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Edge-connectivity and the orientation of a graph

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Abstract. Let G be a k-edge-connected graph and let L denote the subset of all vertices having odd degree in G. For every subset $K = \{u_1, u_2, \ldots, u_k\}$ of L with $|K| \leq \frac{|L|}{2}$, and for every function h defined on K having the property that $h(u_i) \in \left\{ \left\lceil \frac{d_G(u_i)}{2} \right\rceil, \left\lfloor \frac{d_G(u_i)}{2} \right\rfloor \right\}$ for all $u_i \in K$, there exists an orientation D of G such that $d_D^+(x) = h(x)$ when $x \in K$ and $\left\lfloor \frac{d_G(x)}{2} \right\rfloor \leq d_D^+(x) \leq \left\lceil \frac{d_G(x)}{2} \right\rceil$ when $x \in V(G) - K$.

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§1. Introduction

All graphs considered are simple and finite. We refer the reader to [1] for standard graph theoretic terms not defined in this paper.

Let G be a graph. The degree $d_G(u)$ of a vertex u in G is the number of edges of G incident with u. For any subset S of vertices of G, we define the neighbourhood of S in G to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_G(S)$. If $S \subseteq V(G)$, the set $V(G) - S$ will be denoted by $\overline{S}$. The subgraph of G whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S is called the subgraph of G induced by S and will be denoted by $G[S]$.

If S and T are disjoint subsets of vertices of G, we write $E_G(S,T)$ and $e_G(S,T)$ for the set and the number respectively of the edges of G joining S to T. If e is an edge of G having u and v as end-vertices, it will be denoted by uv. The edge-connectivity $k'(G)$ of G is the minimum number of edges whose removal from G results in a disconnected graph or a trivial graph. We say that G is k-edge-connected if $k'(G) \geq k$.

If we replace the edges of G by arcs, we will get a digraph D which is called an orientation of G. An edge e of G is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends. Note that the internal vertex of this path is a new vertex. If the edges of a walk W in G are distinct, W is called a trail. A closed trail that traverses every edge of G is called an Euler trail. We will say that G is Eulerian if it contains an Euler trail. Let $f(x)$ and $g(x)$ be integer valued functions on the vertex set $V(G)$ such that $0 \leq g(x) \leq f(x)$ for each vertex $x \in V(G)$. Then a spanning subgraph F of G is called a $(g, f)-$factor of G if $g(x) \leq d_F(x) \leq f(x)$ for each vertex $x \in V(G)$.

Let D be a digraph. The indegree $d_D^-(u)$ of a vertex u in D is the number of arcs with head u, and the outdegree $d_D^+(u)$ of u is the number of arcs with tail u.

The following Proposition appears in many textbooks on Graph Theory.

Proposition 1. *For every graph G, there exists an orientation D such that*

$$\left\lfloor \frac{d_G(x)}{2} \right\rfloor \leq d_D^+(x) \leq \left\lceil \frac{d_G(x)}{2} \right\rceil \quad \textit{for all} \quad x \in V(G).$$

Proof. We first assume that G is a connected graph. Let $L = \{v_1, v_2, \ldots, v_{2r}\}$ be the set of vertices of G, which have odd degree and let G^* be the graph obtained from G by adding the independent edges $v_1v_2, v_3v_4, \ldots, v_{2r-1}v_{2r}$. Since all the vertices of G^* have clearly even degree in G^*, G^* has a closed Eulerian trail T^* [2]. We follow T^* and we orient the edges of G^* in the same direction as that of the Eulerian trail. The above orientation give us a digraph D^* such that

$$\frac{d_{G^*}(x)}{2} = d_{D^*}^+(x) = d_{D^*}^-(x) \text{ for every vertex } x \text{ of } D^*.$$

Now we delete from D^* the arcs arising from the orientation of the edges $v_1v_2, \ldots, v_{2r-1}v_{2r}$. The resulting digraph D is clearly an orientation of G having the following property,

$$\frac{d_G(x)}{2} = d_D^+(x) = d_D^-(x) \text{ when } x \in V(G) - L$$

and

$$|d_D^+(x) - d_D^-(x)| = 1 \text{ when } x \in L.$$

If G is a disconnected graph, we will get a proof by applying the same arguments to every component of G. $\square$

In the following theorem which is the main result of this paper we prove that if the edge-connectivity of G is sufficiently high then G has an orientation D having the property mentioned in Proposition 1 and additionally some of the vertices of odd degree can have the prescribed outdegrees in D.

