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Kuramoto-Sivashinsky type equations on a half-line

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Abstract. We study the initial-boundary value problem for a general class of nonlinear dissipative equations on a half-line

$$\begin{cases} u_t + \mathbb{N}(u,u_x) + \mathbb{K}u = f, \quad (x,t) \in \mathbf{R}^+ \times \mathbf{R}^+, \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^+, \\ \partial_x^{j-1} u(0,t) = h_j\,(t) \quad \text{for } j = 1,...,M, \end{cases} \tag{0.1}$$

where the nonlinear term $\mathbb{N}(u,u_x)$ depends on the unknown function u and its derivative u_x and satisfies the estimate

$$|\mathbb{N}(u,v)| \leq C\,|u|^\rho\,|v|^\sigma$$

with $\rho, \sigma \geq 0$ and the linear operator $\mathbb{K}(u)$ is defined as follows

$$\mathbb{K}(u) = a_n \partial_x^n + a_m \partial_x^m,$$

where the constants $a_n, a_m \in \mathbf{R}$, n, m are integers, $m > n, n \leq M+1, n$ is an even integer.

The aim of this paper is to prove the global existence of solutions to the initial-boundary value problem (0.1). We find the main term of the asymptotic representation of solutions.

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§1. Introduction

We consider the initial-boundary value problem on a half-line for nonlinear equation

$$\begin{cases} u_t + \mathbb{N}(u,u_x) + \mathbb{K}(u) = f, \quad t > 0, x > 0, \\ u(x,0) = u_0(x),\ x > 0; \\ \partial_x^{j-1} u\,(0,t) = h_j\,(t)\,,\ t > 0, j = 1,...,M, \end{cases} \tag{1.1}$$

where the nonlinear term $\mathbb{N}(u, u_x)$ depends on the unknown function u and its derivative u_x and satisfies the estimate

$$|\mathbb{N}(u, v)| \leq C\,|u|^{\rho}\,|v|^{\sigma}$$

with $\sigma \geq 0, \rho \geq 2$. The linear operator $\mathbb{K}(u)$ is defined as follows

$$\mathbb{K}u = a_n \partial_x^n + a_m \partial_x^m,$$

where the constants $a_n, a_m \in \mathbf{R}$, n, m are integers. The number M of the boundary data depends essentially on the operator $\mathbb{K}$ (see [8]).

Equation (1.1) is a simple universal model, which appears as the first approximation in the description of the dispersive dissipative nonlinear waves (see [16]), so that a great number of physical problem have dealt with problem (1.1). We do not even attempt to provide a complete review of these problem, we give a list of some well-known equations, leading to nonlinear equation (1.1). The famous Kortweg-de Vries-Burgers equation

$$u_t + u_x u + \alpha u_{xxx} - \nu u_{xx} = 0$$

appears in the theory of nonlinear acoustics for fluids with gas bubbles. The Kuramoto-Sivashinsky equation

$$u_t + \frac{1}{2} u_x^2 + u_{xx} + \alpha u_{xxxx} = 0$$

is applied, for instance, in the theory of combustion to model a flame front and also in the study of two-dimensional turbulence. Finally, we mention the Kawahara equation

$$u_t + u_x u + \alpha u_{xxx} - u_{xxxxx} = 0,$$

which describes propagation of signals in transmission lines, propagation of long waves under ice cover in liquids depth, and also gravity waves on the surface of a liquid with surface tension. In conclusion, we emphasize that the description of all above-mentioned numerous and various examples of physical problems are described in a unified way by nonlinear equation (1.1). Thus, the study of nonlinear equation (1.1) enables one to proceed from the analysis of individual equations to the investigation of wide classes of nonlinear equations that are of great interest for physical application.

Note that the operator $\mathbb{K}u$ has the symbol

$$K(p) = a_n p^n + a_m p^m.$$

In this paper we assume the dissipation condition $\operatorname{Re} K(p) > 0$ for $\operatorname{Re} p = 0$, $p \neq 0$, and also let $m > n$, $n \leq M+1$, n be an even integer. The number of the boundary data is $M = \left[\frac{m}{2}\right]$. If m is an odd integer the number of boundary data depends on the sign of a_m. If $a_m > 0$ we need to put $\left[\frac{m}{2}\right]$ boundary data and, then $a_m < 0$ the number of boundary data equals to $\left[\frac{m+1}{2}\right]$ (see [8]). The main goal of this paper is to find the large time asymptotics of solutions to the problem (1.1). A great number of publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of this publications, we do list some known results [1], [2], [3], [4], [5], [11], [12], [17] and [18], where there were obtained optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations.

Some results on the decay estimates of the solutions in different norms to the Cauchy problems for the Korteweg-de Vries-Burgers type equations were obtained in papers [14], [15], [16]. A general theory of nonlinear nonlocal equations on a half-line was developed in book [8], where it was introduced the pseudodifferential operator $\mathbb{K}$ with homogeneous symbol $K(p) = Cp^{\beta}$ and it was shown that the number of the boundary data which are necessary for the well-posedness of the problem is equal to $\left[\frac{\beta}{2}\right]$ except the case, when β is an odd integer. As far as we know the initial-boundary value problem (1.1) for nonlinear equations with general nonhomogeneous operator $\mathbb{K}$ were not studied previously. In the present paper we fill this gap, considering as example the equation (1.1) with a polynomial $K(p) = a_n p^n + a_m p^m$. To construct the Green operator for problem (1.1) we can not use the methods of book [8] directly, also we need to obtain additional estimates for the Green functions, which have different analyticity properties comparing with the case of homogeneous symbol $K(p)$. Another difficulty which we overcome in the present paper is in evaluating the contribution of the boundary data into the large time asymptotic behavior of solutions, which can be completely different comparing with the case of the corresponding Cauchy problem. Indeed as we will see below the solution of the initial-boundary value problem (1.1) obtains an additional time decay due to boundary data comparing with the corresponding Cauchy problem. As a result the nonlinear term in the initial-boundary value problem (1.1) is super critical in the contrary to the corresponding Cauchy problem. In particular, the nonlinearity of the shallow water type uu_x in the Korteweg-de Vries-Burgers equations is critical in the case of the Cauchy problem, however it is super critical in the case of the Dirichlet initial-boundary value problem (see [9]).

Denote by $\mathbf{H}_p^{s,k} = \left\{ f \in \mathbf{L}^p ; \|f\|_{\mathbf{H}_p^{s,k}} = \| \langle x \rangle^k \langle i\partial_x \rangle^s f \|_{\mathbf{L}^p} < \infty \right\}$ the weighted

Sobolev space and define

$$\mathbf{Y}(\beta_1,\beta_2) = \{\varphi(t) \in C(0,\infty), \|\phi\|_{\mathbf{Y}} < \infty, \}$$
$$\text{with } \|\phi\|_{\mathbf{Y}} {=} \sup_{t>0} \{t\}^{\beta_1} \sum_{k=1}^{m-1} \left\| \frac{d^k}{dt^k}\phi \right\|_{\mathbf{L}^\infty} + \langle t\rangle^{\beta_2} \|\phi\|_{\mathbf{L}^\infty},$$

$\beta_1 \in [0,1)\,, \beta_2 > 1$. Here and below $\langle x\rangle = \sqrt{1+x^2}$, $\{x\} = \frac{x}{\langle x\rangle}$. By the same letter C we denote different positive constants.

In the next theorem we give sufficient conditions for the global existence of solutions of the initial-boundary value problem (1.1). Denote $Q = \frac{n}{2}$.

