(1+\varepsilon\xi^2)^{c-\frac{1}{2}}|x-y| \\ &\le 2c\sqrt{\varepsilon}(1+4|x-y|^2)^{c-\frac{1}{2}}|x-y| \\ &\le 2c\sqrt{\varepsilon}(1+4|x-y|^2)^c \\ &\le 2^{1+2c}c\sqrt{\varepsilon}(1+|x-y|^2)^c.\end{aligned}$$

Thus, $|w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)-1| \le 2^{1+2c}c\sqrt{\varepsilon}(1+|x-y|^2)^c$ for all $\varepsilon \in [0,1]$ and $x,y,z \in \mathbb{R}^N$. $\square$

The following lemma is of use together with Proposition 3.6 in the proof of Proposition 4.6.

Lemma 4.4 (*cf.* [6, Lemma 3.4])**.** *Let p and q be in $[1,\infty]$ with $p \le q$. Then, for any $\eta > 0$, there exists a constant $C_\eta = C_\eta(p,q) > 0$ such that*

$$\|w_{\varepsilon,z}^{-1}e^{-t(-\Delta)^\alpha}w_{\varepsilon,z}\|_{p,q} \le C_\eta t^{-\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})}e^{\eta t}$$

for all $t > 0, \varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$.

Proof. For all $u \in \mathcal{S}$, it is easy to see that

$$\big(w_{\varepsilon,z}^{-1}e^{-t(-\Delta)^\alpha}w_{\varepsilon,z}u\big)(x) = \int_{\mathbb{R}^N} w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)K_\alpha(t,x-y)u(y)\,dy$$

for all $x \in \mathbb{R}^N$. By the estimate (2.1), Lemma 4.3 (i) and Young's inequality, we obtain for $r \in [1,\infty)$ defined by $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$ and $t > 0$,

$$
\begin{aligned}
&\|w_{\varepsilon,z}^{-1} e^{-t(-\Delta)^\alpha} w_{\varepsilon,z}\|_{p,q}^r \\
&\quad \le 2^c C_\alpha \int_{\mathbb{R}^N} \left[(1+|x|^2)^c \cdot \frac{t}{(t^{\frac{1}{\alpha}} + |x|^2)^{\frac{N}{2}+\alpha}} \right]^r dx \\
&\quad = C t^{-\frac{N}{2\alpha}(r-1)} \int_{\mathbb{R}^N} \left[(1 + t^{\frac{1}{\alpha}} |x|^2)^c \cdot \frac{1}{(1+|x|^2)^{\frac{N}{2}+\alpha}} \right]^r dx \\
&\quad = C t^{-\frac{N}{2\alpha}(r-1)} \Big(\int_{|x|\le t^{-\frac{1}{2\alpha}}} + \int_{|x|\ge t^{-\frac{1}{2\alpha}}} \Big) \Bigg[(1 + t^{\frac{1}{\alpha}}|x|^2)^c \\
&\qquad\qquad\qquad\qquad \times \frac{1}{(1+|x|^2)^{\frac{N}{2}+\alpha}} \Bigg]^r dx \\
&\quad \le C t^{-\frac{N}{2\alpha}(r-1)} \Bigg(2^{cr} \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^{(\frac{N}{2}+\alpha)r}}\, dx \\
&\qquad\qquad\qquad + 2^{cr} t^{\frac{cr}{\alpha}} \int_{\mathbb{R}^N} \frac{|x|^{2cr}}{(1+|x|^2)^{(\frac{N}{2}+\alpha)r}}\, dx \Bigg) \\
&\quad \le C t^{-\frac{N}{2\alpha}(r-1)} (1 + t^{\frac{cr}{\alpha}}).
\end{aligned}
$$

Hence, for any $\eta > 0$ there exists a constant $C_\eta > 0$ such that

$$
\|w_{\varepsilon,z}^{-1} e^{-t(-\Delta)^\alpha} w_{\varepsilon,z}\|_{p,q} \le C_\eta t^{-\frac{N}{2\alpha}(\frac{1}{p}-\frac{1}{q})} e^{\eta t}
$$

for all $t > 0, \varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$. □

By using the following lemma and propositions, we can prove Theorem 4.2.

Lemma 4.5 (*cf.* [6, Lemma 3.6]). *Let $r \in \left(1, \frac{\alpha}{c}\right)$ and r' be the conjugate exponent of r. Assume that V satisfies the assumption* (4.1). *Then, $r'V$ is U_α-admissible and for any $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$,*

$$
\|w_{\varepsilon,z}^{-1} e^{-tH_{\alpha,2,V}} w_{\varepsilon,z}\|_{p,q} \le \|w_{\varepsilon,z}^{-r} e^{-t(-\Delta)^\alpha} w_{\varepsilon,z}^r\|_{p,q}^{\frac{1}{r}} \|e^{-tH_{\alpha,2,r'V}}\|_{p,q}^{\frac{1}{r'}}.
$$

Proof. Although the way of the proof is similar as in the proof of [6, Lemma 3.6], we will reproduce it here for the reader's convenience. We first verify that $r'V$ is U_α-admissible. Under the assumption (4.1), $r'V_- \in \hat{K}_{N,\alpha}$ and $c_{N,\alpha}(r'V_-) = r'c_{N,\alpha}(V_-) = 0$, hence by Theorem 2.14, $r'V_-$ is U_α-admissible. On the other hand, by [18, Remark 2.3 (a)], $r'V_+$ is U_α-admissible. Thus, $r'V$ is U_α-admissible.

Let $\varepsilon \in [0,1], z \in \mathbb{R}^N$ and $u \in \mathcal{S}, v := w_{\varepsilon,z}u$ $(\in \mathcal{S})$. By using the Feynman-Kac formula, the inequality

$$\begin{aligned}
\big|\big(e^{-tH_{\alpha,2,V}}v\big)(x)\big| &= \Big|\int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big)v(\omega(t))\,dW_x^\alpha(\omega)\Big| \\
&\le \int_\Omega \exp\Big(-\int_0^t V\big(\omega(s)\big)\,ds\Big)\big|u\big(\omega(t)\big)\big|^{\frac{1}{r'}} \\
&\qquad \times \big|w_{\varepsilon,z}\big(\omega(t)\big)^r u\big(\omega(t)\big)\big|^{\frac{1}{r}}\,dW_x^\alpha(\omega) \\
&\le \Big(\int_\Omega \exp\Big(-r'\int_0^t V\big(\omega(s)\big)\,ds\Big)\big|u\big(\omega(t)\big)\big|\,dW_x^\alpha(\omega)\Big)^{\frac{1}{r'}} \\
&\qquad \times \Big(\int_\Omega w_{\varepsilon,z}\big(\omega(t)\big)^r\big|u\big(\omega(t)\big)\big|\,dW_x^\alpha(\omega)\Big)^{\frac{1}{r}} \\
&\qquad\qquad\qquad\qquad \text{(by Hölder's inequality)} \\
&= \big(\exp(-tH_{\alpha,2,r'V})|u|\big)(x)^{\frac{1}{r'}}\big(e^{-t(-\Delta)^\alpha}(w_{\varepsilon,z}^r|u|)\big)(x)^{\frac{1}{r}}
\end{aligned}$$

holds for a.e. $x \in \mathbb{R}^N$. Multiplying by $w_{\varepsilon,z}$, taking q-th powers and integrating, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\big|w_{\varepsilon,z}(x)^{-1}\big(e^{-tH_{\alpha,2,V}}(w_{\varepsilon,z}u)\big)(x)\big|^q\,dx \\
&\quad \le \int_{\mathbb{R}^N}\big(\exp(-tH_{\alpha,2,r'V})|u|\big)(x)^{\frac{q}{r'}} \\
&\qquad\qquad \times \big(w_{\varepsilon,z}^{-r}e^{-t(-\Delta)^\alpha}(w_{\varepsilon,z}^r|u|)\big)(x)^{\frac{q}{r}}\,dx \\
&\quad \le \Big(\int_{\mathbb{R}^N}\big(\exp(-tH_{\alpha,2,r'V})|u|\big)(x)^q\,dx\Big)^{\frac{1}{r'}} \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}^N}\big(w_{\varepsilon,z}^{-r}e^{-t(-\Delta)^\alpha}(w_{\varepsilon,z}^r|u|)\big)(x)^q\,dx\Big)^{\frac{1}{r}},
\end{aligned}$$

which implies

$$\begin{aligned}
\|w_{\varepsilon,z}^{-1}e^{-tH_{\alpha,2,V}}w_{\varepsilon,z}u\|_q &\le \|\exp(-tH_{\alpha,2,r'V})\|_{p,q}^{\frac{1}{r'}}\|u\|_p^{\frac{1}{r'}} \\
&\qquad \times \|w_{\varepsilon,z}^{-r}e^{-t(-\Delta)^\alpha}w_{\varepsilon,z}^r\|_{p,q}^{\frac{1}{r}}\|u\|_p^{\frac{1}{r}} \\
&= \|w_{\varepsilon,z}^{-r}e^{-t(-\Delta)^\alpha}w_{\varepsilon,z}^r\|_{p,q}^{\frac{1}{r}}\|\exp(-tH_{\alpha,2,r'V})\|_{p,q}^{\frac{1}{r'}}\|u\|_p.
\end{aligned}$$

Thus, the estimate of this lemma is proved. □

Proposition 4.6 (*cf.* [6, Proposition 3.7])**.** *Let* $1 \le p \le q \le \infty$ *and* $n > \frac{N}{2\alpha}\big(\frac{1}{p} - \frac{1}{q}\big)$ $(n \in \mathbb{N})$. *Assume that* V *satisfies the assumption* (4.1). *Then, there exists a constant* $C > 0$ *such that*

$$\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \lambda)^{-n}w_{\varepsilon,z}\|_{p,q} \le C$$

for real and sufficiently negative λ and all $\varepsilon \in [0,1], z \in \mathbb{R}^N$.