Theorem 1. *Let G be a k-edge-connected graph and L the set of all vertices with degree odd in G.*

For every subset $K = \{u_1, u_2, \ldots, u_k\}$ of L with $|K| \leq \frac{|L|}{2}$, and for every function h defined on K having the property that $h(u_i) \in \left\{ \left\lceil \frac{d_G(u_i)}{2} \right\rceil, \left\lfloor \frac{d_G(u_i)}{2} \right\rfloor \right\}$ for all $u_i \in K$, there exists an orientation D of G such that $d^+_D(x) = h(x)$ when $x \in K$ and $\left\lfloor \frac{d_G(x)}{2} \right\rfloor \leq d^+_D(x) \leq \left\lceil \frac{d_G(x)}{2} \right\rceil$ when $x \in V(G) - K$.

§2. Proof of Theorem 1

For the proof of Theorem 1, we will use the following Lemmas.

Lemma 1 ([3])**.** *A bipartite graph G has a $(g, f)-$factor if and only if for every set $S \subseteq V(G)$,*

$$\sum_{x \in \overline{S}} \max\{0, g(x) - d_{G-S}(x)\} \leq \sum_{x \in S} f(x).$$

Lemma 2. *Let G be a graph and let $f : V(G) \to Z^+$ and $g : V(G) \to Z^+$ be functions such that $g(x) \leq f(x)$. We subdivide every edge of G and define f and g, to be both 1 for the new vertices. The resulting graph G^* has a $(g, f)-$factor if and only if G has an orientation D such that $g(x) \leq d^+_D(x) \leq f(x)$ for every $x \in V(D)$.*

Proof. Suppose first that G^* has a $(g, f)-$factor F. Clearly every edge of G^* has an end-vertex in $V(G)$ and the other in $V(G^*) - V(G)$. Define S to be the set of edges belonging to F and $S' = E(G^*) - E(F)$. We orient the elements of S in the following way: the tail of every arc belongs to $V(G)$ and the head belongs to $V(G^*) - V(G)$. We also orient the elements of S' as follows: the tail of every arc belongs to $V(G^*) - V(G)$ and the head belongs to $V(G)$. By considering such an orientation of G^*, we get a digraph D^* having the following properties:

$$d^+_{D^*}(x) = 1 \quad \text{when } x \in V(G^*) - V(G) \text{ and}$$
$$g(x) \leq d^+_{D^*}(x) = d_F(x) \leq f(x) \quad \text{when } x \in V(G).$$

Now we apply the following procedure to every vertex of $V(G^*) - V(G)$. For $u \in V(G^*) - V(G)$, let a_1 be the arc of D^* having u as a tail and let a_2 be the arc having u as a head. Let v_1 also be the tail of a_2 and v_2 the head of a_1.

We delete u, a_1, a_2 from D^* and we add an arc having v_1 as a tail and v_2 as a head.

The resulting digraph D is an orientation of G satisfying $g(x) \leq d^+_{D^*}(x) = d^+_D(x) \leq f(x)$ for every $x \in V(D)$.

By reversing the argument we can prove easily that if G has an orientation D such that $g(x) \leq d^+_D(x) \leq f(x)$ for all $x \in V(D)$, then G^* has a $(g,f)-$factor. $\square$

For the proof of Lemma 2, we used ideas and techniques mentioned in [4].

