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Chaotic or hypercyclic semigroups on a function space $C_0(I, \mathbb{C})$ or $L^p(I, \mathbb{C})$

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Abstract. To investigate conditions for strongly continuous semigroups to be chaotic or hypercyclic, we consider a strongly continuous semigroup $\{T_t\}$ on a function space $C_0([0, \infty), \mathbb{C})$ or $L^p([0, \infty), \mathbb{C})$ expressed by $T_t f(x) = g(x, t)f(x+t)$. We also consider a strongly continuous semigroup $\{T_t\}$ on a function space $C_0([0, 1], \mathbb{C})$ or $L^p([0, 1], \mathbb{C})$ expressed by $T_t f(x) = q(x, t)f(e^{\gamma t}x)$ with $\gamma < 0$, which have the relation to the solution semigroups to an initial value problem.

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§1. Introduction

A strongly continuous semigroup $\{T_t\}$ on a Banach space X is called *hypercyclic* if there exists $x \in X$ such that the set $\{T_t(x) | t \geq 0\}$ is dense in X. $\{T_t\}$ is called *chaotic* if it is hypercyclic and the set of periodic points is dense in X. (An element $f \in X$ is called periodic if there exists some $t > 0$ such that $T_t f = f$.)

As for strongly continuous semigroups on Banach spaces the conditions to be hypercyclic or chaotic have been investigated by many people. T. Bermudez et al. [1] showed that every separable infinite dimensional complex Banach space admits a hypercyclic uniformly continuous semigroup and there exist Banach spaces admitting no chaotic strongly continuous semigroups. Desch et al. [2] considered weighted function spaces on $[0, \infty)$ and they gave a necessary and sufficient condition to be hypercyclic for translation semigroups on weighted function spaces. We examined necessary and sufficient conditions for a strongly continuous semigroup to be chaotic [5] and applied these results to partial differential equations [7]. A. Lasota et al.([3],[4]) investigated the dynamics of a population of cells undergoing simultaneous proliferation and maturation and showed that the solution semigroup to a partial differential equation describing the dynamics, is chaotic by using the theory of Wiener process.

In this paper, we investigate conditions for a strongly continuous semigroup $\{T_t\}$ on $C_0(I, \mathbb{C})$ or $L^p(I, \mathbb{C})$ to be hypercyclic or chaotic more deeply than the results ([5], [6],[7], [8]) and also consider a strongly continuous semigroup $\{T_t\}$ on $C_0([0, \infty), \mathbb{C})$ or $L^p([0, \infty), \mathbb{C})$ expressed as $T_t f(x) = g(x, t)f(x + t)$ and a strongly continuous semigroup $\{S_t\}$ on $C_0([0, 1], \mathbb{C})$ or $L^p([0, 1], \mathbb{C})$ expressed as $S_t f(x) = q(x, t)f(e^{\gamma t}x)$ with $\gamma < 0$.

In section 2, we treat a strongly continuous semigroup on a function space on $[0, \infty)$. By using a former result by the author et al.(Theorem A), we show a condition of a partial differential equation for the solution semigroup to be hypercyclic or chaotic (Theorem 2.1). As an extension of a strongly continuous semigroup $\{T_t\}$ in Theorem 2.1 expressed as $T_t f(x) = \frac{\rho(x)}{\rho(x+t)} f(x + t)$, we consider a strongly continuous semigroup $\{T_t\}$ on $C_0([0, \infty), \mathbb{C})$ or $L^p([0, \infty), \mathbb{C})$ expressed as $T_t f(x) = g(x, t)f(x + t)$ with $g(x, t) \in C^1([0, \infty) \times [0, \infty), \mathbb{C})$ and obtain a condition of the function g for the strongly continuous semigroup to be hypercyclic or chaotic (Theorem 2.3). We examine the relation among strongly continuous semigroups $\{T_t\}$ defined in several ways (Proposition 2.4).

Section 3 is devoted to an investigation of a strongly continuous semigroup on a function space on [0,1]. On such a function space, the translation semigroup cannot be considered, since $x + t$ goes outside of [0,1] for $x \in [0, 1]$ and $t > 0$. So by considering a map $\psi : [0, \infty) \to (0, 1]$ defined by $\psi(x) = e^{\gamma x}$ with $\gamma < 0$, we investigate a strongly continuous semigroup $\{S_t\}$ on $C_0([0, 1], \mathbb{C})$ or $L^p([0, 1], \mathbb{C})$ in contrast to $T_t f(x) = g(x, t)f(x + t)$ on $C_0([0, \infty), \mathbb{C})$ or $L^p([0, \infty), \mathbb{C})$. We introduce an admissible weight function on (0,1] induced from an admissible weight function on $[0, \infty)$ and obtain a condition of an admissible weight function for a strongly continuous semigroup on $C_{0,\rho}([0, 1], \mathbb{C})$ to be hypercyclic or chaotic (Theorem 3.1). By using the map ψ and Theorems 2.1 and 2.3, we investigate the solution semigroup to an initial value problem (Theorem 3.2) and a strongly continuous semigroup $\{S_t\}$ (Theorem 3.4). As for the space $L^p([0, 1], \mathbb{C})$, an admissible weight function on (0,1] does not work well and so by using the spectral property of an infinitesimal generator, we get a condition of the function q for a strongly continuous semigroup to be chaotic (Theorem 3.5). We examine the relation among strongly continuous semigroups $\{S_t\}$ defined in several ways (Proposition 2.4).