Proof. For all $u \in \mathcal{S}$, we have

$$w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \lambda)^{-n} w_{\varepsilon,z} u = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{\lambda t} w_{\varepsilon,z}^{-1} e^{-tH_{\alpha,2,V}} w_{\varepsilon,z} u \, dt$$

for all $\varepsilon \in [0,1], z \in \mathbb{R}^N$ and $n \in \mathbb{N}$, and hence

$$\begin{aligned}
&\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \lambda)^{-n} w_{\varepsilon,z} u\|_q \\
&\quad = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{\lambda t} \|w_{\varepsilon,z}^{-1} e^{-tH_{\alpha,2,V}} w_{\varepsilon,z}\|_{p,q} \, dt \cdot \|u\|_p.
\end{aligned}$$

We have already proved in Lemma 4.5, that

$$\|w_{\varepsilon,z}^{-1} e^{-tH_{\alpha,2,V}} w_{\varepsilon,z}\|_{p,q} \le \|w_{\varepsilon,z}^{-r} e^{-t(-\Delta)^\alpha} w_{\varepsilon,z}^r\|_{p,q}^{\frac{1}{r}} \|e^{-tH_{\alpha,2,r'V}}\|_{p,q}^{\frac{1}{r'}}$$

for all $\varepsilon \in [0,1], z \in \mathbb{R}^N, r \in \left(1, \frac{\alpha}{c}\right)$ and the conjugate exponent r'. $w_{\varepsilon,z}^r$ coincides with $w_{\varepsilon,z}$ replaced its exponent c with cr in the definition of $w_{\varepsilon,z}$. Since $\frac{1}{2} < c < cr < \alpha$, we can use the estimate of Lemma 4.4 and hence for any $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$\|w_{\varepsilon,z}^{-r} e^{-t(-\Delta)^\alpha} w_{\varepsilon,z}^r\|_{p,q} \le C_\eta t^{-\frac{N}{2\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} e^{\eta t}$$

for all $t > 0, \varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$. On the other hand, as stated in the proof of Lemma 4.5, $r'V_- \in \hat{K}_{N,\alpha}, c_{N,\alpha}(r'V_-) = 0$ and $r'V_+$ is U_α-admissible, hence by Proposition 3.6, there exist constants $M > 0, b > 0$ such that

$$\|\exp(-tH_{\alpha,2,r'V})\|_{p,q} \le M t^{-\frac{N}{2\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} e^{tb}$$

for all $t > 0$. Thus,

$$\begin{aligned}
\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \lambda)^{-n} w_{\varepsilon,z} u\|_q &\le C'_\eta M' \int_0^\infty t^{n - \frac{N}{2\alpha}\left(\frac{1}{p} - \frac{1}{q}\right) - 1} e^{(\lambda + \frac{\eta}{r} + \frac{b}{r'})t} \, dt \cdot \|u\|_p \\
&= C\|u\|_p,
\end{aligned}$$

provided that $n > \frac{N}{2\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)$ and λ is sufficiently negative. $\square$

Proposition 4.7 (*cf.* [6, Proposition 3.3])**.** *Assume that V satisfies the assumption* (4.1)*. Let K be any compact subset of $\rho(H_{\alpha,2,V})$. Then there exist constants $\varepsilon_0 = \varepsilon_0(K) \in [0,1]$ and $C = C(\varepsilon_0, K) > 0$ such that, for all $\zeta \in K, \varepsilon \in [0, \varepsilon_0]$ and $z \in \mathbb{R}^N$, the operator $w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \zeta)^{-1} w_{\varepsilon,z}$ with domain $\mathcal{S}$ has an extension $R_{\alpha,\varepsilon,z}(\zeta) \in \mathcal{L}(L^2)$ satisfying the following estimate: For all $\zeta \in K, \varepsilon \in [0, \varepsilon_0]$ and $z \in \mathbb{R}^N$,*

$$\|R_{\alpha,\varepsilon,z}(\zeta)\|_{2,2} \le C.$$

We prove this proposition in the next subsection since the proof is lengthy. For the time being, we admit this proposition, and prove Theorem 4.2 (*cf.* the proof of [6, Proposition 3.3]).

Proof of Theorem 4.2. For any V satisfying the assumption of this theorem, V is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$ by Proposition 2.13 (i) and Theorem 2.14.

Let us pick an integer $n > \frac{N}{4\alpha}$. Then, by Proposition 4.6, there exists a constant $C > 0$ such that

$$\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\lambda)^{-n}w_{\varepsilon,z}\|_{1,2},\ \|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\lambda)^{-n}w_{\varepsilon,z}\|_{2,\infty} \le C$$

for real and sufficiently negative $\lambda, \varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$. We fix such a λ.

On the other hand, let K be any compact subset of $\rho(H_{\alpha,2,V}), \zeta \in K$ and ε_0 be the same as in Proposition 4.7, then the estimate

$$\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}\|_{2,2} \le C$$

holds for all $\varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$ by Proposition 4.7 (if necessary, take a larger constant C). Hence, for the n above, we have

$$\begin{aligned}
&\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-2n}w_{\varepsilon,z}\|_{1,\infty} \\
&\le \sum_{j=0}^{2n}\binom{2n}{j}|\zeta-\lambda|^j\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\lambda)^{-n}w_{\varepsilon,z}\|_{2,\infty} \\
&\qquad\qquad \times \|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}\|_{2,2}^j\|w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\lambda)^{-n}w_{\varepsilon,z}\|_{1,2} \\
&\le C
\end{aligned}$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$ (if necessary, take a larger constant C). Hence, for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$, $w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-2n}w_{\varepsilon,z}$ is an integral operator and its integral kernel $G_{n,\varepsilon,z}(\zeta;x,y)$ satisfies the estimate

$$|G_{n,\varepsilon,z}(\zeta;x,y)| \le C$$

for all $\zeta \in K$ and a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$ (see [1, Proposition 6.2]). Since $G_{n,0,z}$ is independent of $z \in \mathbb{R}^N$ (in fact, $G_{n,0,z}(\zeta;x,y)$ is the integral kernel of $(H_{\alpha,2,V}-\zeta)^{-2n}$), we may write simply G_n. It is easy to verify that

$$G_{n,\varepsilon,z}(\zeta;x,y) = w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)G_n(\zeta;x,y)$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}^N$ and a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Hence,

$$|G_n(\zeta;x,y)| \le Cw_{\varepsilon,z}(x)w_{\varepsilon,z}(y)^{-1} \tag{4.2}$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}^N$ and a.e. $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$. Now we take a countable dense subset $S := \{z_n \in \mathbb{R}^N | n \in \mathbb{N}\}$ of $\mathbb{R}^N$. Then, there exists a null set $\mathscr{N} \subset \mathbb{R}^N \times \mathbb{R}^N$ such that the estimate (4.2) holds for all $\zeta \in K, \varepsilon \in [0, \varepsilon_0], z \in S$ and $(x, y) \in (\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathscr{N}$. For an arbitrary $(x, y) \in (\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathscr{N}$, we can take a sequence $\{z_{n_j}\}_j$ in S such that $z_{n_j} \to x$ as $j \to \infty$. By substituting $z = z_{n_j}$ for (4.2) and taking the limit as $j \to \infty$, we obtain the estimate

$$|G_n(\zeta; x, y)| \leq C(1 + \varepsilon_0 |x - y|^2)^{-c} \tag{4.3}$$

for all $\zeta \in K$ and a.e. $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(The argument above holds also under the assumption (4.1). However, we need $N = 1$ from here.) Let $p \in [1, \infty)$. By using the estimate (4.3), we can define the function $G_{n,p} \colon \rho(H_{\alpha,2,V}) \to \mathcal{L}(L^p)$ by

$$\big(G_{n,p}(\zeta)u\big)(x) := \int_{\mathbb{R}^N} G_n(\zeta; x, y) u(y)\, dy$$

for all $\zeta \in \rho(H_{\alpha,2,V}), u \in L^p$ and a.e. $x \in \mathbb{R}^N$. The function $G_{n,p}$ is holomorphic on $\rho(H_{\alpha,2,V})$. In fact, for all $u \in L^p(\mathbb{R}) \cap L^2(\mathbb{R})$ and $v \in L^{p'}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, the function $\zeta \mapsto \langle G_{n,p}(\zeta)u, v\rangle = \langle G_{n,2}(\zeta)u, v\rangle = \langle (H_{\alpha,2,V} - \zeta)^{-1}u, v\rangle$ is holomorphic on $\rho(H_{\alpha,2,V})$, where $\langle \phi, \psi\rangle := \int_{\mathbb{R}^N} \phi(x)\psi(x)\, dx$. Furthermore, we see that the function $G_{n,p}$ is locally bounded in $\rho(H_{\alpha,2,V})$. Hence, the function $G_{n,p}$ is weakly holomorphic, hence holomorphic on $\rho(H_{\alpha,2,V})$. On the other hand, $G_{n,p}(\zeta)$ coincides with $(H_{\alpha,p,V} - \zeta)^{-2n}$ for real and sufficiently negative ζ, since $e^{-tH_{\alpha,2,V}}$ and $e^{-tH_{\alpha,p,V}}$ are consistent. Hence, by unique continuation, the function $\zeta \mapsto (H_{\alpha,p,V} - \zeta)^{-2n}$ is holomorphic on $\rho(H_{\alpha,2,V})$ (note that $\rho(H_{\alpha,2,V})$ is a connected open subset, since $H_{\alpha,2,V}$ is self-adjoint and bounded below). Thus $\rho(H_{\alpha,2,V})$ is included in $\rho(H_{\alpha,p,V})$, the domain of holomorphy of $(H_{\alpha,p,V} - \zeta)^{-2n}$. $\square$

We give a sufficient condition for a potential to satisfy the assumption of Theorem 4.2 and close this subsection.