Proof of Theorem 1.
Let G^* be the graph obtained from G by subdividing its edges. By Lemma 2, G will have an orientation D if and only if G^* has a $(g,f)-$factor having the following properties:

$$g(x) = f(x) = h(x) \text{ for every } x \in K;$$
$$g(x) = \left\lfloor \frac{d_G(x)}{2} \right\rfloor, f(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil \text{ for every } x \in V(G) - K;$$

and $g(x) = f(x) = 1$ for every $x \in V(G^*) - V(G) = R$ (We note here that R consists of all the inserted vertices of degree 2).

Suppose that G^* has no $(g,f)-$factor having the above properties. Clearly G^* is a bipartite graph with bipartition (X,Y) where $X = V(G)$ and $Y = V(G^*) - V(G) = R$. Then by Lemma 1, there exists $S \subseteq V(G^*)$ such that

$$\sum_{x\in\overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} > \sum_{x\in S} f(x). \tag{2.1}$$

Define

$$\begin{array}{ll} S \cap Y = S_y, & S \cap X = S_x, \\ \overline{S} \cap Y = \overline{S}_y, & \overline{S} \cap X = \overline{S}_x, \\ \left.\begin{array}{l} S_{y_i} = \{u \in S_y \| N_{G^*}(u) \cap S_x| = i\} \\ \overline{S}_{y_i} = \{u \in \overline{S}_y \| N_{G^*}(u) \cap \overline{S}_x| = i\} \end{array}\right\} & \text{for } i = 0,1,2, \\ K_S = K \cap S_x, \quad \text{and} & K_{\overline{S}} = K \cap \overline{S}_x. \end{array}$$

We assume that S is minimal with respect to (2.1). We will prove that $S_{y_2} = \emptyset$ and $S_{y_1} = \emptyset$.

Suppose that $S_{y_2} \neq \emptyset$ and let $v \in S_{y_2}$. Define $S' = S - \{v\}$. Then

$$\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} > \sum_{x\in S'} f(x)$$

since

$$\sum_{x\in S'} f(x) = \sum_{x\in S} f(x) - 1$$

and

$$\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} = \sum_{x\in\overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} + 1.$$

This contradicts the fact S is minimal with respect to (2.1).

Similarly suppose that $S_{y_1} \neq \emptyset$ and let $v \in S_{y_1}$. Define $S' = S - \{v\}$. Then

$$\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} > \sum_{x\in S'} f(x)$$

since $\sum_{x\in S'} f(x) = \sum_{x\in S} f(x) - 1$, and $\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} \geq \sum_{x\in\overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} - 1$. This is also a contradiction because S is minimal with respect to (2.1).

Now let $v \in S_{y_0}$ and suppose that $N_{G^*}(v) = \{w_1, w_2\}$. It is obvious that $w_1, w_2 \in \overline{S}_x$. We will prove that $g(w_1) > d_{G^*-S}(w_1)$ and $g(w_2) > d_{G^*-S}(w_2)$. Without loss of generality, we may assume that $g(w_1) \leq d_{G^*-S}(w_1)$. Define $S' = S - \{v\}$. We have

$$\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} > \sum_{x\in S'} f(x)$$

since $\sum_{x\in S'} f(x) = \sum_{x\in S} f(x) - 1$ and $\sum_{x\in\overline{S}'} \max\{0, g(x) - d_{G^*-S'}(x)\} \geq \sum_{x\in\overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} - 1$. This is a contradiction because S is minimal with respect to (2.1).

Define $M = \left\{x \in \overline{S}_x | N_{G^*}(x) \cap S_{y_0} \neq \emptyset\right\}$. In fact we have just proved that

$$d_{G^*-S}(x) \leq g(x) - 1 \text{ for every } x \in M. \tag{2.2}$$

At this point we consider the following cases:

<u>Case 1</u>: $M = V(G)$

In this case $S_x = \emptyset$, $\overline{S}_x - M = \emptyset$, $\overline{S}_{y_1} = \emptyset$, and $\overline{S}_{y_0} = \emptyset$.