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§2. Translation semigroup on $I = [0, \infty)$

By an *admissible weight function* on $[0, \infty)$ we mean a measurable function $\rho : [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(i) $\rho(x) > 0$ for all $x \in [0, \infty)$;

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{\omega t}\rho(t + x)$ for all $x \in [0, \infty)$ and $t > 0$.

With an admissible weight function ρ, we construct the following function spaces:

$$C_{0,\rho}([0, \infty), \mathbb{C}) = \left\{ f : [0, \infty) \to \mathbb{C} \mid f \text{ continuous}, \lim_{x\to\infty} \rho(x)f(x) = 0 \right\}$$
$$\text{with } \|f\|_\rho = \sup_{\tau\in[0,\infty)} |f(\tau)\rho(\tau)|,$$
$$L^p_\rho([0, \infty), \mathbb{C}) = \left\{ f : [0, \infty) \to \mathbb{C} \mid f \text{ measurable}, \int_0^\infty |f(\tau)\rho(\tau)|^p \, d\tau < \infty \right\}$$
$$\text{with } \|f\|_{p,\rho} = \left(\int_0^\infty |f(\tau)\rho(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \qquad (p \geq 1)$$

and consider a (forward) *translation semigroup* $\{\widetilde{T}_t\}$ with parameter $t \geq 0$ defined by

$$\widetilde{T}_t f(x) = f(x + t) \qquad \text{for } f \in C_{0,\rho}([0, \infty), \mathbb{C}) \text{ or } L^p_\rho([0, \infty), \mathbb{C}). \tag{2.1}$$

Desch et al. [2] defined the space $L^p_\rho([0, \infty))$ by using $\|f\|_\rho = \left(\int_0^\infty |f(\tau)|^p \rho(\tau) \, d\tau \right)^{\frac{1}{p}}$, instead of $\|f\|_\rho = \left(\int_0^\infty |f(\tau)\rho(\tau)|^p \, d\tau \right)^{\frac{1}{p}}$. However in order to extend the following results in Theorem A to an initial value problem, the norm $\|f\|_\rho = \left(\int_0^\infty |f(\tau)\rho(\tau)|^p \, d\tau \right)^{\frac{1}{p}}$ is better, since the following equation (2.4) is obtained by using this norm.

As for the translation semigroup $\{\widetilde{T}_t\}$, the following has been obtained.

Theorem A ([2], [6], [5]). *Let $\widetilde{X}$ be $C_{0,\rho}([0, \infty), \mathbb{C})$ or $L^p_\rho([0, \infty), \mathbb{C})$ with an admissible weight function ρ and consider the translation semigroup $\{\widetilde{T}_t\}$ on $\widetilde{X}$. Then*

(1) *$\{\widetilde{T}_t\}$ is hypercyclic if and only if $\liminf_{t\to\infty} \rho(t) = 0$;*

(2) *if $\widetilde{X}$ is $C_{0,\rho}([0, \infty), \mathbb{C})$, then $\{\widetilde{T}_t\}$ is chaotic if and only if $\lim_{\tau\to\infty} \rho(\tau) = 0$;*

(3) *if* $\widetilde{X}$ *is* $L^p_\rho([0,\infty),\mathbb{C})$, *then* $\{\widetilde{T}_t\}$ *is chaotic if and only if for all* $\varepsilon > 0$ *and for all* $l > 0$, *there exists* $P > 0$ *such that*

$$\sum_{n=1}^{\infty}(\rho(l+nP))^p < \varepsilon.$$

Let X be a function space on an interval I and $u(t,x)$ be the solution of the following initial value problem:

$$\text{(2.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)) \end{cases}$$

for $f \in X$. Let T_t $(t \geq 0)$ be defined by $T_t f(x) = u(t,x)$ for $f \in X$ and $x \in I$. When T_t is a strongly continuous semigroup on X, we shall call $\{T_t\}$ the *solution semigroup* to an initial value problem (2.2).