Proposition 4.8. *Assume that $V^2 \in \hat{K}_{N,\alpha}$, then the following assertions hold:*
(i) *$V \in \hat{K}_{N,\alpha}$ and $c_{N,\alpha}(V) = 0$,*
(ii) *V is $(-\Delta)^\alpha$-bounded with relative bound 0.*

Remark 4.9. For an arbitrary dimension N, this proposition is proved.

To prove this proposition, we prepare the following lemma concerning the resolvent $\big(\lambda + (-\Delta)^\alpha\big)^{-1}$ $(\lambda > 0)$. For all $\alpha \in (0, 1]$ and $\lambda > 0$, the resolvent $\big(\lambda + (-\Delta)^\alpha\big)^{-1}$ is an integral operator and its integral kernel $G_\alpha(\lambda; x - y)$ is given by

$$G_\alpha(\lambda; x) = \int_0^\infty e^{-\lambda t} K_\alpha(t, x)\, dt$$

for all $\lambda > 0$ and $x \in \mathbb{R}^N \setminus \{0\}$ (see [14, Lemma 3.7]). For each $\lambda > 0$, the function $x \mapsto G_\alpha(\lambda; x)$ is integrable on $\mathbb{R}^N$ (see [14, Lemma 3.7, 3.8 and 3.9]).

Lemma 4.10. *For each $0 < \alpha < 1$, the function G_α satisfies the following properties:*

(i) $G_\alpha(\lambda; x) = \lambda^{-1+\frac{N}{2\alpha}} G_\alpha(1; \lambda^{\frac{1}{2\alpha}} x)$ *for all* $\lambda > 0$ *and* $x \in \mathbb{R}^N \setminus \{0\}$,

(ii) *There exists a constant* $M_1 > 0$ *such that*

$$\|G_\alpha(\lambda; \cdot)\|_1 = \frac{M_1}{\lambda}$$

for all $\lambda > 0$,

(iii) *There exists a constant* $M_2 > 0$ *such that*

$$0 \le G_\alpha(\lambda; x) \le \frac{M_2}{\lambda^2} \cdot \frac{1}{|x|^{N+2\alpha}}$$

for all $\lambda > 0$ *and* $x \in \mathbb{R}^N \setminus \{0\}$,

(iv) *There exists a constant* $M_{N,\alpha} > 0$ *such that*

$$0 \le G_\alpha(\lambda; x) \le \begin{cases} M_{N,\alpha} \lambda^{-1+\frac{N}{2\alpha}} & \left(\frac{N}{2} < \alpha\right), \\ \dfrac{1}{\pi}\left(1 + \frac{1}{2}\log 2 + |\log \lambda|\right) + |g_{N,\alpha}(x)| & \left(\frac{N}{2} = \alpha\right), \\ g_{N,\alpha}(x) & \left(\frac{N}{2} > \alpha\right) \end{cases}$$

for all $\lambda > 0$ *and* $0 < |x| \le 1$.

Remark 4.11. In the case of $\alpha = 1$, assertions (i) and (ii) remain true, and the following modified assertions (iii) and (iv) hold.

(iii) There exist constants $M > 0$ and $\kappa > 0$ such that

$$0 \le G_1(\lambda; x) \le M \lambda^{-1+\frac{N}{2}} e^{-\kappa\sqrt{\lambda}|x|}$$

for all $\lambda > 0$ and $|x| \ge 1$ ($x \in \mathbb{R}^N$).

(iv) There exists a constant $M_N > 0$ such that

$$0 \le G_1(\lambda; x) \le \begin{cases} M_N \lambda^{-1+\frac{N}{2}} & (N = 1), \\ M_N\left(1 + \log|\lambda|\right) + |g_N(x)| & (N = 2), \\ g_N(x) & (N \ge 3) \end{cases}$$

for all $\lambda > 0$ and $0 < |x| \le 1$ ($x \in \mathbb{R}^N$), where g_N is the fundamental solution of Δ:

$$g_N(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| & (N = 2), \\ \dfrac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}(N-2)} |x|^{-N+2} & (N \ge 3) \end{cases}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$.

Proof. (i) is proved as follows:

$$\begin{aligned}
G_\alpha(\lambda;x) &= \int_0^\infty e^{-\lambda t}K_\alpha(t,x)\,dt \\
&= \lambda^{-1}\int_0^\infty e^{-t}K_\alpha(\lambda^{-1}t,x)\,dt \\
&= \lambda^{-1+\frac{N}{2\alpha}}\int_0^\infty e^{-t}K_\alpha(t,\lambda^{\frac{1}{2\alpha}}x)\,dt \\
&\qquad \text{(by Proposition 2.2 (ii)-(a))} \\
&= \lambda^{-1+\frac{N}{2\alpha}}G_\alpha(1;\lambda^{\frac{1}{2\alpha}}x).
\end{aligned}$$

(ii) is proved as follows:

$$\begin{aligned}
\|G_\alpha(\lambda;\cdot)\|_1 &= \lambda^{-1+\frac{N}{2\alpha}}\int_{\mathbb{R}^N} G_\alpha(1;\lambda^{\frac{1}{2\alpha}}x)\,dx \\
&= \lambda^{-1}\int_{\mathbb{R}^N} G_\alpha(1;x)\,dx.
\end{aligned}$$

(iii) Since there exits a constant $M_2 > 0$ such that

$$0 \le K_\alpha(t,x) \le M_2 \cdot \frac{t}{|x|^{N+2\alpha}} \quad (t>0, x\ne 0)$$

by (2.1), we have

$$0 \le G_\alpha(1;x) = \int_0^\infty e^{-t}K_\alpha(t,x)\,dx \le \frac{M_2}{|x|^{N+2\alpha}}.$$

This inequality and (i) imply (iii).

(iv) We have only to prove the following estimate

$$G_\alpha(1;x) \le \begin{cases} M_{N,\alpha} & \left(\frac{N}{2} < \alpha\right) \\ \dfrac{1}{\pi}\big(1+\log 2 + \big|\log|x|\big|\big) & \left(\frac{N}{2} = \alpha\right) \\ g_{N,\alpha}(x) & \left(\frac{N}{2} > \alpha\right) \end{cases}$$

for all $x \in \mathbb{R}^N \setminus \{0\}$, since assertion (iv) follows from this inequality and (i).

First case: $\frac{N}{2} < \alpha$ (i.e. $N = 1, \frac{1}{2} < \alpha < 1$). For all $x \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned}
G_\alpha(1;x) &= \int_0^\infty e^{-t}K_\alpha(t,x)\,dt \\
&\le C_\alpha \int_0^\infty e^{-t}\cdot\frac{t}{(t^{\frac{1}{\alpha}}+|x|^2)^{\frac{1}{2}+\alpha}}\,dt \quad \text{(by (2.1))} \\
&\le C_\alpha \int_0^\infty e^{-t}t^{-\frac{1}{2\alpha}}\,dt < \infty.
\end{aligned}$$

(Since $\frac{1}{2\alpha} < 1$, the last integral is finite.)

Second case: $\frac{N}{2} = \alpha$ (i.e. $N = 1, \alpha = \frac{1}{2}$). In this case, as is well known, $K_\alpha(t,x)$ is the Poisson kernel:

$$K_\alpha(t,x) = \frac{1}{\pi} \cdot \frac{t}{t^2 + x^2}.$$

Hence, for all $x \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned}
\pi G_\alpha(1;x) &= \pi \int_0^\infty e^{-t} K_\alpha(t,x)\,dt \\
&= \int_0^\infty e^{-t} \cdot \frac{t}{t^2+x^2}\,dt \\
&\le \int_0^1 \frac{t}{t^2+x^2}\,dt + \int_1^\infty e^{-t} t^{-1}\,dt \\
&\le \frac{1}{2} \log(1 + |x|^{-2}) + 1.
\end{aligned}$$

Since the inequality

$$\log(|x|^{-2} + 1) - \bigl|\log |x|^{-2}\bigr| \le \log 2$$

holds, we obtain the estimate

$$G_\alpha(1;x) \le \frac{1}{\pi}\bigl(\bigl|\log|x|\bigr| + 1 + \tfrac{1}{2}\log 2\bigr).$$

Third case: $\frac{N}{2} > \alpha$. For details of the proof, see the proof of Lemma 2.17. For all $x \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned}
G_\alpha(1;x) &= \int_0^\infty e^{-t} K_\alpha(t,x)\,dt \\
&\le \int_0^\infty K_\alpha(t,x)\,dt \\
&= \alpha |x|^{-N+2\alpha} \int_0^\infty \tau^{\frac{N}{2}-\alpha-1} K_\alpha(1, \tau^{\frac{1}{2}} \boldsymbol{e})\,d\tau \\
&= g_{N,\alpha}(x).
\end{aligned}$$

Thus, the lemma is proved. □

Proof of Proposition 4.8. (i) For all $0 < \rho \le 1$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\int_{|x-y|<\rho} |g_{N,\alpha}(x-y)||V(y)|\,dy \\
&\qquad \le \Bigl(\int_{|x-y|<\rho} |g_{N,\alpha}(x-y)|\,dy\Bigr)^{\frac{1}{2}} \Bigl(\int_{|x-y|<\rho} |g_{N,\alpha}(x-y)||V(y)|^2\,dy\Bigr)^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad \text{(by Schwarz's inequality)}
\end{aligned}$$

$$\leq \Big(\int_{|y|<\rho} |g_{N,\alpha}(y)|\, dy \Big)^{\frac{1}{2}} \|V^2\|_{\hat{K}_{N,\alpha}}.$$

Hence, $V \in \hat{K}_{N,\alpha}$ and

$$c_{N,\alpha}(V) \leq \lim_{\rho\downarrow 0}\Big(\int_{|y|<\rho} |g_{N,\alpha}(y)|\, dy \Big)^{\frac{1}{2}} \|V^2\|_{\hat{K}_{N,\alpha}} = 0,$$

since $g_{N,\alpha} \in L^1\big(B(0,1)\big)$.