So from (2.1), we have

$$\sum_{x\in\overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} > |S_{y_0}|.$$

By $g(x) - d_{G^*-S}(x) < 0$ for each $x \in \overline{S}_{y_2}$, the above inequality implies

$$\sum_{x\in M} \max\{0, g(x) - d_{G^*-S}(x)\} > |S_{y_0}|.$$

Since $d_{G^*-S}(x) \le g(x) - 1$ for every $x \in M$, we have

$$\sum_{x \in M} g(x) - \sum_{x \in M} d_{G^*-S}(x) > |S_{y_0}|.$$

This inequality together with $\sum_{x \in M} d_{G^*-S}(x) = 2|\overline{S}_{y_2}|$ yields

$$\sum_{x \in M} g(x) > 2|\overline{S}_{y_2}| + |S_{y_0}|.$$

Moreover, it follows from $|V(G)| \ge |L| \ge 2|K|$ that $\frac{1}{2}\sum_{x \in V(G)} d_G(x) \ge \sum_{x \in M} g(x)$. Hence $\frac{1}{2}\sum_{x \in M} d_G(x) > 2|\overline{S}_{y_2}| + |S_{y_0}|$. This contradicts the fact $|\overline{S}_{y_2}| + |S_{y_0}| = |E(G)| = \frac{1}{2}\sum_{x \in M} d_G(x)$. This completes the proof of this case.

<u>Case 2</u>: $M \ne V(G)$
We have from (2.1),

$$\sum_{x \in \overline{S}_x} \max\{0, g(x) - d_{G^*-S}(x)\} + |\overline{S}_{y_0}| > \sum_{x \in S_x} f(x) + |S_{y_0}|.$$

So

$$\sum_{x \in K_{\overline{S}}} \max\{0, g(x) - d_{G^*-S}(x)\} + \sum_{x \in \overline{S}_x - K_{\overline{S}}} \max\{0, g(x) - d_{G^*-S}(x)\} + |\overline{S}_{y_0}|$$
$$> \sum_{x \in K_S} f(x) + \sum_{x \in S_x - K_S} f(x) + |S_{y_0}|.$$

For any $x \in \overline{S}_x - M$, $d_{G^*-S}(x) = d_{G^*}(x)$ holds. Thus the previous relation implies

$$\sum_{x \in K_{\overline{S}} \cap M} \max\{0, g(x) - d_{G^*-S}(x)\} + \sum_{x \in M - K_{\overline{S}}} \max\{0, g(x) - d_{G^*-S}(x)\} + |\overline{S}_{y_0}|$$
$$> \sum_{x \in K_S} f(x) + \sum_{x \in S_x - K_S} f(x) + |S_{y_0}|.$$

Now from (2.2), we have

$$g(x) - d_{G^*-S}(x) \ge 1 \text{ for every } x \in M.$$

If we let $g(x) - d_{G^*-S}(x) = \theta(x)$ for every $x \in M$, then the above can be written as

$$(2.3) \quad \sum_{x\in K_{\overline{S}}\cap M} \theta(x) + \sum_{x\in M-K_{\overline{S}}} \theta(x) + |\overline{S}_{y_0}| > \sum_{x\in K_S} f(x) + \sum_{x\in S_x-K_S} f(x) + |S_{y_0}|.$$

Since

$$\sum_{x\in M\cap K_{\overline{S}}} (d_{G^*}(x) - d_{G^*-S}(x)) + \sum_{x\in M-K_{\overline{S}}} (d_{G^*}(x) - d_{G^*-S}(x)) = 2|S_{y_0}|,$$

we have

$$\sum_{x\in K_{\overline{S}}\cap M} (d_{G^*}(x) - g(x) + \theta(x)) + \sum_{x\in M-K_{\overline{S}}} (d_{G^*}(x) - g(x) + \theta(x)) = 2|S_{y_0}|.$$

So

$$\sum_{x\in K_{\overline{S}}\cap M} \left(\left\lfloor \frac{d_{G^*}(x)}{2} \right\rfloor + \theta(x)\right) + \sum_{x\in M-K_{\overline{S}}} \left(\left\lceil \frac{d_{G^*}(x)}{2} \right\rceil + \theta(x)\right) \leq 2|S_{y_0}|.$$