The translation semigroup $\{\widetilde{T}_t\}$ on a weighted function space $\widetilde{X}$ is the solution semigroup to the following initial value problem:

$$\text{(2.3)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)) \end{cases}$$

for $f \in \widetilde{X}$. Let X be the space $C_0([0,\infty),\mathbb{C}) = \{f \in C([0,\infty),\mathbb{C}) \mid \lim_{x\to\infty} f(x) = 0\}$ or $L^p([0,\infty),\mathbb{C})$ and consider the strongly continuous semigroup $\{T_t\}$ on X defined by

$$\text{(2.4)} \qquad T_t f(x) = \frac{\rho(x)}{\rho(x+t)} f(x+t) \qquad \text{for } f \in X.$$

Let

$$\text{(2.5)} \qquad \begin{array}{rcl} \phi : C_{0,\rho}([0,\infty),\mathbb{C}) & \to & C_0([0,\infty),\mathbb{C}) \\ \big[\text{resp. } L^p_\rho([0,\infty),\mathbb{C}) & \to & L^p([0,\infty),\mathbb{C})\big] \end{array}$$

be defined by $\phi(f)(x) = \rho(x)f(x)$ for $f \in C_{0,\rho}([0,\infty),\mathbb{C})$[resp. $L^p_\rho([0,\infty),\mathbb{C})$]. Then ϕ is isomorphic and $\phi(\widetilde{T}_t f)(x) = T_t\phi(f)(x)$ holds. Since $\{T_t\}$ is the solution semigroup to the following initial value problem [8]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \dfrac{\rho'(x)}{\rho(x)}u & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)), \end{cases}$$

we consider the the following initial value problem:

$$\text{(2.6)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)), \end{cases}$$

where h is a bounded continuous function on $[0,\infty)$ and $f \in X$. A modification of [8, Theorems 2.1, 2.2 and 2.7] is the following

Theorem 2.1. *Let X be $C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$. Consider an initial value problem :*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)), \end{cases}$$

where $h \in C([0,\infty),\mathbb{C})$ is bounded and $f \in X$.

Then the solution semigroup $\{T_t\}_{t\geq 0}$ $\left(T_t f(x) = e^{\int_x^{x+t} h(s)ds} f(x+t)\right)$ is a strongly continuous semigroup on X. Moreover

(1) *$\{T_t\}$ is hypercyclic if and only if*

$$\limsup_{x\to\infty} \int_0^x \Re h(s)ds = \infty;$$

(2) *if $X = C_0([0,\infty),\mathbb{C})$, then $\{T_t\}_{t\geq 0}$ is chaotic if and only if*

$$\int_0^\infty \Re h(s)ds = \infty;$$

(3) *if $X = L^p([0,\infty),\mathbb{C})$ and $h(x) = \dfrac{a}{x+1}$ with $a > \dfrac{1}{p}$, then $\{T_t\}$ is chaotic.*

Proof. Put $\kappa(x) = \exp\left\{-\int_0^x h(s)\ ds\right\}$. Then $\rho(x) = |\kappa(x)|$ is an admissible weight function on $[0,\infty)$. Consider the space $\widetilde{X} = C_{0,\rho}([0,\infty),\mathbb{C})$ and the translation semigroup $\{\widetilde{T}_t\}$ defined by $\widetilde{T}_t f(x) = f(x+t)$ for $f \in \widetilde{X}$. Then by [7, Proposition 3], $\{T_t\}$ is hypercyclic [resp. chaotic] if and only if $\{\widetilde{T}_t\}$ is hypercyclic [resp. chaotic]. Hence (1) and (2) follows from Theorem A.
(3) follows from [8, Theorem 2.2 (2)]. □

The above theorem is concerned with the strongly continuous semigroup of the form

$$T_t f(x) = \frac{\kappa(x)}{\kappa(x+t)} f(x+t),$$

where $\rho(x) = |\kappa(x)|$ is an admissible weight function on $[0,\infty)$. As a generalization we consider the strongly continuous semigroup $\{T_t\}$ expressed as

$$T_t f(x) = g(x,t) f(x+t),$$

with $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$ and consider the condition for $\{T_t\}$ to be hypercyclic or chaotic.

The property of $g(x,t)$ is shown in the following

Lemma 1. *Let X be $C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$ and $\{T_t\}$ be a strongly continuous semigroup on X expressed as*