(ii) For all $\lambda > 0$ and $u \in L^2$,

$$\begin{aligned}
(4.4) \qquad & \int_{\mathbb{R}^N} V(x)^2 \big((\lambda + (-\Delta)^\alpha)^{-1} u \big)(x)^2\, dx \\
&= \int_{\mathbb{R}^N} V(x)^2 \Big(\int_{\mathbb{R}^N} G_\alpha(\lambda; x-y) u(y)\, dy \Big)^2 dx \\
&\leq \|G_\alpha(\lambda;\cdot)\|_1 \int_{\mathbb{R}^N} V(x)^2 \Big(\int_{\mathbb{R}^N} G_\alpha(\lambda; x-y) u(y)^2\, dy \Big) dx \\
&\qquad\qquad \text{(by Schwarz's inequality)} \\
&= \frac{M_1}{\lambda} \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} G_\alpha(\lambda; x-y) V(x)^2\, dx \Big) u(y)^2\, dy.
\end{aligned}$$

by Lemma 4.10 (ii) and Fubini's theorem. To estimate $\int_{\mathbb{R}^N} G_\alpha(\lambda; x-y)V(y)^2\, dy$ (for convenience, we change x for y and y for x), we define

$$I_{N,\alpha}(\lambda; x) := \int_{|x-y|\geq 1} G_\alpha(\lambda; x-y) V(y)^2\, dy$$

and

$$J_{N,\alpha}(\lambda; x) := \int_{|x-y|< 1} G_\alpha(\lambda; x-y) V(y)^2\, dy$$

for all $\lambda > 0$ and $x \in \mathbb{R}^N$, and estimate $I_{N,\alpha}$ and $J_{N,\alpha}$. We first estimate $I_{N,\alpha}$. In a similar way as in the proof of (2.24), we have

$$\begin{aligned}
(4.5) \qquad I_{N,\alpha}(\lambda; x) &\leq \frac{M_2}{\lambda^2} \int_{|x-y|\geq 1} V(x-y)^2 |y|^{-N-2\alpha}\, dy \\
&\qquad\qquad \text{(by Lemma 4.10 (iii))} \\
&\leq \frac{M_2'}{\lambda^2} \|V^2\|_{\hat{K}_{N,\alpha}}
\end{aligned}$$

for all $\lambda > 0$ and a.e. $x \in \mathbb{R}^N$.

We next estimate $J_{N,\alpha}$.

First case: $\frac{N}{2} < \alpha$ (i.e. $N = 1, \frac{1}{2} < \alpha < 1$).

$$
\begin{aligned}
J_{N,\alpha}(\lambda; x) &\le M_{N,\alpha}\lambda^{-1+\frac{N}{2\alpha}} \int_{|x-y|<1} V(y)^2\,dy \qquad (4.6)\\
&\qquad\qquad \text{(by Lemma 4.10 (iv))}\\
&\le M_{N,\alpha}\lambda^{-1+\frac{N}{2\alpha}} \|V^2\|_{1,loc,unif}\\
&\le M'_{N,\alpha}\lambda^{-1+\frac{N}{2\alpha}} \|V^2\|_{\hat{K}_{N,\alpha}}
\end{aligned}
$$

for all $\lambda > 0$ and a.e. $x \in \mathbb{R}^N$.

Second case: $\frac{N}{2} = \alpha$ (i.e. $N = 1, \alpha = \frac{1}{2}$).

$$
\begin{aligned}
J_{N,\alpha}(\lambda; x) &\le \frac{1}{\pi}\Big\{\big(1 + \tfrac{1}{2}\log 2 + |\log \lambda|\big) \int_{|x-y|<1} V(y)^2\,dy \qquad (4.7)\\
&\qquad + \int_{|x-y|<1} |g_{N,\alpha}(x-y)| V(y)^2\,dy\Big\}\\
&\le \frac{1}{\pi}\big(1 + \tfrac{1}{2}\log 2 + |\log \lambda|\big)\|V^2\|_{1,loc,unif} + \|V^2\|_{\hat{K}_{N,\alpha}}\\
&\le C\big(1 + \tfrac{1}{2}\log 2 + |\log \lambda|\big)\|V^2\|_{\hat{K}_{N,\alpha}}
\end{aligned}
$$

for all $\lambda > 0$ and a.e. $x \in \mathbb{R}^N$, where $C > 0$ is a constant which is independent of λ, x and V.

Third case: $\frac{N}{2} > \alpha$.

$$
\begin{aligned}
J_{N,\alpha}(\lambda; x) &\le \int_{|x-y|<1} g_{N,\alpha}(x-y) V(y)^2\,dy \qquad (4.8)\\
&\le \|V^2\|_{\hat{K}_{N,\alpha}}
\end{aligned}
$$

for all $\lambda > 0$ and a.e. $x \in \mathbb{R}^N$.

By the inequalities (4.4) through (4.8), we obtain that $H^{2\alpha} \subset D(V)$ and

$$
\lim_{\lambda \to \infty} \big\| V\big(\lambda + (-\Delta)^\alpha\big)^{-1} \big\| = 0.
$$

Hence, V is $(-\Delta)^\alpha$-bounded with relative bound 0. □

4.2. The proof of Proposition 4.7

We prove Proposition 4.7 which was stated without a proof in the previous subsection. We divide the proof into several lemmas and propositions. Our plan of the proof is that we first prove the proposition in the case of $\alpha = 1$, and by using this result, we prove the proposition in the general case.

4.2.1. ∇-boundedness of $w_{\varepsilon,z}^{-1}(-\Delta)w_{\varepsilon,z} - (-\Delta)$

Lemma 4.12. *For any $\varepsilon \in [0,1]$ and $z \in \mathbb{R}^N$, the operator $w_{\varepsilon,z}^{-1}(-\Delta)w_{\varepsilon,z} - (-\Delta)$ with domain $\mathcal{S}$ has an extension $T_{\varepsilon,z}$ with domain H^1 satisfying the following ∇-boundedness: There exists a constant $C > 0$ such that*

$$\|T_{\varepsilon,z}u\|_2 \le C\big(\sqrt{\varepsilon}\|\nabla u\|_2 + \varepsilon\|u\|_2\big)$$

for all $\varepsilon \in [0,1], z \in \mathbb{R}^N$ and $u \in H^1$, where $\nabla u = (\partial_{x_1}u, \dots, \partial_{x_N}u)$ and $\|\nabla u\|_2 = \big(\sum_{j=1}^N \|\partial_{x_j}u\|_2^2\big)^{\frac{1}{2}}$.

Proof. The statement of this proposition is proved by a straightforward calculation. In fact, it is easily verified that

$$\begin{aligned} &w_{\varepsilon,z}^{-1}(-\Delta)w_{\varepsilon,z} - (-\Delta) \\ &\quad = -4c\varepsilon(1+\varepsilon|x-z|^2)^{-1}\sum_{j=1}^N (x_j - z_j)\partial_{x_j} \\ &\qquad - 2c\varepsilon(1+\varepsilon|x-z|^2)^{-2}\big\{N + (N+2c-2)\varepsilon|x-z|^2\big\}. \end{aligned}$$

The right-hand side of this equality defines an operator with domain H^1 since $(1+\varepsilon|x-z|^2)^{-1}(x_j - z_j)$ is bounded for $j = 1, \dots, N$. Let $T_{\varepsilon,z}$ denote this operator. Then there exists a constant $C > 0$ such that

$$\begin{aligned} \|T_{\varepsilon,z}u\|_2 &\le 4c\sqrt{\varepsilon}\sum_{j=1}^N \sup_{x\in\mathbb{R}^N} \sqrt{\varepsilon}(1+\varepsilon|x|^2)^{-1}|x_j| \cdot \|\partial_{x_j}u\|_2 \\ &\qquad + 2c\varepsilon\big\{N + (N+2c-2)\sup_{x\in\mathbb{R}^N}(1+\varepsilon|x|^2)^{-2}\varepsilon|x|^2\big\} \\ &\le C\big(\sqrt{\varepsilon}\|\nabla u\|_2 + \varepsilon\|u\|_2\big) \end{aligned}$$

for all $\varepsilon \in [0,1], z \in \mathbb{R}^N$ and $u \in H^1$. □

4.2.2. L^2-bounded extension of $w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}$

Lemma 4.13. *There exists an $\varepsilon_0' \in (0,1]$ such that for any $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$, the operator $w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}$ with domain $\mathcal{S}$ has an extension $R_{\varepsilon,z}(s) \in \mathcal{L}(L^2)$ satisfying the following estimate:*

$$\|R_{\varepsilon,z}(s)\|_{2,2} \le \frac{2}{s}$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$.

Proof. To prove the assertion of this lemma, note that

$$\|\nabla(s-\Delta)^{-1}\|_{2,2} \le \frac{\sqrt{N}}{2} \cdot \frac{1}{\sqrt{s}}$$

for all $s \in (0,\infty)$ (by using Fourier transform). Hence, by Lemma 4.12, there exists a constant $C > 0$ such that $\|T_{\varepsilon,z}(s-\Delta)^{-1}\|_{2,2} \le C\big(\frac{\sqrt{\varepsilon}}{\sqrt{s}} + \frac{\varepsilon}{s}\big)$ for all $\varepsilon \in [0,1], z \in \mathbb{R}^N$ and $s \in (0,\infty)$, and hence there exists an $\varepsilon_0' \in (0,1]$ such that

$$\|T_{\varepsilon,z}(s-\Delta)^{-1}\|_{2,2} \le \frac{1}{2} \tag{4.9}$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$.