Theorem 1. *Suppose* $u_0 \in \mathbf{H}_1^{0,Q+\delta}(\mathbf{R}^+) \cap \mathbf{H}_2^{1,0}(\mathbf{R}^+)$,

$$\{t\}^{\nu_1} \langle t\rangle^{\nu_2} f \in \mathbf{L}^\infty\left(0,\infty; \mathbf{H}_1^{0,Q+\delta}\left(\mathbf{R}^+\right)\right),\ \sum_{k=1}^{M} h_k \in \mathbf{Y}(\beta_1,\beta_2)$$

with $\nu_1 \in \left(0, 1-\frac{1}{2m}\right)$, $\nu_2 > 1$, $\delta \in (0,1)$, $\beta_1 < \frac{1}{2m}$ *and the norm*

$$\|u_0\|_{\mathbf{H_1}^{0,\frac{Q+\delta}{2}}} + \|u_0\|_{\mathbf{H_2}^{1,0}} + \sup_{t>0}\left(\{t\}^{\nu_1} \langle t\rangle^{\nu_2} \|f\|_{\mathbf{H_1}^{0,\frac{Q+\delta}{2}}} + \left\| \sum_{k=1}^{M} h_k \right\|_{\mathbf{Y}} \right) \le \epsilon,$$

where $\epsilon > 0$ *is small enough. Then under the condition*

$$(Q+1)(\rho+\sigma-1) > n$$

there exists a unique solution

$$u \in \mathbf{C}\left([0,\infty); \mathbf{H}_2^{0,\frac{Q+\delta}{2}}\left(\mathbf{R}^+\right)\right) \cap \mathbf{C}\left(0,\infty; \mathbf{H}_2^{n-1,0}\left(\mathbf{R}^+\right)\right)$$

of the initial-boundary value problem (1.1). *Moreover there exists a constant* A *such that the solution has the following asymptotics*

$$u(x,t) = t^{-\frac{Q+1}{n}} A\Phi\left(\frac{x}{\sqrt[n]{t}}\right) + O\left(t^{-\frac{Q+1+\mu}{n}}\right),$$

for $t \to \infty$ *uniformly with respect to* $x > 0$, *where*

$$\mu = \min(\delta, (Q+1)(\rho+\sigma-1) + \sigma - n, \nu_2 - 1, m - M - Q + n\beta_2 - n)$$

and

$$\Phi(q) = \int_{-i\infty}^{i\infty} e^{zq - a_n z^n} z^Q dz.$$

Remark 1. *Note that the decay rate* $t^{-\frac{Q+1}{n}}$ *of solutions to the problem* (1.1) *obtained in Theorem* 1 *is more rapid in comparison with the case of the Cauchy problem, where the decay rate is* $t^{-\frac{1}{n}}$. *This is due to the influence of the boundary data as in the case of the heat equation.*

Remark 2. *The restrictions of the dissipation condition and* n *to be an even integer are technical ones. We believe that our method could be applied for more general equations.*

Remark 3. *For a general type nonlinearity the blow up phenomena is possible (see* [13]*) so we restrict our attention to the case of small initial data.*

We organize our paper as follows. In Section 2 we consider the linear initial-boundary value problem corresponding to (1.1). We construct the Green function of the solution of the linear problem and formulate Theorem 2 on the existence and uniqueness of the solution. In Section 3 we obtain asymptotic formula for Green function. In Section 4 we prove some preliminary estimates. Section 5 is devoted to the proof of Theorem 2 for the linear problem. In Section 6 we prove Theorem on the local existence of solutions to the nonlinear problem (1.1). Theorem 1 is proved in Section 7.

§2. Linear problem

We consider the linear initial-boundary value problem corresponding to (1.1)

$$\tag{2.1} \begin{cases} u_t + \mathbb{K}u = f(x,t), \quad t>0, x>0, \\ u(x,0) = u_0(x), \quad x>0, \\ \partial_x^{j-1} u(0,t) = h_j(t), \ t>0, \ \text{for} \quad j = 0,...,M, \end{cases}$$

where integer number M depends on order m. By virtue result of the book [8] we need to put into initial boundary value problem (2.1) $M = \left[\frac{m}{2}\right]$ boundary data for its correct solvability.

If m is an odd integer the number of boundary data depends on the sign of a_m. If $a_m > 0$ we need to put $\left[\frac{m}{2}\right]$ boundary data and, then $a_m < 0$ the number of boundary data equals to $\left[\frac{m+1}{2}\right]$. We define symbol of operator $\mathbb{K}$ as

$$K(p) = a_n p^n + a_m p^m,$$

where $n \le M+1, a_n \ne 0, a_m \ne 0$. We denote by $\phi_j(\xi) = K^{-1}(-\xi)$, $j = 1,...,N = m - M$, different roots of equation for equation $K(p) = -\xi$, such that

$$\operatorname{Re} \phi_j(\xi) > 0$$

for all $\xi \in D, \operatorname{Re} \xi > 0$. Here D is domain of analyticity of functions $\phi_j(\xi)$ with boundary Γ

$$\Gamma = \left\{ (-i\infty, -i0) \bigcup_k \left(\left[-i0, K(p_k)e^{i2\pi}\right] \cup [K(p_k), i0] \right) \cup (i0, i\infty) \right\},$$

where $K'(p_k) = 0$. Also we define matrices $\mathbb{A}$

$$\mathbb{A} = \begin{pmatrix} \phi_1^{N-1} & \phi_1^{N-2} & \phi_1^{N-3} & \cdots & 1 \\ \phi_2^{N-1} & \phi_2^{N-2} & \phi_2^{N-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \phi_N^{N-1} & \phi_N^{N-2} & \phi_N^{N-3} & \cdots & 1 \end{pmatrix}.$$

and vector $\overrightarrow{\mathbf{B}}$

$$\overrightarrow{\mathbf{B}} = \begin{pmatrix} e^{-\phi_1(\xi)} \\ e^{-\phi_2(\xi)} \\ \dots \\ e^{-\phi_N(\xi)} \end{pmatrix}.$$

In this section we follow the method of the book [8] to obtain the explicit formula for the solution of the linear problem (2.1) under the condition

$$u_0 \in \mathbf{L}^1\left(\mathbf{R}^+\right), f \in \mathbf{L}^q\left(0, T; \mathbf{L}^1\left(\mathbf{R}^+\right)\right)$$

with $q > 2$. From the book [8] we have that solution of problem (2.1) has the following form

$$\begin{aligned} u(x,t) = & \int_0^{+\infty} u_0(y) G(x,y,t) dy \\ & + \int_0^t d\tau \int_0^{+\infty} f(x,y,\tau) G(x,y,t-\tau) dy \\ & + \mathcal{H}\left[h_1, ..., h_M\right](x,t), \end{aligned}$$

where

$$G(x,y,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px} H(p,y,t) dp,$$

$$H(p,y,t) = e^{-py-K(p)t} - \sum_{j=1}^{N} p^{N-j} \frac{1}{2\pi i} \int_\Gamma \frac{e^{\xi t}}{K(p)+\xi} \left(\mathbb{A}^{-1}\overrightarrow{\mathbf{B}}\right)_j,$$

and the function $\mathcal{H}\left[h_1, ..., h_M\right](x,t)$ has the following form

$$\mathcal{H}\left[\overrightarrow{h}\right](x,t) = \frac{1}{2\pi i} \sum_{j=1}^{[\alpha]} \int_0^t d\tau H_j(\tau) \int_{-i\infty}^{i\infty} e^{px-K(p)(t-\tau)} K(p) p^{-j} dp.$$

The functions H_j are defined as

$$H_j(t) = h_j(t)$$

for $j = 1, ..., M$ and

$$H_{j+M}(t) = -\frac{1}{2\pi i}\sum_{l=1}^{M}\int_{-i\infty}^{i\infty} e^{\xi t}\widehat{h}_l(\xi)\xi^{-\frac{\alpha-M-j}{\alpha}}\left(\mathbb{C}^{-1}\begin{pmatrix}\phi_1^{\alpha-l}(\xi)\\ ....\\ \phi_m^{\alpha-l}(\xi)\end{pmatrix}\right)_j d\xi$$

for $j = 1, ..., N$. Since for $\alpha \geq j$

$$p^{\alpha-j} = \lim_{y\to+0}\overline{\partial_y^{\alpha-j}}e^{-py},$$

where

$$\overline{\partial_x^w f} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}(-p)^w\left(\widehat{f} - \sum_{j=1}^{[w]}\frac{\partial_x^{j-1}f(0)}{p^j}\right)dp,$$

we can rewrite operator $\mathcal{H}\left[\overrightarrow{h}\right]$ in the following form

$$\mathcal{H}\left[h_1, ..., h_M\right](x,t) = \sum_{j=n,m}\sum_{k=1}^{M}a_j\int_0^t d\tau h_k(\tau)\partial_y^{(j-k)}G(x,0,t-\tau).$$

Denote

$$\widetilde{H}(p,\phi_1,\phi_2,...,\phi_N,y) = \sum_{k=1}^{N}p^{N-k}\left(\mathbb{A}^{-1}\overrightarrow{B}\right)_k, \tag{2.2}$$

We write function $\widetilde{H}$ in the form

$$\widetilde{H} = \sum_{k=1}^{N}p^{N-k}\sum_{j=1}^{N}e^{-\phi_j y}\left(\mathbb{A}^{-1}\overrightarrow{E_j}\right)_k,$$

where $\overrightarrow{E_j}$ is vector with component $e_l, l = 1, .., N$

$$e_l = \begin{cases} 1, & l = j\\ 0, & l \neq j.\end{cases}$$

We have

$$\left(\mathbb{A}^{-1}\overrightarrow{E_j}\right)_k = (-1)^{k+N}\sigma_{k-1}(\phi_1,\phi_2...\phi_{k-1},\phi_{k+1},...,\phi_N)\prod_{l=1,l\neq k}(\phi_k-\phi_l)^{-1}, \tag{2.3}$$

where σ_k are symmetrical polynomials, such that

$$\sigma_0(\phi_2, ..., \phi_N) = 1,$$
$$\sigma_1(\phi_2, ..., \phi_N) = \sum_{j=2}^{N} \phi_j,$$
$$\sigma_2(\phi_2, ..., \phi_N) = \sum_{j \neq k} \phi_j \phi_k,$$
$$.....$$
$$\sigma_N(\phi_2, ..., \phi_N) = \prod_{j=2}^{N} \phi_j.$$