$$T_t f(x) = g(x,t) f(x+t),$$

with $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$. Then

(1) *$g(x,s+t) = g(x,s)g(x+s,t)$ for any $x,s,t \in [0,\infty)$,*

(2) *$g(x,0) = 1$ for any $x \in [0,\infty)$,*

(3) *$g(x,s) \neq 0$ for any $(x,s) \in [0,\infty)\times[0,\infty)$,*

(4) *$g(x,t) = \dfrac{g(0,x+t)}{g(0,x)}$.*

Proof. (1) By the relations $T_{s+t}f(x) = g(x,s+t)f(x+s+t)$ and $T_s(T_t f(x)) = g(x,s)T_t f(x+s) = g(x,s)g(x+s,t)f(x+s+t)$, we have $g(x,s+t) = g(x,s)g(x+s,t)$.
(2) By the definition, $f(x) = g(x,0)f(x)$ holds for any $f \in X$. So $g(x,0) = 1$ holds for any $x \in [0,\infty)$.
(3) Suppose there exists $(x,s) \in [0,\infty)\times[0,\infty)$ satisfying $g(x,s) = 0$. Then $g(x,s+t) = g(x,s)g(x+s,t)$ implies $g(x,t) = 0$ for any $t \geq s$. Let $s_0 = \min\{s \mid g(x,s) = 0\}$. If $s_0 > 0$, then for $t (0 \leq t \leq s_0)$, $g(x+t, s_0-t) = 0$ holds by the relation $g(x,s_0) = g(x,t)g(x+t,s_0-t)$. So $g(x+s_0,0) = 0$, which contradicts (2). Hence $g(x,s) \neq 0$ for any $(x,s) \in [0,\infty)\times[0,\infty)$.
(4) By (3), $\dfrac{g(0,x+t)}{g(0,x)}$ is well defined and (4) follows from (1). □

Proposition 2.2. *Let X be $C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$ and $\{T_t\}$ be a strongly continuous semigroup on X expressed as*

$$T_t f(x) = g(x,t) f(x+t),$$

where $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$ with $\left\|\dfrac{g_t(x,t)}{g(x,t)}\right\|_\infty < \infty$.

(1) *If we put $\rho(x) = \dfrac{1}{|g(0,x)|}$, then ρ is a continuous admissible weight function on $[0,\infty)$.*

(2) *Let $\widetilde{X}$ be the space $C_{0,\rho}([0,\infty),\mathbb{C})$ or $L^p_\rho([0,\infty),\mathbb{C})$ and $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ be the translation semigroup on $\widetilde{X}$. Then*

   (i) *$\{T_t\}_{t\geq 0}$ is hypercyclic on X iff $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is hypercyclic on $\widetilde{X}$.*

(ii) $\{T_t\}_{t\geq 0}$ *is chaotic on* X *iff* $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ *is chaotic on* $\widetilde{X}$.

Proof. (1) By Lemma 1 (3), $\rho(x)$ is well-defined. By the assumption $\left\|\frac{g_t(x,t)}{g(x,t)}\right\|_\infty = c < \infty$ and the relation $\rho(\tau) = \frac{1}{|g(0,\tau)|} = |\exp\{-\log g(0,\tau) + \log g(0,0)\}| = \left|\exp\{-\int_0^\tau \frac{g_t(0,s)}{g(0,s)} ds\}\right|$, we have

$$\begin{aligned}\rho(\tau) &= \left|\exp\left\{-\int_0^{t+\tau} \frac{g_t(0,s)}{g(0,s)} ds\right\} \exp\left\{\int_\tau^{t+\tau} \frac{g_t(0,s)}{g(0,s)} ds\right\}\right| \\ &\leq \rho(\tau+t) \exp\left\{\int_\tau^{t+\tau} \frac{|g_t(0,s)|}{|g(0,s)|} ds\right\} \leq \rho(\tau+t)e^{ct}.\end{aligned}$$

So ρ is an admissible weight function on $[0,\infty)$.
(2) Define an operator $\varphi : \widetilde{X} \to X$ as $\varphi(f)(x) = \rho(x)f(x)$ for $f \in \widetilde{X}$ and for $x \in [0,\infty)$. Then φ is an isomorphism of $\widetilde{X}$ to X and $T_t \circ \varphi = \varphi \circ \widetilde{T}_t$ holds. So we get the conclusion. □

Theorem 2.3. *Let* X *be* $C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$ *and* $\{T_t\}$ *be a strongly continuous semigroup on* X *expressed as*

$$T_t f(x) = g(x,t)f(x+t),$$

where $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$ *with* $\left\|\frac{g_t(x,t)}{g(x,t)}\right\|_\infty < \infty$. *Then*

(1) *the semigroup* $\{T(t)\}$ *is hypercyclic if and only if*

$$\limsup_{\tau\to\infty} |g(0,\tau)| = \infty;$$

(2) *if* $X = C_0([0,\infty),\mathbb{C})$, *then* $\{T(t)\}$ *is chaotic if and only if* $\lim_{\tau\to\infty} |g(0,\tau)| = \infty$;

(3) *if* $X = L^p([0,\infty),\mathbb{C})$ *and* $g(x,t) = \left(1+\frac{t}{x+1}\right)^b$ *with* $b > \frac{1}{p}$, *then* $\{T(t)\}$ *is chaotic.*

Proof. Put $\rho(\tau) = \frac{1}{|g(0,\tau)|}$. Then $\liminf_{\tau\to\infty} \rho(\tau) = 0$ [resp. $\lim_{\tau\to\infty} \rho(\tau) = 0$] is equivalent to $\limsup_{\tau\to\infty} |g(0,\tau)| = \infty$ [resp. $\lim_{\tau\to\infty} |g(0,\tau)| = \infty$] . So (1) and (2) follows from Theorem A (1), (2) and Proposition 2.2.
(3) If $g(x,t) = \left(1+\frac{t}{x+1}\right)^b$, then $g(x,s)g(x+s,t) = g(x,s+t)$ holds. So

$T_tf(x) = g(x,t)f(x+t)$ with $g(x,y) = \left(1+\dfrac{t}{x+1}\right)^b$ is a strongly continuous semigroup.