Next, for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N, s \in [1,\infty)$ and $u \in \mathcal{S}$,

$$w_{\varepsilon,z}^{-1}(s-\Delta)w_{\varepsilon,z}u = \big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)(s-\Delta)u.$$

By the estimate (4.9) above, the operator of the right-hand side of this equality is invertible in $\mathcal{L}(L^2)$. Since this equality holds and $w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}v \in \mathcal{S}$ for all $v \in \mathcal{S}$, we have the following equality

$$\begin{aligned} v &= w_{\varepsilon,z}^{-1}(s-\Delta)w_{\varepsilon,z}\big[w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}v\big] \\ &= \big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)(s-\Delta)\big[w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}v\big], \end{aligned}$$

hence,

$$(s-\Delta)^{-1}\big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)^{-1}v = w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}v$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N, s \in [1,\infty)$ and $v \in \mathcal{S}$.

Now we define

$$R_{\varepsilon,z}(s) := (s-\Delta)^{-1}\big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)^{-1} \in \mathcal{L}(L^2)$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$. This operator is an extension of $w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}$ and satisfies the estimate

$$\|R_{\varepsilon,z}(s)\|_{2,2} \le \big\|(s-\Delta)^{-1}\big\|_{2,2}\big\|\big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)^{-1}\big\|_{2,2} \le \frac{2}{s}$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$. □

4.2.3. The second resolvent equations

Lemma 4.14. *Let $T_{\varepsilon,z}$ be the same as in Lemma 4.12 and ε_0' and $R_{\varepsilon,z}$ be the same as in Lemma 4.13. Then, for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $s \in [1,\infty)$, the equality*

$$(s-\Delta)^{-1} - R_{\varepsilon,z}(s) = R_{\varepsilon,z}(s)T_{\varepsilon,z}(s-\Delta)^{-1}$$

holds on L^2.

Proof. Since the domain of $T_{\varepsilon,z}$ is H^1 including the domain of Δ, we have the equality

$$s - \Delta + T_{\varepsilon,z} = \big(1 + T_{\varepsilon,z}(s-\Delta)^{-1}\big)(s-\Delta)$$

on H^2 for all $\varepsilon \in [0, \varepsilon_0']$, $z \in \mathbb{R}^N$ and $s \in [1,\infty)$. As was proved in Lemma 4.13, the right-hand side operator in this equality is invertible in $\mathcal{L}(L^2)$ and its inverse operator is $R_{\varepsilon,z}(s)$. Hence, an arbitrary $s \in [1,\infty)$ belongs to the resolvent set of the operator sum $\Delta - T_{\varepsilon,z}$ defined on H^2 for all $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$, and the resolvent $(s - \Delta + T_{\varepsilon,z})^{-1}$ coincides with $R_{\varepsilon,z}(s)$. Now, it is clear that the equality in this lemma holds, since the equality is nothing but the second resolvent equation concerning $(s-\Delta)^{-1}$ and $(s-\Delta+T_{\varepsilon,z})^{-1}$. □

4.2.4. ∇-boundedness of $w_{\varepsilon,z}^{-1}(-\Delta)^{\alpha}w_{\varepsilon,z} - (-\Delta)^{\alpha}$

Proposition 4.15. *Let ε_0' be the same as in Lemma 4.13. For any $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$, the operator $w_{\varepsilon,z}^{-1}(-\Delta)^{\alpha}w_{\varepsilon,z} - (-\Delta)^{\alpha}$ with domain $\mathcal{S}$ has an extension $T_{\alpha,\varepsilon,z}$ with domain H^1 satisfying the following ∇-boundedness: There exists a constant $C > 0$ such that*

$$\|T_{\alpha,\varepsilon,z}u\|_2 \le C\sqrt{\varepsilon}\big(\|\nabla u\|_2 + \|u\|_2\big)$$

for all $\varepsilon \in [0,\varepsilon_0']$, $z \in \mathbb{R}^N$ and $u \in H^1$.

Proof. The assertion in the case of $\alpha = 1$ is the result in Lemma 4.12, and so we assume $\alpha \in (0,1)$. Let $\varepsilon_0', R_{\varepsilon,z}(s)$ and $T_{\varepsilon,z}$ be the same as in Lemma 4.13 and Lemma 4.12, respectively, for all $\varepsilon \in [0,\varepsilon_0']$, $z \in \mathbb{R}^N$ and $s \in [1,\infty)$. We will use the well-known formula

$$(-\Delta)^{\alpha}u = \frac{\sin \pi\alpha}{\pi}\int_0^{\infty} s^{\alpha-1}(s-\Delta)^{-1}(-\Delta)u\,ds$$

for all $u \in \mathcal{S}$ (see the formula (4) in [20, Chapter IX, Section 11] or [9, (5.13)]), where the integrand of the right-hand side of this equality is Bochner integrable on $[0,\infty)$. By this formula, the following equality holds: For all $\varepsilon \in [0,\varepsilon_0']$, $z \in \mathbb{R}^N$ and $u \in \mathcal{S}$,

$$\begin{aligned}
(4.10)\quad & \frac{\pi}{\sin \pi\alpha}\big(w_{\varepsilon,z}^{-1}(-\Delta)^{\alpha}w_{\varepsilon,z}u - (-\Delta)^{\alpha}u\big) \\
&= w_{\varepsilon,z}^{-1}\int_0^{\infty} s^{\alpha-1}(s-\Delta)^{-1}(-\Delta)w_{\varepsilon,z}u\,ds \\
&\qquad\qquad - \int_0^{\infty} s^{\alpha-1}(s-\Delta)^{-1}(-\Delta)u\,ds \\
&= \int_0^{\infty} s^{\alpha-1}w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}(-\Delta)w_{\varepsilon,z}u\,ds \\
&\qquad\qquad - \int_0^{\infty} s^{\alpha-1}(s-\Delta)^{-1}(-\Delta)u\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 s^{\alpha-1}\Big(w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}(-\Delta)w_{\varepsilon,z} - (s-\Delta)^{-1}(-\Delta)\Big)u\,ds \\
&\qquad + \int_1^\infty s^{\alpha-1}\Big(w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}(-\Delta)w_{\varepsilon,z} - (s-\Delta)^{-1}(-\Delta)\Big)u\,ds.
\end{aligned}
$$

We can execute a calculation on the first term of the rightmost side of (4.10) as follows:

$$
\begin{aligned}
&\int_0^1 s^{\alpha-1}\Big(w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}(-\Delta)w_{\varepsilon,z} - (s-\Delta)^{-1}(-\Delta)\Big)u\,ds \\
&\qquad = \int_0^1 s^{\alpha-1}\Big(w_{\varepsilon,z}^{-1}\big(1-s(s-\Delta)^{-1}\big)w_{\varepsilon,z} - \big(1-s(s-\Delta)^{-1}\big)\Big)u\,ds \\
&\qquad = \int_0^1 s^{\alpha}\Big((s-\Delta)^{-1} - w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}\Big)u\,ds.
\end{aligned}
$$

In addition, $\big(1-w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)\big)G_1(s;x-y)$ is the integral kernel of $(s-\Delta)^{-1} - w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}$ for all $\varepsilon \in [0,\varepsilon_0']$, $z \in \mathbb{R}^N$ and $s>0$. Hence, for all $\varepsilon \in [0,\varepsilon_0']$, $z \in \mathbb{R}^N$ and $u \in \mathcal{S}$, the inequality

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Big(\Big[\int_0^1 s^\alpha\big((s-\Delta)^{-1} - w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}\big)u\,ds\Big](x)\Big)^2 dx \\
&\quad = \int_{\mathbb{R}^N}\Big(\int_0^1 s^\alpha\big[\big((s-\Delta)^{-1} - w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}\big)u\big](x)\,ds\Big)^2 dx \\
&\quad = \int_{\mathbb{R}^N}\Big(\int_0^1 s^\alpha\Big(\int_{\mathbb{R}^N}\big(1-w_{\varepsilon,z}(x)^{-1}w_{\varepsilon,z}(y)\big)G_1(s;x-y)u(y)\,dy\Big)\,ds\Big)^2 dx \\
&\quad \le \big(2^{1+2c}c\sqrt{\varepsilon}\big)^2\int_{\mathbb{R}^N}\Big(\int_0^1 s^\alpha\Big(\int_{\mathbb{R}^N}(1+|x-y|^2)^c \\
&\qquad\qquad\qquad \times G_1(s;x-y)|u(y)|\,dy\Big)\,ds\Big)^2 dx \\
&\qquad\qquad\qquad \text{(by Lemma 4.3 (ii))} \\
&\quad = \big(2^{1+2c}c\sqrt{\varepsilon}\big)^2\int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}(1+|x-y|^2)^c \\
&\qquad\qquad\qquad \times\Big(\int_0^1 s^\alpha G_1(s;x-y)\,ds\Big)|u(y)|\,dy\Big)^2 dx \\
&\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&\quad \le \big(2^{1+2c}c\sqrt{\varepsilon}\big)^2\Big(\int_{\mathbb{R}^N}(1+|x|^2)^c\Big(\int_0^1 s^\alpha G_1(s;x)\,ds\Big)\,dx\Big)^2\cdot\|u\|_2^2 \\
&\qquad\qquad\qquad \text{(by Young's inequality)}
\end{aligned}
$$