Hence

$$(2.4) \quad \sum_{x\in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil + \sum_{x\in M} \theta(x) \leq 2|S_{y_0}| + |M \cap K_{\overline{S}}|.$$

By (2.3) and (2.4),

$$2|S_{y_0}| + |M\cap K_{\overline{S}}| - \sum_{x\in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil + |\overline{S}_{y_0}| > \sum_{x\in K_S} f(x) + \sum_{x\in S_x-K_S} f(x) + |S_{y_0}|,$$

which implies

$$(2.5) \quad |S_{y_0}| + |M \cap K_{\overline{S}}| - \sum_{x\in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil + |\overline{S}_{y_0}| > \sum_{x\in S_x} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil - |K_S|.$$

At this point, we consider the following two subcases:

<u>Case 2a</u>: $S_x \neq \emptyset$

We first point out that $S_x \neq V(G)$. If $S_x = V(G)$, then by $S_{y_0} = \emptyset$ and (2.1),

$$|E(G)| = |\overline{S}_{y_0}| > \sum_{x\in S} f(x) \geq \sum_{x\in V(G)} \frac{d_G(x)}{2}.$$

This is a contradiction.

Since G is k-edge-connected and the number of edges in G joining two vertices of S_x is $|\overline{S}_{y_0}|$, we have $k \le e_G(S_x, V(G) - S_x) = \sum_{x \in S_x} d_G(x) - 2|\overline{S}_{y_0}|$. Hence

$$2|\overline{S}_{y_0}| + k \le \sum_{x \in S_x} d_{G^*}(x)$$

and so

$$|\overline{S}_{y_0}| \le \sum_{x \in S_x} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil - \frac{k}{2} - \frac{|K_S|}{2}. \tag{2.6}$$

By (2.5) and (2.6), we have

$$|S_{y_0}| + |M \cap K_{\overline{S}}| - \sum_{x \in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil > \frac{k}{2} - \frac{|K_S|}{2}. \tag{2.7}$$

We also should point out here that $M \ne \emptyset$. If $M = \emptyset$, then $|K_S| > k$ holds by (2.7). But this is a contradiction since $|K_S| \le |K| \le k$. Therefore $M \ne \emptyset$. By (2.7), $|M \cap K_{\overline{S}}| + |K_S| \le |K| = k$ and

$$\sum_{x \in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil \ge \frac{1}{2} \sum_{x \in M} d_{G^*}(x),$$

we have $|S_{y_0}| + \frac{k}{2} > \frac{1}{2} \sum_{x \in M} d_{G^*}(x)$ and hence $2|S_{y_0}| + k > \sum_{x \in M} d_{G^*}(x)$. On the other hand, by the edge-connectivity of G,

$$\sum_{x \in M} d_{G^*}(x) = \sum_{x \in M} d_G(x) = 2|E(G[M])| + e_G(M, V(G) - M) \ge 2|S_{y_0}| + k.$$

This is a contradiction.

Case 2b: $S_x = \emptyset$
We have from (2.5),

$$|S_{y_0}| + |M \cap K_{\overline{S}}| > \sum_{x \in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil \tag{2.8}$$

since $\overline{S}_{y_0} = \emptyset$ and $K_S = \emptyset$ when $S_x = \emptyset$.

We should point out here that $M \ne \emptyset$, since otherwise (2.8) give us a contradiction. By $|E(G[M])| \ge |S_{y_0}|$ and

$$\sum_{x \in M} \left\lceil \frac{d_{G^*}(x)}{2} \right\rceil = \sum_{x \in M} \frac{d_{G^*}(x)}{2} + \frac{|M \cap L|}{2} \ge \sum_{x \in M} \frac{d_{G^*}(x)}{2} + \frac{|M \cap K_{\overline{S}}|}{2},$$

(2.8) implies

$$|E(G[M])| + \frac{|M \cap K_{\overline{S}}|}{2} > \sum_{x\in M} \frac{d_{G^*}(x)}{2}.$$