Put $\rho(\tau) = \dfrac{1}{g(0,\tau)} = (1+\tau)^{-b}$. For any $\varepsilon > 0$ and any $l > 0$, we have

$$\sum_{n=1}^{\infty}(\rho(l+nP))^p < \sum_{n=1}^{\infty}\frac{1}{(nP)^{bp}} < \frac{1}{P^{bp}}\left(\frac{bp}{bp-1}\right) < \varepsilon$$

for $P > \left(\dfrac{bp}{\varepsilon(bp-1)}\right)^{\frac{1}{bp}}$. Then the translation semigroup $\{\widetilde{T}_t\}$ on a weighted function space $L^p_\rho([0,\infty),\mathbb{C})$ is chaotic by Theorem A (3).

Since $g(x,t) = \left(1+\dfrac{t}{x+1}\right)^b$, $\dfrac{g_t(x,t)}{g(x,t)} = \dfrac{b}{x+t+1}$ means $\left\|\dfrac{g_t(x,t)}{g(x,t)}\right\|_\infty = b$. So by Proposition 2.2, $\{T_t\}$ is chaotic. □

As for the relation among the strongly continuous semigroups $\{T_t\}$ mentioned above, we have

Proposition 2.4. *Consider the following strongly continuous semigroups* (1)–(4) :

(1) *The strongly continuous semigroup* $\{T_t\}$ *on* $C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$ *expressed as* $T_tf(x) = g(x,t)f(x+t)$ *where* $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$ *satisfies* $\left\|\dfrac{g_t(x,t)}{g(x,t)}\right\|_\infty < \infty$;

(2) *The strongly continuous semigroup* $\{T_t\}$ *on* $C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$ *expressed as* $T_tf(x) = \dfrac{\kappa(x)}{\kappa(x+t)}f(x+t)$, *where* $\rho(x) = |\kappa(x)|$ *is an admissible weight function on* $[0,\infty)$;

(2') *The strongly continuous semigroup* $\{T_t\}$ *on* $C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$ *expressed as* $T_tf(x) = \dfrac{\rho(x)}{\rho(x+t)}f(x+t)$, *where* $\rho(x)$ *is an admissible weight function on* $[0,\infty)$;

(3) *The solution semigroup* $\{T_t\}$ *to the following initial value problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u & (x\in[0,\infty),\ t>0) \\ u(x,0) = f(x) & (x\in[0,\infty)), \end{cases}$$

where h *is a complex-valued bounded continuous function on* $[0,\infty)$ *and* $f \in C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$;

(4) *The translation semigroup* $\{\widetilde{T}_t\}$ *on* $C_{0,\rho}([0,\infty),\mathbb{C})$ *or* $L^p_\rho([0,\infty),\mathbb{C})$ *with an admissible weight function* ρ*.*

Then (1) $\Leftrightarrow$ (3) $\Rightarrow$ (2) *and* (2') $\Leftrightarrow$ (4) *holds, which means that there is a bijection between* (1) *and* (3)[*resp.* (2') *and* (4)] *and any* T_t *defined by* (1) *or* (3) *corresponds to some* T_t *defined by* (2)*.*

If we replace (2) *by the following ;*

(2") *The strongly continuous semigroup* $\{T_t\}$ *on* $C_0([0,\infty),\mathbb{C})$ *or* $L^p([0,\infty),\mathbb{C})$ *expressed as* $T_tf(x) = \dfrac{\kappa(x)}{\kappa(x+t)}f(x+t)$*, where* $\rho(x) = |\kappa(x)|$ *is a differentiable admissible weight function on* $[0,\infty)$ *satisfying* $\left\|\dfrac{\kappa'(t)}{\kappa(t)}\right\|_\infty < \infty$*,*

then there is a one-to-one onto correspondence among (1), (2") *and* (3)*.*

Proof. (1) $\Rightarrow$ (2): For $g(x,t)$ defined in (1), put $\kappa(x) = \frac{1}{g(0,x)}$. Then $\rho(x) = |\kappa(x)|$ is an admissible weight function on $[0,\infty)$.
(1) $\Rightarrow$ (3): For $g(x,t)$ defined in (1), put $h(x) = \frac{g_t(0,x)}{g(0,x)}$. Then the solution semigroup $\{T_t\}$ is obtained by $T_tf(x) = \exp\left\{\displaystyle\int_x^{x+t}\frac{g_t(0,x)}{g(0,x)}ds\right\}f(x+t) = g(x,t)f(x+t)$ by Lemma 1.(4).
(3) $\Rightarrow$ (1): For h defined in (3), put $g(x,t) = \exp\left\{\displaystyle\int_x^{x+t}h(s)ds\right\}$. Then $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$ and $\left\|\dfrac{g_t(x,t)}{g(x,t)}\right\|_\infty < \infty$, since $\frac{g_t(x,t)}{g(x,t)} = h(x+t)$ holds.
(2') $\Leftrightarrow$ (4) follows from the equation (2.5).
(2") $\Rightarrow$ (3): For $\kappa(x)$ defined in (2"), put $h(x) = -\frac{\kappa'(x)}{\kappa(x)}$. Then h is a bounded continuous function. $\square$