We prove formula (2.3) by induction. We can see that

$$(\mathbb{A}_3^{-1}\overrightarrow{E})_k = (-1)^{k+3} \frac{\sigma_{k-1}(\phi_2, \phi_3)}{(\phi_2 - \phi_1)(\phi_3 - \phi_1)},$$

where

$$\mathbb{A}_3 = \begin{pmatrix} \phi_1^2 & \phi_1 & 1 \\ \phi_2^2 & \phi_2 & 1 \\ \phi_3^2 & \phi_3 & 1 \end{pmatrix}.$$

In the case $k = 1$ we directly obtain (2.3). We have for $k > 1$

$$\begin{aligned}
&(\mathbb{A}^{-1}\overrightarrow{E})_k \\
&= (-1)^{k+1}(-1)^{2N+1} \left((-1)^k \phi_2 (\mathbb{A}_{N-1}^{-1}\overrightarrow{E})_{k-1} + (-1)^{k+1} (\mathbb{A}_{N-1}^{-1}\overrightarrow{E})_k \right) \\
&\quad \times \frac{(\phi_3 - \phi_2)(\phi_4 - \phi_2)...(\phi_N - \phi_2)}{(\phi_2 - \phi_1)...(\phi_m - \phi_1)} \\
&= (-1)^{k+1}(-1)^{3N} \frac{(-1)^{2k-1}\phi_2\sigma_{k-2}(\phi_3, ..., \phi_N) + (-1)^{2k+1}\sigma_{k-1}(\phi_3, ..., \phi_N)}{\prod_{l=3}^{N}(\phi_l - \phi_2)} \\
&\quad \times \frac{\prod_{l=3}^{N}(\phi_l - \phi_2)}{\prod_{l=2}^{N}(\phi_l - \phi_1)} = (-1)^k(-1)^N \frac{\sigma_{k-1}(\phi_2, ..., \phi_N)}{(\phi_2 - \phi_1)(\phi_3 - \phi_1)...(\phi_N - \phi_1)}.
\end{aligned}$$

Thus by induction we have (2.3). Therefore using (2.3) and Viett Theorem by direct calculation we obtain

$$\widetilde{H} = \sum_{j=1}^{N} \frac{e^{-\phi_j y} P_N(p)}{P_N'(\phi_j)(p - \phi_j)}, \tag{2.4}$$

where by $P_N(p)$ we denote

$$P_N(p) = \prod_{l=1}^{N} (p - \phi_l). \tag{2.5}$$

We easily see that function $\widetilde{H}(p,\phi_1,\phi_2,...,\phi_N,y)$ is symmetrical with respect to variables $\phi_j(\xi)$, such that, for example,

$$\widetilde{H}(p,\phi_1,\phi_2,...,\phi_N,y)=\widetilde{H}(p,\phi_2,\phi_1,...,\phi_N,y).$$

Since $\mathrm{Re}\,K(p)>0$ for all $\mathrm{Re}\,p=0$, $p\neq 0$ and the function $\widetilde{H}(p,\phi_1,...,\phi_N,y)$ is analytic in the domain D and has the estimate

$$\left|\widetilde{H}(p,\phi_1,...,\phi_N,y)\right|\le Ce^{-Cy\sqrt[m]{|\xi|}} \tag{2.6}$$

for $|\xi|\to\infty$, $\mathrm{Re}\,\xi>0$ (see asymptotic formulas ((3.1) below), therefore by the Cauchy theorem and symmetrical properties of function $\widetilde{H}$ we can change the contour of integration Γ to the imaginary axis $(-i\infty,i\infty)$ to get

$$\int_\Gamma \widetilde{H}\frac{e^{\xi t}}{K(p)+\xi}d\xi=\int_{-i\infty}^{i\infty}\widetilde{H}\frac{e^{\xi t}}{K(p)+\xi}d\xi,$$

(since $\mathrm{Re}\,\phi_j(\xi)>0$ for all $\mathrm{Re}\,\xi=0$, and taking into account inequality (2.6), we see that the last integral in the above formula is converges absolutely). Applying the identities

$$K'(\phi_l)=-\frac{1}{\phi_l'(\xi)}$$

and using the theory of residues, we obtain

$$\int_{-i\infty}^{i\infty}\frac{\widetilde{H}(p,\phi_1,...,\phi_N,y)}{K(p)+\xi}e^{px}dp=2\pi i\sum_{l=1}^{M}\widetilde{H}(\phi_{l+N},\phi_1,...,\phi_N,y)e^{\phi_{l+N}(\xi)x}\phi_{l+N}'(\xi),$$

where ϕ_{l+N} are "negative" roots of equation $K(p)=-\xi$, such that

$$\mathrm{Re}\,\phi_j(\xi)<0,\ \ \mathrm{Re}\,\xi>0.$$

Making the change of variable $p=\phi_{l+N}(\xi)$, using that for $\phi_k(\xi)$, $k=1,...,N$ there exists some function $\phi_j(\xi)$, $j=1,...,N$, such that for $\mathrm{Re}\,\xi=0$ $\phi_k(-\xi)=\phi_j(\xi)$ and taking into account symmetrical properties of function $\widetilde{H}$ we get

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{-i\infty}^{i\infty}e^{px}\int_{-i\infty}^{i\infty}\sum_{j=1}^{N}p^{N-j}\int_\Gamma\frac{e^{\xi t}}{K(p)+\xi}\left(\mathbb{A}^{-1}\overrightarrow{\mathbf{B}}\right)_j \\
&=-\frac{1}{2\pi i}\sum_{l=1}^{M}\int_{-i\infty}^{i\infty}e^{\xi t}\widetilde{H}(\phi_{l+N},\phi_1,\phi_2,...,\phi_N,y)e^{\phi_{l+N}(\xi)x}\phi_{l+N}'(\xi)d\xi \\
&=-\frac{1}{2\pi i}\sum_{l=1}^{M}\int_{\Gamma_l}e^{px-K(p)t}\widetilde{H}(p,\phi_1(K(p)),\phi_2,...,\phi_N,y)dp \\
&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{px-K(p)t}\widetilde{H}(p,\phi_1(K(p)),\phi_2,...,\phi_N,y)dp,
\end{aligned} \tag{2.7}$$

where

$$\Gamma_l = \{p = \phi_{l+N}(\xi), \operatorname{Re}\xi = 0\}.$$

Therefore taking into account (2.7), (2.4) we obtain the following integral representation for Green function $G(x, y, t)$ of problem (2.1)

$$G(x, y, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{px - K(p)t} H(p, y) dp, \tag{2.8}$$

$$H(p, y) = e^{-py} + \sum_{j=1}^{N} \frac{e^{-\phi_j(K(p))y} P_N(p)}{P'_N(\phi_j)(p - \phi_j)},$$

where the function P_N is defined in (2.5).

We have

$$\lim_{y \to +0} \partial_y^{(w)} H(p, y) = (-1)^w p^w \left(1 + \frac{1}{p^w} \sum_{j=1}^{N} \frac{\phi_j^w P_N(p)}{P'_N(\phi_j)(p - \phi_j)}\right).$$

Since for $w = 1, ..., N - 1$

$$\sum_{j=1}^{N} \frac{\phi_j^w P_N(p)}{P'_N(\phi_j)(p - \phi_j)} = \frac{P_N(p)}{2\pi i} \int_{-i\infty, u \neq p}^{i\infty} \frac{u^w}{P_N(u)(p - u)} dp = -p^w \tag{2.9}$$

we get

$$\lim_{y \to +0} \partial_y^{(w)} H(p, y) = 0, w = 1, ..., N - 1.$$

Whence we obtain the following integral representation for solution $u(x, t)$ of the problem (2.1)

$$\begin{aligned} u(x, t) = & \int_0^{+\infty} u_0(y) G(x, y, t) dy \\ & + \int_0^t d\tau f(x, y, \tau) G(x, y, t - \tau) \\ & + \sum_{j=n,m} \sum_{k=1}^{\min(M, j-N)} (-1)^{j-k} a_j \int_0^t h_k(\tau) \partial_y^{j-k} G(x, 0, t - \tau) d\tau, \end{aligned} \tag{2.10}$$

where function $G(x, y, t)$ is defined by formula (2.8).