holds. Thus, for all $\varepsilon \in [0, \varepsilon_0']$ and $z \in \mathbb{R}^N$, we have

$$\begin{aligned}
&\Big\| \int_0^1 s^{\alpha-1} \Big(w_{\varepsilon,z}^{-1} (s-\Delta)^{-1} (-\Delta) w_{\varepsilon,z} - (s-\Delta)^{-1}(-\Delta) \Big)\, ds \Big\|_{2,2} \\
&\quad \le 2^{1+2c} c \sqrt{\varepsilon} \int_{\mathbb{R}^N} (1+|x|^2)^c \Big(\int_0^1 s^{\alpha} G_1(s;x)\, ds \Big)\, dx \\
&\quad = C\sqrt{\varepsilon} \int_{\mathbb{R}^N} (1+|x|^2)^c \Big(\int_0^1 s^{\alpha+\frac{N}{2}-1} G_1(1; s^{\frac{1}{2}} x)\, ds \Big)\, dx \\
&\quad = C\sqrt{\varepsilon} \int_0^1 s^{\alpha+\frac{N}{2}-1} \Big(\int_{\mathbb{R}^N} (1+|x|^2)^c G_1(1; s^{\frac{1}{2}} x)\, dx \Big)\, ds \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Fubini's theorem)} \\
&\quad = C\sqrt{\varepsilon} \int_0^1 s^{\alpha-1} \Big(\int_{\mathbb{R}^N} (1+s^{-1}|x|^2)^c G_1(1;x)\, dx \Big)\, ds \\
&\quad = C\sqrt{\varepsilon} \int_0^1 s^{\alpha-1} \Big[\Big(\int_{|x|<1} + \int_{|x|\ge 1} \Big) (1+s^{-1}|x|^2)^c G_1(1;x)\, dx \Big]\, ds \\
&\quad \le C\sqrt{\varepsilon} \int_0^1 s^{\alpha-1} \Big[(1+s^{-1})^c \int_{|x|<1} G_1(1;x)\, dx \\
&\qquad\qquad\qquad\qquad + (1+s^{-1})^c \int_{|x|\ge 1} |x|^{2c} G_1(1;x)\, dx \Big]\, ds \\
&\quad = C\sqrt{\varepsilon} \int_0^1 s^{\alpha-1} (1+s^{-1})^c\, ds \\
&\qquad\quad \times \Big(\int_{|x|<1} G_1(1;x)\, dx + \int_{|x|\ge 1} |x|^{2c} G_1(1;x)\, dx \Big).
\end{aligned}$$

The integrals of the right-hand side of the last equality are finite. More precisely, the function $s \mapsto s^{\alpha-1}(1+s^{-1})^c$ is integrable on $(0,1)$ by $c < \alpha$, and the functions $x \mapsto G_1(1;x)$ and $x \mapsto |x|^{2c} G_1(1;x)$ are integrable on $B(0,1)$ and $B(0,1)^c$ respectively by Remark 4.11. Hence, the operator of the leftmost side of this equality can be extended to an operator belonging to $\mathcal{L}(L^2)$ for all $\varepsilon \in [0, \varepsilon_0']$ and $z \in \mathbb{R}^N$. In addition, there exists a constant $C > 0$ such that the $\mathcal{L}(L^2)$-norm of this operator is not greater than $C\sqrt{\varepsilon}$ for all $\varepsilon \in [0, \varepsilon_0']$ and $z \in \mathbb{R}^N$.

Next, we can execute a calculation on the second term of the rightmost side of (4.10) as follows:

$$\begin{aligned}
(4.11)\quad &\int_1^\infty s^{\alpha-1} \Big(w_{\varepsilon,z}^{-1} (s-\Delta)^{-1} (-\Delta) w_{\varepsilon,z} - (s-\Delta)^{-1}(-\Delta) \Big) u\, ds \\
&\quad = \int_1^\infty s^{\alpha-1} \Big(w_{\varepsilon,z}^{-1} (s-\Delta)^{-1} w_{\varepsilon,z} \cdot w_{\varepsilon,z}^{-1} (-\Delta) w_{\varepsilon,z} \\
&\qquad\qquad\qquad\qquad - (s-\Delta)^{-1}(-\Delta) \Big) u\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_1^\infty s^{\alpha-1}\Big(w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}\big(w_{\varepsilon,z}^{-1}(-\Delta)w_{\varepsilon,z}-(-\Delta)\big) \\
&\qquad\qquad + \big(w_{\varepsilon,z}^{-1}(s-\Delta)^{-1}w_{\varepsilon,z}-(s-\Delta)^{-1}\big)(-\Delta)\Big)u\,ds \\
&= \int_1^\infty s^{\alpha-1}\Big(R_{\varepsilon,z}(s)T_{\varepsilon,z}-R_{\varepsilon,z}(s)T_{\varepsilon,z}(s-\Delta)^{-1}(-\Delta)\Big)u\,ds \\
&= \int_1^\infty s^{\alpha-1}\Big(R_{\varepsilon,z}(s)T_{\varepsilon,z}-R_{\varepsilon,z}(s)T_{\varepsilon,z}\big(1-s(s-\Delta)^{-1}\big)\Big)u\,ds
\end{aligned}
$$

for all $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$. (To obtain (4.11), note that $(w_{\varepsilon,z}^{-1}(-\Delta)w_{\varepsilon,z} - (-\Delta))u \in \mathcal{S}$ for all $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$ and use Lemma 4.12, 4.13 and 4.14.)

It can be shown that the rightmost side of this equality defines an operator with domain H^1 and satisfies the following estimate by Lemma 4.12 and Lemma 4.13: Indeed, for any $u \in H^1$,

$$
\begin{aligned}
&\Big\|\int_1^\infty s^{\alpha-1}\Big(R_{\varepsilon,z}(s)T_{\varepsilon,z}-R_{\varepsilon,z}(s)T_{\varepsilon,z}\big(1-s(s-\Delta)^{-1}\big)\Big)u\,ds\Big\|_2 \\
&\quad\le \int_1^\infty s^{\alpha-1}\cdot\frac{2}{s}\,ds \times C\big(\sqrt{\varepsilon}\|\nabla u\|_2+\varepsilon\|u\|_2\big) \\
&\qquad + \int_1^\infty s^{\alpha-1}\cdot\frac{2}{s}\cdot C\Big(\sqrt{\varepsilon}\big\|\nabla\big(1-s(s-\Delta)^{-1}\big)u\big\|_2 \\
&\qquad\qquad\qquad + \varepsilon\big\|\big(1-s(s-\Delta)^{-1}\big)u\big\|_2\Big)\,ds \\
&\quad\le C\big(\sqrt{\varepsilon}\|\nabla u\|_2+\varepsilon\|u\|_2\big)
\end{aligned}
$$

for all $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$.

Thus, for any $\varepsilon \in [0,\varepsilon_0']$ and $z \in \mathbb{R}^N$, the operator $w_{\varepsilon,z}^{-1}(-\Delta)^\alpha w_{\varepsilon,z} - (-\Delta)^\alpha$ has an extension $T_{\alpha,\varepsilon,z}$ with domain H^1 satisfying the following estimate: There exists a constant $C > 0$ such that

$$
\|T_{\alpha,\varepsilon,z}u\|_2 \le C\sqrt{\varepsilon}\big(\|\nabla u\|_2+\|u\|_2\big)
$$

for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $u \in H^1$. □

4.2.5. $H_{\alpha,2,V}$-boundedness of ∇

Lemma 4.16. *Under the assumption* (4.1), ∇ *is* $H_{\alpha,2,V}$*-bounded with relative bound* 0.

Proof. Since $(-\Delta)^\alpha + V \subset H_{\alpha,2,V}$ by [6, Corollary 2.7] and both of the operators are self-adjoint, $H_{\alpha,2,V} = (-\Delta)^\alpha + V$. Furthermore, since ∇ is $(-\Delta)^\alpha$-bounded with relative bound 0 and V is $(-\Delta)^\alpha$-bounded with relative bound < 1, the assertion of this lemma holds. □

4.2.6. Completion of the proof

Proof of Proposition 4.7. Let K be any compact subset of $\rho(H_{\alpha,2,V})$. As stated in the proof of Lemma 4.16, $H_{\alpha,2,V} = (-\Delta)^\alpha + V$. (It is implied in this equality that $D(H_{\alpha,2,V}) = H^{2\alpha}$.) By this equality and Proposition 4.15 and Lemma 4.16, for ε_0' and $T_{\alpha,\varepsilon,z}$ which are the same as in Proposition 4.15, the estimate

$$\|T_{\alpha,\varepsilon,z}u\|_2 \le C\sqrt{\varepsilon}\big(\|H_{\alpha,2,V}u\|_2 + \|u\|_2\big)$$

holds for all $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $u \in H^1$, where C is a constant which is independent of $\varepsilon \in [0,\varepsilon_0'], z \in \mathbb{R}^N$ and $u \in H^1$. Hence, there exists an $\varepsilon_0 \in (0,\varepsilon_0']$ such that

$$\big\|T_{\alpha,\varepsilon,z}(H_{\alpha,2,V} - \zeta)^{-1}\big\|_{2,2} \le \frac{1}{2}$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$. On the other hand, the equality

$$w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \zeta)w_{\varepsilon,z}u = \big(1 - T_{\alpha,\varepsilon,z}(H_{\alpha,2,V} - \zeta)^{-1}\big)(H_{\alpha,2,V} - \zeta)u$$

holds for all $\varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}^N$ and $u \in \mathcal{S}$.