If we consider real-valued functions $g(x,t)$, $h(x)$ and we assume $\rho(x)$ is differentiable, then we have

Corollary. There is a one-to-one onto correspondence among the following strongly continuous semigroups (1) – (4) :

(1) The strongly continuous semigroup $\{T_t\}$ on $C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$ expressed as $T_tf(x) = g(x,t)f(x+t)$ where $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{R})$ with $\left\|\dfrac{g_t(x,t)}{g(x,t)}\right\|_\infty < \infty$;

(2) The strongly continuous semigroup $\{T_t\}$ on $C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$ expressed as $T_t f(x) = \dfrac{\rho(x)}{\rho(x+t)} f(x+t)$, where $\rho(x)$ is a differentiable admissible weight function on $[0,\infty)$ satisfying $\left\|\dfrac{\rho'(t)}{\rho(t)}\right\|_\infty < \infty$;

(3) The solution semigroup $\{T_t\}$ to the following initial value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u & (x \in [0,\infty),\ t > 0) \\ u(x,0) = f(x) & (x \in [0,\infty)), \end{cases}$$

where h is a real-valued bounded continuous function on $[0,\infty)$ and $f \in C_0([0,\infty),\mathbb{C})$ or $L^p([0,\infty),\mathbb{C})$;

(4) The translation semigroup $\{\widetilde{T}_t\}$ on $C_{0,\rho}([0,\infty),\mathbb{C})$ or $L^p_\rho([0,\infty),\mathbb{C})$ with a differentiable admissible weight function ρ satisfying $\left\|\dfrac{\rho'(t)}{\rho(t)}\right\|_\infty < \infty$.

§3. Transformation semigroup on $I = [0,1]$

Now we shall consider the case of $I = [0,1]$ and a strongly continuous semigroup $\{S_t\}$ on the function space $C_0([0,1],\mathbb{C}) = \{f \in C([0,1],\mathbb{C}) \mid f(0) = 0\}$ with sup norm or $L^p([0,1],\mathbb{C})$.

Consider a map $\psi : [0,\infty) \to (0,1]$ defined by

$$\psi(x) = e^{\gamma x} \tag{3.1}$$

with $\gamma < 0$.

By using an admissible weight function ρ on $[0,\infty)$, we shall consider a measurable function $\eta : (0,1] \to \mathbb{R}$ defined by

$$\eta(x) = \rho(\psi^{-1}(x)) \qquad \text{for } x \in (0,1]. \tag{3.2}$$

Then η satisfies the following conditions:

(i) $\eta(x) > 0$ for $x \in (0,1]$;

(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\eta(x) \leq M e^{\omega t} \eta(e^{\gamma t} x)$ for all $x \in (0,1]$ and $t > 0$.

We shall call η an *admissible weight function* on $(0,1]$.

With an admissible weight function η on $(0,1]$, we construct the following function spaces:

$$C_{0,\eta}((0,1],\mathbb{C}) = \left\{ f : (0,1] \to \mathbb{C} \mid f \text{ continuous on } (0,1],\ \lim_{x\to 0} \eta(x) f(x) = 0 \right\}$$
$$\text{with } \|f\|_\eta = \sup_{\tau\in(0,1]} |f(\tau)\eta(\tau)|,$$
$$L^p_\eta([0,1],\mathbb{C}) = \left\{ f : [0,1] \to \mathbb{C} \mid f \text{ measurable}, \int_0^1 |f(\tau)\eta(\tau)|^p \, d\tau < \infty \right\}$$
$$\text{with } \|f\|_{p,\eta} = \left(\int_0^1 |f(\tau)\eta(\tau)|^p \, d\tau \right)^{\frac{1}{p}} \qquad (p \geq 1).$$

Consider an operator $\varphi : C_{0,\eta}((0,1],\mathbb{C}) \to C_{0,\rho}([0,\infty),\mathbb{C})$ defined by

$$\varphi(f)(x) = f(\psi(x)) \tag{3.3}$$

for $f \in C_{0,\eta}((0,1],\mathbb{C})$, where ψ is defined by (3.1). Then φ is an isomorphism from $C_{0,\eta}((0,1],\mathbb{C})$ to $C_{0,\rho}([0,\infty),\mathbb{C})$.