Now we formulate the following result, which will be proved below in Section 4.

Denote

$$\mathbf{Y}(\beta_1,\beta_2) = \{\varphi(t) \in C(0, \infty), \|\phi\|_{\mathbf{Y}} < \infty, \}$$

$$\text{and } \|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\beta_1} \sum_{k=0}^{m-1} \| \frac{d^k}{dt^k} \phi \|_{\mathbf{L}^\infty} + \langle t \rangle^{\beta_2} \|\phi\|_{\mathbf{L}^\infty}, \beta_1 < 1, \beta_2 > 1.$$

Theorem 2. *Let*

$$u_0 \in \mathbf{H}_1^{0,\delta}\left(\mathbf{R}^+\right) \cap \mathbf{H_2}^{1,0}\left(\mathbf{R}^+\right), t^\nu f \in \mathbf{L}^\infty\left(0,T;\mathbf{H_1}^{0,\delta}\left(\mathbf{R}^+\right)\right), \sum_{k=1}^{M} h_k \in \mathbf{Y}(\beta_1,\beta_2)$$

with $\nu \in \left(0, 1-\frac{1}{2m}\right)$, $\beta_1 < \frac{1}{2m}, \delta \in \left[\frac{1}{2}, Q\right]$. *Then for some* $T > 0$ *there exists a unique solution*

$$u \in \mathbf{C}\left([0,T]\,;\mathbf{H}_2^{0,\delta}\left(\mathbf{R}^+\right)\right) \cap \mathbf{C}\left((0,T]\,;\mathbf{H}_2^{m-1,0}\left(\mathbf{R}^+\right)\right)$$

of the initial-boundary value problem (2.1) *such that*

$$\sup_{t\in(0,T]}\left(\|u(\cdot,t)\|_{\mathbf{H}_2^{0,\delta}} + \sum_{l=1}^{m-1} t^{\frac{l}{m}}\|\partial_x^l u(\cdot,t)\|_{\mathbf{L}^2}\right) \le C\lambda,$$

where

$$\lambda = \|u_0\|_{\mathbf{H}_1^{0,\delta}} + \|u_0\|_{\mathbf{H}^{1,0}} + T^{1-\nu_1}\sup_{t\in[0,T]}\left(t^\nu\|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\|\sum_{k=1}^{M} h_k\right\|_{\mathbf{Y}}\right)$$

and $\nu_1 = \max\left(\nu + \frac{1}{2m}, 1 - \frac{1}{2m} + \beta_1\right) < 1, \gamma > 0.$

§3. Asymptotics of the Green function

Using result of book [8] if $K(p) = a_n p^n + a_m p^m$, $m > n$, then there exist $M_1 = \left[\frac{m+1}{2}\right]$ different inverse functions $\phi_j(\xi) = K^{-1}(-\xi)$, such that for $\xi \in D$

$$\operatorname{Re}\phi_j(\xi) > 0$$

for $j = 1, 2, ..., M_1$. Moreover the asymptotics

$$\phi_l(\xi) = e^{i(\pi+2\pi l)\frac{1}{m}}\left(a_m^{-1}\xi\right)^{\frac{1}{m}} + O\left(\xi^{-\frac{1+\gamma}{m}}\right) \tag{3.1}$$

is true as $\xi \to \infty$. Now we consider case $\xi \to 0$.

We represent

$$p^n = \frac{1}{a_n}\frac{-\xi}{1+O(p)} \quad \text{or} \quad p^{m-n} = \frac{a_n}{a_m}(-1 + \frac{-\xi}{a_n p^n} + O(p)) \text{ for } |p| \le 1.$$

Hence we get the asymptotic representations for $l = 1, ..., Q$

$$\phi_l(\xi) = \xi^{\frac{1}{n}} e^{i\frac{2\pi(l-1)}{n}} + O(|\xi|^{\frac{2}{n}}), \xi \to 0 \tag{3.2}$$

and for $l = Q+1, ..., N$

$$\phi_l(\xi) = \left(\frac{a_n}{a_m}\right)^{\frac{1}{m-n}} e^{i\frac{\pi+2\pi(l-1)}{m-n}} + O(|\xi|^{\frac{2}{n}}), \xi \to 0. \tag{3.3}$$

Here number $Q = \frac{n}{2}$.

We have (see (2.8))

$$G(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t} H(p,y)dp,$$

$$H(p,y) = e^{-py} + \sum_{j=1}^{N} \frac{e^{-\phi_j(K(p))y} P_N(p)}{P_N'(\phi_j)(p-\phi_j)}. \tag{3.4}$$

Since for $l = 0, ..., N-1$ (see (2.9))

$$\sum_{j=1}^{N} \frac{\phi_j^l P_N(p)}{P_N'(\phi_j)(p-\phi_j)} = -p^l$$

we have

$$H(p,y) = e^{-py} - \sum_{l=1}^{N-1} \frac{(-py)^l}{l!} + \sum_{k=1}^{N} B(\phi_j(K(p))) \frac{P_N(p)}{P_N'(\phi_j)\,(p-\phi_j)},$$

where

$$B(z) = e^{-zy} - \sum_{l=0}^{N-1} (-1)^l \frac{(zy)^l}{l!}.$$

Therefore using (3.2)-(3.3) and symmetrical properties of function $H(p,y)$ we obtain for $|p| \ll 1$,

$$\sum_{j=1}^{Q} B(\phi_j(K(p))) \frac{P_N(p)}{P_N'(\phi_j)\,(p-\phi_j)} = O(p^{Q+1} y^{Q+1}) \tag{3.5}$$

and

$$\begin{aligned} &\sum_{k=Q+1}^{N} B(\phi_k(K(p))) \frac{P_N(p)}{P_N'(\phi_k)\,(p-\phi_k)} \\ &= (-1)^{1+N-Q} (\prod_{l=1}^{N-Q} r_l)^{-1} p^Q \prod_{l=1}^{N-Q} r_l \prod_{j=1}^{Q} (1-\beta_j) \sum_{k=1}^{N-Q} B(r_k) \\ &\quad + O(p^{Q+1}(1+y^{Q+1})) \\ &= (-1)^{1+N-Q} p^Q \prod_{j=1}^{Q} (1-\beta_j) \sum_{k=1}^{N-Q} B(r_k) + O(p^{Q+1}(1+y^{Q+1})), \end{aligned} \tag{3.6}$$

where

$$\beta_l = a_n^{\frac{1}{n}} e^{i\frac{2\pi(l-1)}{n}}$$

and

$$r_k = \left(\frac{a_n}{a_m}\right)^{\frac{1}{m-n}} e^{i\frac{2\pi(k-1)}{m-n}}.$$

Substituting (3.5)-(3.6) into (3.4) we easily get

$$H(p,y) = Dp^Q B(y) + O(p^{Q+1}(1+y^{Q+1})),$$

where constant D is defined by

$$D = (-1)^{1+N-Q}\prod_{j=1}^{Q}(1-\beta_j) \neq 0$$

and function $B(y)$

$$B(y) = \sum_{j=1}^{N-Q}\left(e^{-r_j y} - \sum_{l=0}^{Q}(-1)^l \frac{(r_j y)^l}{l!}\right).$$

Note that for $|p| > 1$

$$|H(p,y)| < C.$$

Making the change of variable $p^n t = z^n$ we obtain for Green function

$$\begin{aligned}
(3.7)\qquad & G(x,y,t) \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t} H(p,y)dp \\
&= \frac{1}{2\pi i}\left(\int_{-i}^{i} e^{px-a_n p^n t}\left(Dp^Q B(y) + O(p^{Q+1}(1+y^{Q+1}))\right)dp\right. \\
&\quad + \int_{-i}^{i} e^{px-a_n p^n t}(e^{-Cp^m t}-1)O(p^Q(1+y^{Q+1}))dp \\
&\quad \left. + \int_{-i\infty,|p|>1}^{i\infty} e^{px-K(p)t} H(p,y)dp\right) \\
&= \frac{1}{2\pi i} Dt^{-\frac{Q+1}{n}} B(y)\int_{-i\infty}^{i\infty} e^{zx_1 - a_n z^n} z^Q dz + O(t^{-\frac{Q+2}{n}}(1+y^{Q+1})),
\end{aligned}$$

where $x_1 = xt^{-\frac{1}{n}}$.