By the estimate above, the operator of the right-hand side of this equality is invertible in $\mathcal{L}(L^2)$, hence we can define the operator

$$R_{\alpha,\varepsilon,z}(\zeta) := (H_{\alpha,2,V} - \zeta)^{-1}\big(1 - T_{\alpha,\varepsilon,z}(H_{\alpha,2,V} - \zeta)^{-1}\big)^{-1} \in \mathcal{L}(L^2)$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$. This operator satisfies the equality

$$R_{\alpha,\varepsilon,z}(\zeta)w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \zeta)w_{\varepsilon,z}u = u \tag{4.12}$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}^N$ and $u \in \mathcal{S}$. We can prove that this equality holds for all $u \in w_{\varepsilon,z}^{-1}H^{2\alpha}$, where $w_{\varepsilon,z}^{-1}H^{2\alpha}$ is the image of $H^{2\alpha}$ by the multiplication operator $w_{\varepsilon,z}^{-1}$. In fact, since $\mathcal{S}$ is a core of $(-\Delta)^\alpha$, for any $u \in w_{\varepsilon,z}^{-1}H^{2\alpha}$ there exists a sequence $\{v_n\}_n$ in $\mathcal{S}$ such that $v_n \to w_{\varepsilon,z}u$ and $(-\Delta)^\alpha v_n \to (-\Delta)^\alpha w_{\varepsilon,z}u$ in L^2 as $n \to \infty$. Since $w_{\varepsilon,z}^{-1}v_n \in \mathcal{S}$ for all $n \in \mathbb{N}$, we can substitute $w_{\varepsilon,z}^{-1}v_n$ for u in (4.12) and we have the equality

$$R_{\alpha,\varepsilon,z}(\zeta)w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \zeta)v_n = w_{\varepsilon,z}^{-1}v_n$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}$ and $n \in \mathbb{N}$. As $n \to \infty$, the right-hand side of this equality converges to u in L^2 by the way of taking the sequence $\{v_n\}_n$. On the other hand, the left-hand side of this equality converges to $R_{\alpha,\varepsilon,z}(\zeta)w_{\varepsilon,z}^{-1}(H_{\alpha,2,V} - \zeta)w_{\varepsilon,z}u$ in L^2 as $n \to \infty$ since $(-\Delta)^\alpha v_n \to (-\Delta)^\alpha w_{\varepsilon,z}u$ in L^2 as $n \to \infty$ and also V is $(-\Delta)^\alpha$-bounded with relative bound < 1 by the assumption. Hence, the equality (4.12) holds for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in$

$\mathbb{R}^N$ and $u \in w_{\varepsilon,z}^{-1}H^{2\alpha}$. Now, let v be an arbitrary function in $\mathcal{S}$. Since $w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}v \in w_{\varepsilon,z}^{-1}H^{2\alpha}$, we can substitute $w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}v$ for u in (4.12) and we have the equality

$$R_{\alpha,\varepsilon,z}(\zeta)v = w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}v$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0], z \in \mathbb{R}^N$ and $v \in \mathcal{S}$. Hence $R_{\alpha,\varepsilon,z}(\zeta)$ is an extension of $w_{\varepsilon,z}^{-1}(H_{\alpha,2,V}-\zeta)^{-1}w_{\varepsilon,z}$ and satisfies the estimate

$$\begin{aligned}\|R_{\alpha,\varepsilon,z}(\zeta)\|_{2,2} &\le \|(H_{\alpha,2,V}-\zeta)^{-1}\|_{2,2}\big\|\big(1+T_{\alpha,\varepsilon,z}(H_{\alpha,2,V}-\zeta)^{-1}\big)^{-1}\big\|_{2,2}\\ &\le 2\sup_{\zeta\in K}\|(H_{\alpha,2,V}-\zeta)^{-1}\|_{2,2} < \infty\end{aligned}$$

for all $\zeta \in K, \varepsilon \in [0,\varepsilon_0]$ and $z \in \mathbb{R}^N$. The proof of Proposition 4.7 is thus completed. □

4.3. The case of $e^{-tH^D_{\alpha,p,V}}$ on bounded sets

In this subsection, we prove L^p-spectral independence of a perturbed fractional Dirichlet Laplacian. Let O be a bounded open subset of $\mathbb{R}^N$ and let Δ_D be the Dirichlet Laplacian in $L^2(O)$. i.e., $-\Delta_D$ is the operator associated with the sesquilinear form $a(u,v) := \int_O \nabla u\overline{\nabla v}\,dx$ $\big(u,v \in D(a) = H_0^1(O)\big)$. Since a is positive, closed and symmetric with dense domain, $-\Delta_D$ is self-adjoint and positive definite.

For all $\alpha \in (0,1]$, the C_0-semigroup $\big(e^{-t(-\Delta_D)^\alpha}\big)_{t\ge 0}$ on $L^2(O)$ is positive and satisfies a Gaussian estimate of order α (see (4.13) below). In fact, by the maximal principle [3, Théorème IX.27] (see also the footnote there), $(\lambda - \Delta_D)^{-1} \ge 0$ for all $\lambda > 0$. Hence $e^{t\Delta_D} \ge 0$ $(t \ge 0)$. In addition, by using [15, Proposition 4.2], $0 \le e^{t\Delta_D} \le e^{t\Delta}$ $(t \ge 0)$ (this inequality means that $u \le e^{t\Delta_D}u \le e^{t\Delta}u$ for all positive $u \in L^2(O)$ and $t \ge 0$. Here, the heat semigroup $e^{t\Delta}$ on $L^2(\mathbb{R}^N)$ operates on any $u \in L^2(O)$ identified an element of $L^2(\mathbb{R}^N)$ by considering u to have value 0 on $\mathbb{R}^N \setminus O$). It follows from this inequality and the formula (2) in [20, Chapter IX, Section 11] that

$$0 \le e^{-t(-\Delta_D)^\alpha} \le e^{-t(-\Delta)^\alpha} \quad (t \ge 0). \tag{4.13}$$

Hence, as is proved in [14, Proposition 3.5], there exists a positive C_0-semigroup $U^D_{\alpha,p} = \big(U^D_{\alpha,p}(t)\big)_{t\ge 0}$ on $L^p(O)$ for each $p \in [1,\infty)$ such that $U^D_{\alpha,p}(t)$ and $U^D_{\alpha,q}(t)$ are consistent for all $t \ge 0$ and $p,q \in [1,\infty)$ and $U^D_{\alpha,2}(t) = e^{-t(-\Delta_D)^\alpha}$ for all $t \ge 0$ (by the consistency condition, $U^D_{\alpha,p}$ is unique for each $p \in [1,\infty)$). Since $(-\Delta_D)^\alpha$ is self-adjoint in $L^2(\mathbb{R}^N)$, $U^D_{\alpha,2}(t)(= e^{-t(-\Delta_D)^\alpha})$ is self-adjoint for all

$t \geq 0$. Using the term in Definition 2.7, the family $\{U^D_{\alpha,p}; p \in [1,\infty)\}$ is self-adjoint and consistent. By the consistency above and the self-adjointness of $U^D_{\alpha,2}$, the equality $U^D_{\alpha,p}(t)' = U^D_{\alpha,p'}(t)$ holds for all $t \geq 0$ and $p \in (1,2) \cup (2,\infty)$ (*cf.* Remark 2.8).

In what follows, for any function $f\colon O \to \mathbb{R}$, we define the function $\widetilde{f}\colon \mathbb{R}^N \to \mathbb{R}$ by $\widetilde{f}(x) = f(x)$ $(x \in O), 0$ $(x \notin O)$. (We write $f^{\sim}$ instead of $\widetilde{f}$ in some cases.) This definition is used in the next proposition, which states a sufficient condition for a potential to be $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$. For a $U^D_{\alpha,p}$-admissible V, in the former notation, the perturbed semigroup should be written as $(U^D_{\alpha,p})_V$. However, we will write simply the perturbed semigroup as $U^D_{\alpha,p,V}$.

Proposition 4.17. *Let $V\colon O \to \mathbb{R}$ be a measurable function. Assume the following conditions:*

(i) $\widetilde{V}_- \in \hat{K}_{N,\alpha}$ *and* $c_{N,\alpha}(\widetilde{V}_-) < 1$,

(ii) $Q\big((-\Delta_D)^\alpha\big) \cap Q(V_+)$ *is dense in* $L^2(O)$,

(iii) $H^\alpha(\mathbb{R}^N) \cap Q(\widetilde{V}_+)$ *is dense in* $L^2(\mathbb{R}^N)$.

Then V is $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$ and $\widetilde{V}$ is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$. In addition, the domination

$$0 \leq U^D_{\alpha,2,V}(t) \leq U_{\alpha,2,\widetilde{V}}(t)$$

holds for all $t \geq 0$.

Proof. As is proved in Theorem 2.14, $\widetilde{V}$ is $U_{\alpha,p}$-admissible for all $p \in [1,\infty)$. It is proved that V_+ is $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$ in a similar way as in [18, Proposition 5.8]. The domination for V_+ is proved as follows. For any $t \geq 0, n, m \in \mathbb{N}$ and positive $u \in L^2(O)$, the inequality

$$0 \leq \Big[\Big(U^D_{\alpha,2}\big(\tfrac{t}{m}\big)e^{-\frac{t}{m}V^{(n)}_+}\Big)^m u\Big]^{\sim}(x) \leq \Big[\Big(U_{\alpha,2}\big(\tfrac{t}{m}\big)e^{-\frac{t}{m}\widetilde{V}^{(n)}_+}\Big)^m \widetilde{u}\Big](x)$$

holds for a.e. $x \in \mathbb{R}^N$. Hence, by using the Trotter product formula, we obtain that for any $n \in \mathbb{N}$,

$$0 \leq \big(U^D_{\alpha,2,V^{(n)}_+}(t)u\big)^{\sim}(x) \leq \big(U_{\alpha,2,\widetilde{V}^{(n)}_+}(t)\widetilde{u}\big)(x)$$

for all $t \geq 0$, positive $u \in L^2(O)$ and a.e. $x \in \mathbb{R}^N$. Hence, the domination

$$0 \leq \big(U^D_{\alpha,2,V_+}(t)u\big)^{\sim}(x) \leq \big(U_{\alpha,2,\widetilde{V}_+}(t)\widetilde{u}\big)(x)$$

holds for all $t \geq 0$, positive $u \in L^2(O)$ and a.e. $x \in \mathbb{R}^N$.