It follows from $\frac{|M \cap K_{\overline{S}}|}{2} \le \frac{|K_{\overline{S}}|}{2} \le \frac{|K|}{2} = \frac{k}{2}$ that

$$2|E(G[M])| + k > \sum_{x\in M} d_{G^*}(x).$$

This is a contradiction, since by the edge-connectivity of G,

$$\sum_{x\in M} d_{G^*}(x) = 2|E(G[M])| + e_G(M, V(G) - M) \ge 2|E(G[M])| + k.$$

□

Next, we will describe a family of graphs, which shows that the connectivity condition imposed on graph G in Theorem 1 is necessary.

Let H_1 and H_2 be two $(k-1)$-edge-connected graphs with $V(H_1) = \{u_1, u_2, \dots, u_{|V(H_1)|}\}$, $V(H_2) = \{v_1, v_2, \dots, v_{|V(H_2)|}\}$ and $\min\{|V(H_1)|, |V(H_2)|\} \ge k+1$. We also assume that H_1 and H_2 have the following properties:

(a) the vertices $u_1, u_2, \dots, u_{k-1}$ and $v_1, v_2, \dots, v_{k-1}$ have even degree in H_1 and H_2 respectively,

(b) the vertices u_k and v_k have odd degree in H_1 and H_2 respectively.

If we add the independent edges $u_1v_1, u_2v_2, \dots, u_{k-1}v_{k-1}$ to $H_1 \cup H_2$, we obtain a graph G which is clearly $(k-1)$-edge-connected having at least $2k$ vertices of odd degree.

However, G has no orientation D such that $d_D^+(x) = \left\lfloor \frac{d_G(x)}{2} \right\rfloor$ when $x \in \{u_1, u_2, \dots, u_k\} = K$ and $\left\lfloor \frac{d_G(x)}{2} \right\rfloor \le d_D^+(x) \le \left\lceil \frac{d_G(x)}{2} \right\rceil$ when $x \in V(G) - K$. In fact, we will show the above claim as follows:

Let G^* be the bipartite graph obtained from G by subdividing all edges. We define functions $f : V(G^*) \to Z^+$, $g : V(G^*) \to Z^+$ such that

(i) $f(x) = \left\lceil \frac{d_G(x)}{2} \right\rceil$ and $g(x) = \left\lfloor \frac{d_G(x)}{2} \right\rfloor$ when $x \in V(G) - K$,

(ii) $f(x) = g(x) = \left\lfloor \frac{d_G(x)}{2} \right\rfloor$ when $x \in K$, and

(iii) $f(x) = g(x) = 1$ when $x \in V(G^*) - V(G)$.

According to Lemma 2, G will have an orientation D having the properties stated before if and only if G^* has a $(g, f)-$factor having properties (i), (ii) and (iii).

We will prove that G^* has no such a $(g, f)-$factor. As in the proof of Theorem 1, G^* has bipartition (X, Y) where $X = V(G)$, $Y = V(G^*) - V(G)$. We use the notation defined in Theorem 1. Let $S_x = K$, $S_y = \emptyset$, $\overline{S}_x = V(G) - K$, and $\overline{S}_y = V(G^*) - V(G)$. We have $\sum_{x \in S} f(x) = \sum_{x \in K} f(x) = \sum_{x \in K} \left\lfloor \frac{d_G(x)}{2} \right\rfloor = \sum_{x \in K} \frac{d_G(x)}{2} - \frac{k}{2}$, $\sum_{x \in \overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} = |\overline{S}_{y_0}| = |E(G[S_x])| = |E(G[K])|$ and $2|E(G[K])| = \sum_{x \in K} d_G(x) - e_G(K, V(G) - K) = \sum_{x \in K} d_G(x) - (k-1)$.
So

$$\sum_{x \in \overline{S}} \max\{0, g(x) - d_{G^*-S}(x)\} > \sum_{x \in S} f(x)$$

and therefore by Lemma 1, G^* has no $(g, f)-$factor.

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