§4. Preliminaries

We introduce the operators

$$\mathbb{G}(t)f = \int_0^{+\infty} G(x,y,t)f(y)dy \tag{4.1}$$

and

$$\mathbb{H}_w(x,t)h = \int_0^t h(\tau)\partial_y^w G(x,0,t-\tau)d\tau, \tag{4.2}$$

where

$$G(x,y,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-K(p)t}H(p,y)dp,$$

$$H(p,y) = e^{-py} + \sum_{j=1}^{N}\frac{e^{-\phi_j(K(p))y}P_N(p)}{P_N'(\phi_j)(p-\phi_j)},$$

$$P_N(p) = \prod_{l=1}^{N}(p-\phi_l).$$

Lemma 1. *The following estimates are valid*

$$\left\|(\cdot)^{\delta}\,\partial_x^{(k)}\mathbb{G}f\right\|_{\mathbf{L}^2} \le Ct^{-\frac{1+2k-2\delta}{2m}-\gamma}\left\|\langle y\rangle^{\delta} f\right\|_{\mathbf{L}^1}, t<1$$

and

$$\left\|(\cdot)^{\delta}\,\partial_x^{(k)}\mathbb{G}f\right\|_{\mathbf{L}^2} \le Ct^{-\frac{1+2k+2Q-2\delta}{2n}+\gamma}\left\|\langle y\rangle^{Q} f\right\|_{\mathbf{L}^1}, t>1$$

and

$$\left\|\partial_x^{(k)}\mathbb{G}f\right\|_{\mathbf{L}^2} < Ct^{-\frac{k}{m}}\left(\|f\|_{\mathbf{L}^2} + \left\|\left(1+y^{-\frac{1}{2}-\gamma}\right)f\right\|_{\mathbf{L}^1}\right),$$

where $\gamma>0,\ \delta\in[0,Q],\ k\in[0,m-1],\ t>0.$ *Moreover the asymptotics for large time is true*

$$\mathbb{G}f = t^{-\frac{Q+1}{n}}A\int_{-i\infty}^{i\infty} e^{zx_1-a_nz^n}z^Q dz + O(t^{-\frac{Q+1+\mu}{n}}\left\|\langle y\rangle^{Q+\mu} f\right\|_{\mathbf{L}^1}), \tag{4.3}$$

where $x_1 = xt^{-\frac{1}{n}}$, $\mu\in[0,1)$ *and constant* A

$$A = (-1)^{1+N-Q}\prod_{j=1}^{Q}(1-(a_n^{-1}e^{i2\pi(j-1)})^{\frac{1}{n}})$$
$$\times\sum_{j=1}^{N-Q}\int_0^{+\infty}\left(e^{-r_jy} - \sum_{l=0}^{Q}(-1)^l\frac{(r_jy)^l}{l!}\right)f(y)dy,$$

where

$$r_j = \left(\frac{a_n e^{i2\pi(j-1)}}{a_m} \right)^{\frac{1}{m-n}}.$$

Proof. We rewrite the Green function

$$G(x,y,t) = F_1(x,y,t) + F_2(x-y,t),$$

where

$$F_1(x,y,t) = \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{-i\infty}^{i\infty} e^{px-K(p)t} \frac{e^{-\phi_j y} P_N(p)}{P_N'(\phi_j)(p-\phi_j)} dp$$

and

$$F_2(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dpe^{px-K(p)t} dp.$$

Now we prove some estimates for the function $F_1(x,y,t)$. From (3.2)-(3.3) and (3.1) we have

$$\left| \sum_{j=1}^{N} \frac{P_N(p)}{P_N'(\phi_j)(p-\phi_j)} \right| < C$$

Changing the contour of integration to the contour

$$\mathcal{C} = \{p \in \mathbf{C},\ \ \operatorname{Re} K(p) > 0, \operatorname{Re} p < 0\}$$

such that $\operatorname{Re} \phi_j(K(p)) > 0$, $j = 1,...,N$ for $p \in \mathcal{C}$ we obtain

$$e^{px} p^k \sum_{1}^{N} e^{-\phi_j y} \frac{P_N(p)}{P_N'(\phi_j)(p-\phi_j)} = O\left(x^{-\mu} |p|^{k-\mu} \right)$$

for $|p| < 1$ and

$$e^{px} p^k \sum_{1}^{N} e^{-\phi_j y} \frac{P_N(p)}{P_N'(\phi_j)(p-\phi_j)} = O\left(x^{-\mu} y^{-\theta_1} |p|^{k-\theta_1} \right), |p| \geq 1,$$

where $k \geq 0$; $\mu \geq 0$, $\theta_1 \geq 0$. Therefore choosing $\theta_1 = 1 - \gamma_1$, $\gamma_1 \in [0,1]$ we get

$$\begin{aligned} \left| F_{1x}^{(k)}(x,y,t) \right| &\leq Cx^{-\mu} \int_{p\in\mathcal{C},|p|<1} e^{-\operatorname{Re} K(p)t} |p|^{k-\mu} \, |dp| \\ &\quad + Cx^{-\mu} y^{\gamma-1} \int_{|p|>1,\xi\in\mathcal{C}} e^{-\operatorname{Re} K(p)t} |p|^{k+\gamma-1-\mu} \, |dp| \\ &= x^{-\mu} \left(O(1) + O\left(y^{\gamma-1} t^{-\frac{k-\mu+\gamma}{m}} \right) \right), \end{aligned}$$

where $\mu \in [0, 1+k)$. So choosing

$$\mu = \frac{1}{2} \pm \gamma,$$

($\gamma > 0$ is small enough) we obtain

$$\left\| F_{1x}^{(k)}(\cdot, y, t) \right\|_{\mathbf{L}^2} \leq C \left(1 + y^{-\frac{1}{2}-\gamma} \right) t^{-\frac{k}{m}}$$

for all $y > 0$, $t \in (0, T]$. Also since

$$\left\| F_{2x}^{(k)}(\cdot, t) \right\|_{\mathbf{L}^1} < C t^{-\frac{k}{m}}.$$

therefore we obtain

$$\left\| \partial_x^k \mathbb{G} f \right\|_{\mathbf{L}^2} < C t^{-\frac{k}{m}} \left(\|f\|_{\mathbf{L}^2} + \left\| \left(1 + y^{-\frac{1}{2}} \right) f \right\|_{\mathbf{L}^1} \right),$$

In another hand we have for $\delta_1 \in [0, Q]$

$$H(p, y) = O(p^{\delta_1}(1 + y^{\delta_1})), |p| < 1$$

and

$$H(p, y) < C, |p| > 1$$

so

$$\begin{aligned} \left| G_x^{(n)}(x, y, t) \right| &\leq C x^{-\mu}(1 + y^{\delta_1}) \int_{p \in \mathcal{C}, |p| < 1} e^{-\operatorname{Re} K(p) t} |p|^{k + \delta_1 - \mu} |dp| \\ &\quad + \frac{1}{(x-y)^{\mu}} \int_{|p| > 1, p \in \mathcal{C}} e^{-\operatorname{Re} K(p) t} |p|^{k - \mu} |dp| \\ &= x^{-\mu}(1 + y^{\delta_1}) O\left(t^{-\frac{k + \delta_1 - \mu + 1}{n}} \right) + (x - y)^{-\mu} O\left(e^{-Ct} t^{-\frac{k+1-\mu}{m}} \right), \end{aligned}$$

where $\mu \in [0, \delta_1 + k + 1)$. Therefore we obtain

$$\left\| (\cdot)^{\delta} \partial_x^{(k)} \mathbb{G} f \right\|_{\mathbf{L}^2} \leq C t^{-\frac{1+2k-2\delta}{2m} - \gamma} \left\| \langle y \rangle^{\delta} f \right\|_{\mathbf{L}^1}, t < 1$$

and

$$\left\| (\cdot)^{\delta} \partial_x^{(k)} \mathbb{G} f \right\|_{\mathbf{L}^2} \leq C t^{-\frac{1+2k+2Q-2\delta}{2n} + \gamma} \left\| \langle y \rangle^{Q} f \right\|_{\mathbf{L}^1}, t > 1$$

where $\gamma > 0, \delta \in [0, Q]$. Using formula (3.7) we easily obtain (4.3). Lemma 1 is proved. $\square$

Denote

$$\mathbf{Y}(\beta_1,\beta_2) = \{\varphi(t) \in C(0, \infty), \|\phi\|_{\mathbf{Y}} < \infty, \}$$

$$\text{and } \|\phi\|_{\mathbf{Y}} = \sup_{t>0} \{t\}^{\beta_1} \sum_{k=0}^{m-1} \|\frac{d^k}{dt^k}\phi\|_{\mathbf{L}^\infty} + \langle t \rangle^{\beta_2} \|\phi\|_{\mathbf{L}^\infty}, \beta_1 < 1, \beta_2 > 1.$$

Lemma 2. *The following estimates are valid*

$$\left\|(\cdot)^{\delta}\,\partial_x^{(k)}\mathbb{H}_w(x,t)h\right\|_{\mathbf{L}^2} < C\,\{t\}^{-\frac{1+2w-2\delta}{2m}-\gamma-\beta_1}\,\langle t\rangle^{-\frac{1+2k+2w-2\delta}{2n}-\beta_2+\gamma}\,t\,\|h\|_{\mathbf{Y}}$$

and

$$\|\mathbb{H}_w(\cdot,t)h\|_{\mathbf{L}^\infty} < t^{-\frac{1+w}{n}+\gamma-\beta_2+1}\,\|h\|_{\mathbf{Y}}\,, t>1$$

for $\gamma>0, N\le w\le m-1,\ \delta\in\left[0,\frac{1}{2}+k+N\right)$.