To prove that $-V_-$ is $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$, it suffice to show that

$$\sup\Big\{\big\|\exp\big(t(H^D_{\alpha,1} + V^{(n)}_-)\big)\big\| \;\Big|\; 0 \leq t \leq 1, n \in \mathbb{N}\Big\} < \infty$$

(see [19, Proposition 2.2] and Proposition 2.9), where $H^D_{\alpha,1}$ is the generator of $U^D_{\alpha,1}$. By a similar way as in the case of V_+, for all $t \ge 0, n \in \mathbb{N}$ and positive $u \in L^2(O)$, the inequality

$$0 \le \big(U^D_{\alpha,2,-V_-^{(n)}}(t)u\big)^{\sim}(x) \le \big(U_{\alpha,2,-\widetilde{V}_-^{(n)}}(t)\widetilde{u}\big)(x)$$

holds for a.e. $x \in \mathbb{R}^N$. Since $U^D_{\alpha,1,-V_-^{(n)}}(t)$ and $U_{\alpha,1,-\widetilde{V}_-^{(n)}}(t)$ are consistent with $U^D_{\alpha,2,-V_-^{(n)}}(t)$ and $U_{\alpha,2,-\widetilde{V}_-^{(n)}}(t)$, respectively, for all $t \ge 0$ and $n \in \mathbb{N}$, by this consistency and the inequality above, we have

$$0 \le \big(U^D_{\alpha,1,-V_-^{(n)}}(t)u\big)^{\sim}(x) \le \big(U_{\alpha,1,-\widetilde{V}_-^{(n)}}(t)\widetilde{u}\big)(x)$$

for all $t \ge 0, n \in \mathbb{N}$, positive $u \in L^1(O) \cap L^2(O)$ and a.e. $x \in \mathbb{R}^N$. Hence the estimate

$$\big\|U^D_{\alpha,1,-V_-^{(n)}}(t)\big\| \le \big\|U_{\alpha,1,-\widetilde{V}_-^{(n)}}(t)\big\| \le \big\|U_{\alpha,1,-\widetilde{V}_-}(t)\big\|$$

holds for all $t \ge 0$ and $n \in \mathbb{N}$. Thus, the boundedness above holds.

Since both V_+ and $-V_-$ are $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$, V is $U^D_{\alpha,p}$-admissible for all $p \in [1,\infty)$. Now it is easy to prove that

$$0 \le U^D_{\alpha,2,V}(t) \le U_{\alpha,2,\widetilde{V}}(t) \tag{4.14}$$

for all $t \ge 0$. In fact, by replacing $U^D_{\alpha,2}, U_{\alpha,2}$ and V_+ with $U^D_{\alpha,2,V_+}, U_{\alpha,2,\widetilde{V}_+}$ and $-V_-$ respectively in the argument in the case of V_+, we have

$$0 \le \big[\big(U^D_{\alpha,2,V_+}\big)_{-V_-^{(n)}}(t)u\big]^{\sim}(x) \le \big[\big(U_{\alpha,2,\widetilde{V}_+}\big)_{-\widetilde{V}_-^{(n)}}(t)\widetilde{u}\big](x)$$

for all $t \ge 0, n \in \mathbb{N}$, positive $u \in L^2(O)$ and a.e. $x \in \mathbb{R}^N$. Since, as stated in Remark 2.6, $U^D_{\alpha,2,V}(t) = \text{s-}\lim_{n\to\infty}(U^D_{\alpha,2,V_+})_{-V_-^{(n)}}(t)$ in $\mathcal{L}\big(L^2(O)\big)$ and $U_{\alpha,2,\widetilde{V}}(t) = \text{s-}\lim_{n\to\infty}(U_{\alpha,2,\widetilde{V}_+})_{-\widetilde{V}_-^{(n)}}(t)$ in $\mathcal{L}\big(L^2(\mathbb{R}^N)\big)$ for all $t \ge 0$, the domination (4.14) holds. □

Theorem 4.18. *Under the same assumptions as in Proposition* 4.17*,*

$$\sigma(H^D_{\alpha,p,V}) = \sigma(H^D_{\alpha,2,V})$$

holds for all $p \in [1,\infty)$*, where* $H^D_{\alpha,p,V}$ *is the generator of* $U^D_{\alpha,p,V}$*.*

Proof. Note that $\big\|U^D_{\alpha,p,V}(t)\big\|_{p,q} \le \big\|U_{\alpha,p,\widetilde{V}}(t)\big\|_{p,q}$ for all $t \ge 0$ and $1 \le p < q \le \infty$, by the domination in Proposition 4.17. Hence, $\sigma(H^D_{\alpha,2,V}) \subset \sigma(H^D_{\alpha,p,V})$ for all $p \in [1,\infty)$ (for the proof of this spectral inclusion, see the proof of

Theorem 4.1). To prove the converse inclusion, recall Proposition 4.6. For all $1 \le p \le q \le \infty$ and $n > \frac{N}{2\alpha}\big(\frac{1}{p} - \frac{1}{q}\big)$, we have

$$\big\|\big(H^D_{\alpha,2,V} - \lambda\big)^{-n}\big\|_{p,q} < \infty$$

for real and sufficiently negative λ. Next, let K be any compact subset of $\rho(H^D_{\alpha,2,V})$. For all $n > \frac{N}{4\alpha}$ and $\zeta \in K$, by taking a real and sufficiently negative λ, the following estimate holds:

$$\begin{aligned}\big\|\big(H^D_{\alpha,2,V} - \zeta\big)^{-2n}\big\|_{1,\infty} &\le \sum_{j=0}^{2n} \binom{2n}{j} |\zeta - \lambda|^j \big\|(H^D_{\alpha,2,V} - \lambda)^{-n}\big\|_{2,\infty} \\ &\qquad \times \big\|(H^D_{\alpha,2,V} - \zeta)^{-1}\big\|_{2,2}^j \big\|(H^D_{\alpha,2,V} - \lambda)^{-n}\big\|_{1,2} \\ &\le C,\end{aligned}$$

where the constant C is independent of $\zeta \in K$. Hence, $(H^D_{\alpha,2,V} - \zeta)^{-2n}$ above is an integral operator and its integral kernel $G_n(\zeta; x, y)$ satisfies the estimate

$$\big\|G_n(\zeta; \cdot, \cdot)\big\|_{L^\infty(O\times O)} \le C$$

for all $\zeta \in K$. By using this estimate together with the assumption that O is bounded, we can define the function $G_{n,p}$ for all $p \in [1, \infty)$ in the same way as in the proof of Theorem 4.2. By the same argument there, we obtain $\sigma\big(H^D_{\alpha,p,V}\big) \subset \sigma\big(H^D_{\alpha,2,V}\big)$ for all $p \in [1, \infty)$. $\square$

Finally, we give a sufficient condition for a potential to satisfy the assumptions (ii) and (iii) in Proposition 4.17.

Proposition 4.19. *Let O be a bounded open subset of $\mathbb{R}^N$ whose boundary ∂O is a set of Lebesgue measure 0 in $\mathbb{R}^N$. Assume that $V \in L^1_{loc}(O)$, then $Q\big((-\Delta_D)^\alpha\big) \cap Q(V)$ is dense in $L^2(O)$ and $H^\alpha(\mathbb{R}^N) \cap Q\big(\widetilde{V}\big)$ is dense in $L^2(\mathbb{R}^N)$.*

Proof. By the assumption, $C_c^\infty(O)$ is included in $Q\big((-\Delta_D)^\alpha\big) \cap Q(V)$, and hence $Q\big((-\Delta_D)^\alpha\big) \cap Q(V)$ is dense in $L^2(O)$. Next, we prove the latter assertion. To prove this, for any $u \in L^2(\mathbb{R}^N)$, we take a sequence $\{u_n\}_n$ in $C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Now we define

$$K_n := \big\{x \in \mathbb{R}^N \;\big|\; d(x, \partial O) \ge \tfrac{1}{n}\big\}$$

for all $n \in \mathbb{N}$, where $d(x, A)$ denotes the distance of a point $x \in \mathbb{R}^N$ from a closed set $A \subset \mathbb{R}^N$. For all $n \in \mathbb{N}$, K_n is closed and satisfies $K_n \subset K_{n+1}^\circ$ (K_{n+1}° denotes the interior of K_{n+1}) and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^N \setminus \partial O$. For all $n \in \mathbb{N}$, we can take a function $\phi_n \in C^\infty(\mathbb{R}^N)$ such that $0 \le \phi_n \le 1$ and

$$\phi_n(x) = \begin{cases} 1 & (x \in K_n), \\ 0 & (x \in \mathbb{R}^N \setminus K_{n+1}^\circ). \end{cases}$$

It is easy to prove that $\phi_n u_n \in H^\alpha(\mathbb{R}^N) \cap Q(\widetilde{V})$ $(n \in \mathbb{N})$ and $\phi_n u_n \to u$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$ (note the assumption that the measure of ∂O is 0). Thus $H^\alpha(\mathbb{R}^N) \cap Q(\widetilde{V})$ is dense in $L^2(\mathbb{R}^N)$. □

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