Proof. We have

$$\partial_y^{(w_1)}G(x,0,t)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}e^{px-K(p)t}\widetilde{H}(p,w)dp$$

where

$$\widetilde{H}(p,w_1)=(-1)^{w_1}p^{w_1}\left(1+\frac{1}{p^{w_1}}\sum_{j=1}^{N}\frac{\phi_j^{w_1}P_N(p)}{P_N'(\phi_j)(p-\phi_j)}\right).$$

Denote

$$F_1(x,t)=\frac{(-1)^{w_1+1}}{2\pi i}\sum_{j=1}^{N}\int_{-i\infty}^{i\infty}e^{px-K(p)t}\frac{(\phi_j)^{w_1}P_N(p)}{P_N'(\phi_j)(p-\phi_j)}dp$$

and

$$F_2(x,t)=\frac{1}{2\pi i}(-1)^{w_1}\int_{-i\infty}^{i\infty}dpe^{px-K(p)t}p^{w_1}dp.$$

Now we prove some estimates for the function $F_1(x,y,t)$. Changing the contour of integration to the contour

$$\mathcal{C}=\{p\in\mathbf{C},\ \operatorname{Re}K(p)>0,\operatorname{Re}p<0\}$$

we obtain

$$\left|e^{px}\sum_{1}^{N}\frac{{\phi_j}^{w_1}P_N(p)}{P_N'(\phi_j)(p-\phi_j)}\right|<Cx^{-\mu}|p|^{w_1-\mu},$$

where $k\ge0$, $\mu\ge0$. Therefore we get

$$\begin{aligned}|F_{1x}(x,t)|&\le Cx^{-\mu}\int_{p\in\mathcal{C}}e^{-\operatorname{Re}K(p)t}|p|^{w_1-\mu}\,|dp|\\&=x^{-\mu}O\left(\{t\}^{-\frac{w_1-\mu+\gamma+1}{m}}\,\langle t\rangle^{-\frac{w_1-\mu+\gamma+1}{n}}\right),\end{aligned}$$

where $\mu\in[0,1+k+w)$. So choosing

$$\mu=\frac{1}{2}\pm\gamma+\delta,$$

($\gamma > 0$ is small enough), $\delta \in \left[0, \frac{1}{2} + w_1\right)$ we obtain

$$\left\| (\cdot)^{\delta} F_{1x}(t) \right\|_{\mathbf{L}^2} \leq C \{t\}^{-\frac{1+2w_1-2\delta}{2m}-\gamma} \langle t \rangle^{-\frac{1+2w_1-2\delta}{2n}+\gamma}$$

for all $t > 0$. In the same way as in the proof of the above estimate we have

$$\left\| (\cdot)^{\delta} F_{2x}^{(n)}(t) \right\|_{\mathbf{L}^2} \leq C \{t\}^{-\frac{1+2w_1-2\delta}{2m}-\gamma} \langle t \rangle^{-\frac{1+2w_1-2\delta}{2n_1}+\gamma}$$

Since

$$\begin{aligned} \partial_x^{(k)} \mathbb{H}_w(x,t)h &= \int_0^t h(\tau) \partial_y^{w+k} G(x,0,t-\tau) d\tau \\ &= \sum_{j=0}^{k-1} (-1)^j (h^{(j)}(0) - h^{(j)}(t)) - \int_0^t h^{(k)}(\tau) \partial_y^w G(x,0,t-\tau) d\tau \end{aligned}$$

we obtain for $k = 0, ..., m-1$

$$\left\| (\cdot)^{\delta} \, \partial_x^{(k)} \mathbb{H}_w(x,t)h \right\|_{\mathbf{L}^2} < C \{t\}^{-\frac{1+2w-2\delta}{2m}-\gamma-\beta_1} \langle t \rangle^{-\frac{1+2k+2w-2\delta}{2n}-\beta_2+\gamma} t \, \|h\|_{\mathbf{Y}}$$

and

$$\|\mathbb{H}_w(\cdot,t)h\|_{\mathbf{L}^\infty} < t^{-\frac{1+w}{n}+\gamma-\beta_2+1} \|h\|_{\mathbf{Y}}, t > 1$$

Lemma 2 is proved. □

§5. Proof of Theorem 2

From Section 2 we see that the solution of problem (2.1) can be rewrite in the following manner

$$u(x,t) = \mathbb{G}(t)u_0 + \int_0^t d\tau \mathbb{G}(t-\tau) f(\tau) + \sum_{j=n,m} \sum_{k=1}^{\min(M,j-N)} (-1)^{j-k} a_j \mathbb{H}_{j-k}(x,t) h_k, \tag{5.1}$$

where the operators $\mathbb{G}$ and $\mathbb{H}_{j-k}$ are defined in (4.1), (4.2). Let us prove the following estimate

$$\|u\|_{\mathbf{X}_T} \leq \lambda, \tag{5.2}$$

where

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in (0,T]} \left(\|u(\cdot,t)\|_{\mathbf{H}_2^{0,\delta}} + \sum_{l=0}^{m-1} t^{\frac{l}{m}} \|\partial_x^l u(\cdot,t)\|_{\mathbf{L}^2} \right),$$

$$\lambda = \|u_0\|_{\mathbf{H}_1^{0,\delta}} + \|u_0\|_{\mathbf{H}^{1,0}} + T^{1-\nu_1}\left(\sup_{t\in[0,T]} t^{\nu}\|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\|\sum_{k=1}^{M} h_k\right\|_{\mathbf{Y}(\beta_1)}\right),$$

$$T > 0, \nu_1 = \max\left(\nu + \frac{1}{2m}, 1 - \frac{1}{2m} + \beta_1\right) < 1, \gamma > 0, \delta \in \left[\frac{1}{2}, Q\right],$$

the norm is taken with respect to the space variable x which is denoted by the dot.

From Lemmas 1-2 we have for $l = 1, ..., m-1$

$$\begin{aligned}
&\|\partial_x^l u(\cdot,t)\|_{\mathbf{L}^2} \\
&\le Ct^{-\frac{l}{m}}\left(\|u_0\|_{\mathbf{L}^2} + \left\|\left(1 + y^{-\frac{1}{2}-\gamma}\right)u_0\right\|_{\mathbf{L}^1}\right) \\
&\quad + C\int_0^t d\tau (t-\tau)^{-\frac{1+2l}{2m}-\gamma}\|f(\tau)\|_{\mathbf{L}^1} \\
&\quad + Ct^{-\frac{l}{m}}\sum_{j=n,m}\sum_{k=1}^{\min(M,j-N)} t^{-\frac{1+2(j-k)}{2m}-\gamma-\beta_k+1+\frac{l}{m}}\|h_k\|_{\mathbf{Y}} \\
&< Ct^{-\frac{l}{m}}\Bigg(\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^2} \\
&\qquad + T^{1-\nu_1}\left(\sup_{t\in[0,T]} t^{\nu}\|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\|\sum_{k=1}^{M} h_k\right\|_{\mathbf{Y}(\beta_1)}\right)\Bigg) \\
&< C\lambda.
\end{aligned} \tag{5.3}$$

Applying Lemmas 1-2 we get

$$\begin{aligned}
&\|(\cdot)^{\delta}u(\cdot,t)\|_{\mathbf{L}^2} \\
&\le C\left\|\langle\cdot\rangle^{\delta}u_0\right\|_{\mathbf{L}^1} + C\sup_{t\in[0,T]} t^{\nu}\|\langle\cdot\rangle^{\delta} f(\cdot,t)\|_{\mathbf{L}^1}\int_0^t \tau^{-\nu}d\tau \\
&\quad + C\sum_{j=n,m}\sum_{k=1}^{M} t^{-\frac{1+2(j-k)}{2m}-\gamma-\beta_k+1}\|h_k\|_{\mathbf{Y}} \\
&\le C\left(\|u_0\|_{\mathbf{H}_1^{0,\delta}} + \|u_0\|_{\mathbf{H}^{1,0}} + T^{1-\nu_1}\left(\sup_{t\in[0,T]} t^{\nu}\|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\|\sum_{k=1}^{M} h_k\right\|_{\mathbf{Y}(\beta_1)}\right)\right) \\
&\le C\lambda.
\end{aligned}$$

Theorem 2 is proved.

§6. Local existence

Theorem 3. *Let*

$$u_0 \in \mathbf{H}_1^{0,\delta}\left(\mathbf{R}^+\right) \cap \mathbf{H}_2^{1,0}\left(\mathbf{R}^+\right), t^{\nu} f \in \mathbf{L}^{\infty}\left(0, T; \mathbf{H}_1^{0,\delta}\left(\mathbf{R}^+\right)\right), \sum_{k=1}^{M} h_k \in \mathbf{Y}\left(\beta_1, \beta_2\right),$$

with $\nu \in \left(0, 1 - \frac{1}{2m}\right)$, $\beta_1 < \frac{1}{2m}, \delta \in \left[\frac{1}{2}, Q\right]$. *Then under condition*

$$\rho + 3\sigma < 2m + 1$$

for some $T > 0$ *there exists a unique solution*

$$u \in \mathbf{C}\left([0, T]; \mathbf{H}_2^{0,\delta}\left(\mathbf{R}^+\right)\right) \cap \mathbf{C}\left((0, T]; \mathbf{H}_2^{m-1,0}\right)$$

of the initial-boundary value problem (1.1).

Proof. We prove the local existence of solutions by the contraction mapping principle in the space

$$\mathbf{X}_{T,r} = \left\{\phi \in \mathbf{L}^2 : \|\phi\|_{\mathbf{X}_T} < r\right\},$$

where

$$\|u\|_{\mathbf{X}_T} = \sup_{t \in (0,T]} \left(\|u(\cdot, t)\|_{\mathbf{H}_2^{0,\delta}} + \sum_{l=0}^{m-1} t^{\frac{l}{m}} \|\partial_x^l u(\cdot, t)\|_{\mathbf{L}^2} \right),$$

Let $u(x, t)$ be a solution of the following linear problem

$$\left\{ \begin{array}{c} u_t + \mathbb{N}(w, w_x) + \mathbb{K}(u) = 0, \quad t > 0, x > 0, \\ u(x, 0) = u_0(x), \ x > 0, \\ \partial_x^j u(0, t) = 0, j = 0, ..., M \ t > 0, \end{array} \right. \tag{6.1}$$

where $\mathbb{N}(w, w_x)$ is well defined since $w \in \mathbf{X}_T$. Note that the initial-boundary value problem (6.1) defines a mapping $\mathbb{M}$ by $u = \mathbb{M}(w)$ and we will show that $\mathbb{M}$ is the contraction mapping from $\mathbf{X}_{T,r}$ into itself for a sufficiently small $T > 0$. Since $w \in \mathbf{X}_{T,r}$ we have

$$\begin{aligned}
&\sup_{t \in [0,T]} \|\mathbb{N}(w, w_x)(\cdot, t)\|_{\mathbf{H}_1^{0,\delta}} \\
&\leq C \sup_{t \in [0,T]} \|w(\cdot, t)\|_{\mathbf{L}^\infty}^{\rho-2} \|w_x(\cdot, t)\|_{\mathbf{L}^\infty}^{\sigma} \| \langle \cdot \rangle^{\frac{\delta}{2}} w(\cdot, t)\|_{\mathbf{L}^2}^2 \\
&\leq C \sup_{t \in [0,T]} \|w(\cdot, t)\|_{\mathbf{L}_2}^{\frac{\rho-2}{2}} \|w_x(\cdot, t)\|_{\mathbf{L}_2}^{\frac{\rho-2+\sigma}{2}} \|w_{xx}(\cdot, t)\|_{\mathbf{L}_2}^{\frac{\sigma}{2}} \| \langle \cdot \rangle^{\frac{\delta}{2}} w(\cdot, t)\|_{\mathbf{L}^2}^2 \\
&\leq C t^{-\frac{\rho+3\sigma-1}{2m}} r^{\rho+\sigma}.
\end{aligned}$$

Via Theorem 2 under condition

$$\rho + 3\sigma < 2m + 1$$

problem (6.1) has a unique solution $u(x,t) \in \mathbf{X}_{T,\rho}$ with the norm

$$\|u\|_{\mathbf{X}_T} \leq C\lambda,$$

where

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in(0,T]} \left(\|u(\cdot,t)\|_{\mathbf{H}_2^{0,\delta}} + \sum_{l=0}^{m-1} t^{\frac{l}{m}} \|\partial_x^l u(\cdot,t)\|_{\mathbf{L}^2} \right),$$

$$\begin{aligned}\lambda =& \|u_0\|_{\mathbf{H}_1^{0,\delta}} + \|u_0\|_{\mathbf{H}^{1,0}} \\ &+ T^{1-\nu_1} \sup_{t\in[0,T]} \left(t^{\nu} \|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + t^{\frac{\rho+3\sigma-1}{2m}} \|\mathbb{N}(w,w_x)(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\| \sum_{k=1}^{M} h_k \right\|_{\mathbf{Y}(\beta_1)} \right),\end{aligned}$$

$$\begin{aligned}&T > 0, \nu_1 = \max\left(\nu + \frac{1}{2m}, 1 - \frac{1}{2m} + \beta_1, \frac{\rho + 3\sigma - 1}{2m} \right) < 1, \\ &\gamma > 0, \delta \in \left[\frac{1}{2}, Q \right].\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|u\|_{\mathbf{X}_T} \leq& C\|u_0\|_{\mathbf{H}^{1,0}} + \|u_0\|_{\mathbf{H}_1^{\delta,0}} \\ &+ CT^{1-\nu_1} \left(\sup_{t\in[0,T]} t^{\nu} \|f(\cdot,t)\|_{\mathbf{H}_1^{0,\delta}} + \left\| \sum_{k=1}^{M} h_k \right\|_{\mathbf{Y}(\beta_1)} \right) \\ &+ CT^{1-\nu_1} r^{\rho+\sigma},\end{aligned} \tag{6.2}$$

whence we get $\|u\|_{\mathbf{X}_T} \leq r$ if $T < 1$. Thus the mapping $\mathbb{M}$ transforms the closed ball $\mathbf{X}_{T,r}$ with a center at the origin and a radius r into itself. Analogously we can prove the estimate $\sup_{t\in[0,T]} \|u - \widetilde{u}\|_{\mathbf{X}_{T,r}} < \sup_{t\in[0,T]} \|w - \tilde{w}\|_{\mathbf{X}_{T,r}}$ for $T < 1$. Therefore the mapping $\mathbb{M}$ is a contraction mapping in $\mathbf{X}_{T,r}$ and there exists a unique solution $u(x,t) \in \mathbf{X}_{T,r}$ of the initial-value problem (1.1). Theorem 3 is proved. □

Remark 4. *By* (6.2) *we see that if the norm of the initial data* u_0*, source* f *and boundary data are sufficiently small, then for some time* $T > 1$ *there exist a unique solution* u *such that* $\|u\|_{\mathbf{X}_T} < C\epsilon$*.*

§7. Large time asymptotics

We consider the initial-boundary value problem (1.1) with small initial data

$$\|u_0\|_{\mathbf{H}_1^{0,\frac{Q+\delta}{2}}} + \|u_0\|_{\mathbf{H}_2^{1,0}} + \left(\sup_{t\in[0,T]} \{t\}^{\nu_1} \langle t\rangle^{\nu_2} \|f(\cdot,t)\|_{\mathbf{H}_1^{0,\frac{Q+\delta}{2}}} + \left\| \sum_{k=1}^{M} h_k \right\|_{\mathbf{Y}(\beta_1,\beta_2)} \right) < \epsilon_1,$$

where $\epsilon_1 > 0$ is sufficiently small, $\beta_1 < \frac{1}{2m}, \beta_2 > 0, \nu_1 < 1, \nu_2 > 1$.

Let us prove the estimate

$$\sup_{t>1} \left(\sum_{l=0}^{n-1} t^{\frac{2Q+1}{2n}-\gamma+\frac{l}{n}} \left\| u_x^{(l)}(\cdot,t) \right\|_{\mathbf{L}^2} + t^{\frac{Q-\delta+1}{2n}-\gamma} \left\| (\cdot)^{\frac{Q+\delta}{2}} u(\cdot,t) \right\|_{\mathbf{L}^2} \right) < \varepsilon, \tag{7.1}$$

where $\gamma, \varepsilon > 0$ are small enough, $\delta \in (0,1)$. We prove this estimate by the contradiction. We assume that there exists some $T > 1$ such that

$$\sup_{t\in[1,T]} \left(\sum_{l=0}^{n-1} t^{\frac{2Q+1}{2n}-\gamma+\frac{l}{n}} \left\| u_x^{(l)}(\cdot,t) \right\|_{\mathbf{L}^2} + t^{\frac{Q-\delta+1}{2n}-\gamma} \left\| (\cdot)^{\frac{Q+\delta}{2}} u(\cdot,t) \right\|_{\mathbf{L}^2} \right) = \varepsilon. \tag{7.2}$$

Therefore we get for $t \in [1,T]$

$$\begin{aligned} \|\mathbb{N}(u,u_x)\|_{\mathbf{H}_1^{0,Q}} &\le C\|u(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\rho-2}{2}} \|u_x(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\rho-2+\sigma}{2}} \|u_{xx}(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\sigma}{2}} \| \langle\cdot\rangle^{\frac{Q}{2}} u(\cdot,t)\|_{\mathbf{L}^2}^2 \\ &< C\varepsilon^{\rho+\sigma} t^{-\frac{Q+1}{n}(\rho+\sigma-1)-\frac{\sigma}{n}+\gamma} \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{N}(u,u_x)\|_{\mathbf{L}_1} &\le C\|u(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\rho}{2}+1} \|u_x(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\rho-2+\sigma}{2}} \|u_{xx}(\cdot,t)\|_{\mathbf{L}_2}^{\frac{\sigma}{2}} \\ &< C\varepsilon^{\rho+\sigma} t^{-\frac{Q+1}{n}(\rho+\sigma)-\frac{\sigma-1}{n}+\gamma}. \end{aligned}$$

Therefore from Lemma 1 and Lemma 2 under conditions

$$(Q+1)(\rho+\sigma-1) > n$$

$$\beta_2 > 1, \nu_2 > 1$$

we obtain

$$\begin{aligned}
\|\partial_x^{(l)} u(\cdot,t)\|_{\mathbf{L}^2} \le C t^{-\frac{2l+2Q+1}{2n}+\gamma} \|u_0\|_{\mathbf{H}_1^{0,Q}} \\
&+ \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{2l+2Q+1}{2n}+\gamma} \|(\mathbb{N}(u,u_x)(\tau)+f(x,\tau))\|_{\mathbf{H}_1^{0,Q}} d\tau \\
&+ \int_{\frac{t}{2}}^{t} \|\mathbb{N}(u,u_x)(\tau)+f(x,\tau)\|_{\mathbf{L}_1} (t-\tau)^{-\frac{l}{n}-\frac{1}{2n}} d\tau \\
&+ t^{-\frac{l+N}{n}-\beta_2+1+\gamma} \left\| \sum_{k=1}^{M} h_k \right\|_Y \\
&\le C t^{-\frac{2l+2Q+1}{2n}+\gamma} (\varepsilon_1+\varepsilon^{\rho+\sigma}) < \frac{\varepsilon}{2} t^{-\frac{2l+2Q+1}{2n}+\gamma}.
\end{aligned}$$

Also we have

$$\begin{aligned}
\| (\cdot)^{\frac{Q+\delta}{2}} u(\cdot,t)\|_{\mathbf{L}^2} \le C t^{-\frac{Q-\delta+1}{2n}+\gamma} \|u_0\|_{\mathbf{H}_1^{0,Q}} \\
&+ \int_0^{t} \left\| \langle\cdot\rangle^{Q} (\mathbb{N}(u,u_x)(\tau)+f(x,\tau)) \right\|_{\mathbf{L}_1} (t-\tau)^{-\frac{Q-\delta+1}{2n}+\gamma} d\tau \\
&+ t^{-\frac{1+2N-Q-\delta}{2n}-\beta_2+1+\gamma} \left\| \sum_{k=1}^{M} h_k \right\|_Y \\
&\le C t^{-\frac{Q-\delta+1}{2n}+\gamma} (\varepsilon_1+\varepsilon^{\rho+\sigma}) < \frac{\varepsilon}{2} t^{-\frac{Q-\delta+1}{2n}+\gamma}.
\end{aligned}$$

So we obtain that

$$\sup_{t>1} \left(\sum_{l=0}^{n-1} t^{\frac{2l+2Q+1}{2n}-\gamma} \left\| u_x^{(l)}(\cdot,t) \right\|_{\mathbf{L}^2} + t^{\frac{Q-\delta+1}{2n}-\gamma} \left\| (\cdot)^{\frac{Q+\delta}{2}} u(\cdot,t) \right\|_{\mathbf{L}^2} \right) < \varepsilon.$$

The contradiction obtained proves (7.1).

Now using estimate (7.1) and Lemmas 1-2 we prove that the solution has the following asymptotics for $t \to \infty$ uniformly with respect to $x > 0$

$$u(x,t) = t^{-\frac{Q+1}{n}} A \Phi \left(\frac{x}{\sqrt[n]{t}} \right) + O\left(t^{-\frac{Q+1+\mu}{n}} \right), \tag{7.3}$$

where $\mu > 0$

$$\mu = \min(\delta, (Q+1)(\rho+\sigma-1)+\sigma-n, \nu_2-1, N-Q+n\beta_2-n)$$

and

$$\Phi(q) = \int_{-i\infty}^{i\infty} e^{zq-a_n z^n} z^Q dz$$

$$A = D\left(\int_0^\infty B\left(y\right)u_0(y)dy + \int_0^\infty d\tau \int_0^\infty B\left(y\right)\mathbb{N}(u,u_y)dy\right) < \infty.$$

Indeed, via Lemma 1 we have

$$u(x,t) = t^{-\frac{Q+1}{n}} A\Phi\left(\frac{x}{\sqrt[n]{t}}\right) + R(x,t), \tag{7.4}$$

where

$$\begin{aligned}|R(x,t)| \le Ct^{-\frac{Q+\delta}{n}}&\left(\|u_0\|_{\mathbf{H}_1^{0,Q+\delta}} + \int_0^t d\tau \left\|(\mathbb{N}(u,u_x)(\tau) + f(x,\tau))\right\|_{\mathbf{H}_1^{0,Q+\delta}}\right)\\ &+ \int_0^t d\tau \left(\mathbb{G}(t-\tau) - \mathbb{G}(t)\right)\left(\mathbb{N}(\tau) + f(\tau)\right)\\ &+ \int_t^\infty d\tau \int_0^\infty \left|B\left(y\right)\left(\mathbb{N}(u,u_y) + f(y,\tau)\right)\right| dy\\ &+ \sum_{j=n,m} \sum_{k=1}^{\min(M,j-N)} \mathbb{H}_{j-k}(x,t)h_k.\end{aligned}$$

Using estimate (7.1) we see that

$$\int_0^t \tau \left\|\langle\cdot\rangle^Q \mathbb{N}(u,u_x)\right\|_{\mathbf{L}_1} d\tau \le Ct^{-\frac{Q+1}{n}(\rho+\sigma-1)-\frac{\sigma}{n}+2+\gamma} \tag{7.5}$$

for $t > 1$.

In the same way as in the proof of Lemma 1 we prove estimate

$$\|\left(\mathbb{G}(t-\tau) - \mathbb{G}(t)\right)\mathbb{N}(\tau)\|_{\mathbf{L}^\infty} \le C\tau t^{-\frac{Q+1}{n}-1}\left\|\mathbb{N}(u,u_x)\right\|_{\mathbf{H}_1^{0,Q}}.$$

Therefore using (7.5) we have

$$\int_0^t d\tau \left\|\left(\mathbb{G}(t-\tau) - \mathbb{G}(t)\right)\mathbb{N}(\tau)\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{Q+1+\mu_1}{n}} \tag{7.6}$$

where

$$\mu_1 < (Q+1)(\rho+\sigma-1) - n.$$

Also since from Lemma 2

$$\begin{aligned}\sum_{j=n,m} \sum_{k=1}^{\min(M,j-N)} \left\|\mathbb{H}_{j-k}(x,t)h_k\right\|_{\mathbf{L}^\infty} &< Ct^{-\frac{1+N}{n}+\gamma-\beta_2+1}\left\|\sum_{k=1}^{M} h_k\right\|_{\mathbf{Y}}\\ &< Ct^{-\frac{Q+1+\mu_2}{n}},\end{aligned}$$

where

$$\mu_2 < N - Q + n\beta_2 - n.$$

Since $\|(\cdot)^{Q+\delta}u_0\|_{\mathbf{L}^1} + \int_0^{+\infty} \left\| \langle\cdot\rangle^{Q+\delta} f(\cdot,\tau) \right\|_{\mathbf{L}_1} \leq C$ using (7.6) we get

$$|R(x,t)| \leq Ct^{-\frac{Q+1+\delta}{n}}. \tag{7.7}$$

From (7.4) - (7.7) we obtain the asymptotics (7.3) for the solution. Theorem 1 is proved.

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