Let $\widetilde{S}_t : C_{0,\eta}((0,1],\mathbb{C}) \to C_{0,\eta}((0,1],\mathbb{C})$ be defined by

$$\widetilde{S}_t(f) = \varphi^{-1} \circ \widetilde{T}_t \circ \varphi(f) \qquad \text{for } f \in C_{0,\eta}((0,1],\mathbb{C}), \tag{3.4}$$

where $\widetilde{T}_t$ is a translation operator on $C_{0,\rho}([0,\infty),\mathbb{C})$ defined by (2.1). Then

$$\widetilde{S}_t f(x) = f(e^{\gamma t} x). \tag{3.5}$$

So $\{\widetilde{S}_t\}$ is hypercyclic or chaotic if and only if $\{\widetilde{T}_t\}$ is hypercyclic or chaotic respectively.

Then the following theorem follows from Theorem A.

Theorem 3.1. *Let η be a continuous admissible weight function on $(0,1]$, $\widetilde{X} = C_{0,\eta}((0,1],\mathbb{C})$ and the strongly continuous semigroup $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ be defined by* (3.5).

Then

(1) $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ *is hypercyclic if and only if* $\liminf_{\tau\to 0} \eta(\tau) = 0$;

(2) $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ *is chaotic if and only if* $\lim_{\tau\to 0} \eta(\tau) = 0$.

As for L^p space, consider $\varphi(g)(x) = g(\psi(x))$ for $g \in L^p_\eta([0,1],\mathbb{C})$, where ψ is defined by (3.1). In this case, $\varphi(g)$ does not necessarily belong to $L^p_\eta([0,\infty),\mathbb{C})$, since

$$\int_0^\infty |\varphi(g)(\tau)\tilde{\eta}(\tau)|^p \, d\tau = \int_0^1 |g(x)\eta(x)|^p \frac{1}{-\gamma x} dx.$$

So we must investigate in a different way and the next Proposition shows that if $\lim_{\tau \to 0} \eta(\tau)$ exists, then the strongly continuous semigroup $\{\widetilde{S}_t\}$ is always hypercyclic.

Proposition B ([8, Proposition 3.3]). *Let $\widetilde{X}$ be $L^p_\eta([0,1],\mathbb{C})$ and $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ on $\widetilde{X}$ be defined by $\widetilde{S}_t g(x) = g(e^{\gamma t}x)$ for $g \in \widetilde{X}$.*

If η is continuous on $(0,1]$ and $\lim_{\tau\to 0} \eta(\tau) = c < \infty$ exists, then $\widetilde{S}_t$ is a bounded linear operator on $\widetilde{X}$ and the strongly continuous semigroup $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ is hypercyclic.

Since the translation semigroup $\{\widetilde{T}_t\}$ is a solution semigroup to the initial value problem (2.3), the strongly continuous semigroup $\{\widetilde{S}_t\}$ is the solution semigroup to the following initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial v}{\partial x} & (x \in [0,1],\ t > 0) \\ v(0,x) = f(x) & (x \in [0,1]) \end{cases}$$

for $f \in X$. Let X be the space $C_0([0,1],\mathbb{C}) = \{f \in C([0,1],\mathbb{C}) \mid f(0) = 0\}$ and consider a strongly continuous semigroup $\{S_t\}$ defined by

$$S_t(f) = \varphi^{-1} \circ T_t \circ \varphi(f) \qquad \text{for } f \in C_0([0,1],\mathbb{C}), \tag{3.6}$$

where T_t is an operator on $C_0([0,\infty),\mathbb{C})$ defined by (2.4). Then

$$S_t f(x) = \frac{\eta(x)}{\eta(e^{\gamma t}x)} f(e^{\gamma t}x). \tag{3.7}$$

For $v \in C([0,1]\times[0,\infty),\mathbb{C})$ with $v(0,t) = 0$, put $u(x,t) = v(\psi(x),t)$. Then $u \in C([0,\infty)\times[0,\infty),\mathbb{C})$ and $\lim_{x\to\infty} u(x,t) = 0$. If u is a solution of the initial value problem (2.6), then v is a solution of the following initial value problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial v}{\partial x} + k(x)v & (x \in [0,1],\ t > 0) \\ v(x,0) = f(x) & (x \in [0,1]), \end{cases} \tag{3.8}$$

where $k(x) = h(\psi^{-1}x)$. In [7, Theorem 1], it is shown that if $\min\{\Re(k(x)) \mid x \in [0,1]\}$ is positive, then the solution semigroup $\{S_t\}_{t\geq 0}$ on $C_0([0,1],\mathbb{C})$ to (3.8) is chaotic by using the spectral property of its infinitesimal generator. However if we use Theorem 2.1, we get a necessary and sufficient condition for $\{S_t\}$ to be chaotic as follows:

Theorem 3.2. *Let X be the space $C_0([0,1],\mathbb{C})$. Consider the following initial value problem :*

$$\left\{\begin{array}{ll} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial v}{\partial x} + k(x)v & (x \in [0,1],\ t>0) \\ v(x,0) = f(x) & (x \in [0,1]), \end{array}\right. \tag{3.9}$$

where $\gamma < 0$, $k \in C([0,1],\mathbb{C})$ and $f \in X$. Then the strongly continuous semigroup $\{S_t\}_{t\geq 0}$ $\left(S_t f(x) = \exp\left\{\int_0^t k(e^{\gamma(t-r)}x)\,dr\right\} f(e^{\gamma t}x)\right)$ is a strongly continuous semigroup on X.

Moreover $\{S_t\}_{t\geq 0}$ is chaotic if and only if $\lim_{x\to 0}\int_x^1 \frac{\Re k(s)}{s}\,ds = \infty$.

Therefore if $\Re k(0) > 0$, then $\{S_t\}_{t\geq 0}$ is chaotic.

Proof. By using $k(x) = h(\psi^{-1}(x))$, an initial value problem (3.9) corresponds to the initial value problem (2.6). By the equation

$$\int_0^\infty \Re h(s)\,ds = \lim_{x\to 0}\int_x^1 \Re h(\psi^{-1}(\tau))\frac{d\tau}{\gamma\tau} = \lim_{x\to 0}\int_x^1 \frac{\Re k(s)}{\gamma s}\,ds,$$

we get that $\{S_t\}_{t\geq 0}$ is chaotic if and only if $\lim_{x\to 0}\int_x^1 \frac{\Re k(s)}{s}\,ds = \infty$ by using Theorem 2.1 □

As for the space $L^p([0,1],\mathbb{C})$, we have

Theorem C ([8, Theorem 3.5])**.** *Let X be the space $L^p([0,1],\mathbb{C})$ with $p \geq 1$. Consider the following initial value problem :*

$$\left\{\begin{array}{ll} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial v}{\partial x} + k(x)v & (x \in [0,1],\ t>0) \\ v(x,0) = f(x) & (x \in [0,1]) \end{array}\right.$$

where $\gamma < 0$, $k \in C([0,1],\mathbb{C})$ and $f \in X$. Then the strongly continuous semigroup $\{S_t\}_{t\geq 0}$ $(S_t f(x) = \exp\left\{\int_0^t k(e^{\gamma(t-r)}x)\,dr\right\} f(e^{\gamma t}x))$ is a strongly continuous semigroup on X. Moreover

1. *if there exists $\delta > 0$ such that $\Re(k(x)) \geq 0$ for $0 \leq \forall x \leq \delta$, then $\{S_t\}_{t\geq 0}$ is hypercyclic ;*
2. *if $\min\{\Re(k(x)) \mid x \in [0,1]\} > \frac{\gamma}{p}$, then $\{S_t\}_{t\geq 0}$ is chaotic.*

By using a strongly continuous semigroup $\{T_t\}$ on $C_0([0,\infty),\mathbb{C})$ expressed as

$$T_t f(x) = g(x,t)f(x+t),$$

with $g(x,t) \in C^1([0,\infty)\times[0,\infty),\mathbb{C})$, we shall consider a strongly continuous semigroup $\{S_t\}$ on $C_0([0,1],\mathbb{C})$ expressed as

$$S_t(f) = \varphi^{-1}\circ T_t \circ \varphi(f) \qquad \text{for } f \in C_0([0,1],\mathbb{C}).$$

Then by putting

$$q(x,t) = g(\psi^{-1}x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C}), \tag{3.10}$$

$S_t f(x) = q(x,t)f(e^{\gamma t}x)$ is a generalization of a strongly continuous semigroup of the form $S_t f(x) = \frac{\eta(x)}{\eta(e^{\gamma t}x)} f(e^{\gamma t}x)$.

By Lemma 1, the property of the function $q(x,t)$ is obtained as follows.

Lemma 2. *Let $\gamma < 0$, $X = C_0([0,1],\mathbb{C})$ or $L^p([0,1],\mathbb{C})$ and $\{S_t\}$ be a strongly continuous semigroup on X expressed as*

$$S_t f(x) = q(x,t)f(e^{\gamma t}x),$$

where $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C})$. Then

(1) *$q(x,s+t) = q(x,s)q(e^{\gamma s}x,t)$ for any $x,s,t \in [0,1]$,*

(2) *$q(x,0) = 1$ for any $x \in [0,1]$,*

(3) *$q(x,s) \neq 0$ for any $(x,s) \in [0,1]\times[0,\infty)$,*

(4) *$q(e^{\gamma s},t) = \dfrac{q(1,s+t)}{q(1,s)}$.*

Proposition 3.3. *Let X be $C_0([0,1],\mathbb{C})$ [resp. $L^p([0,1],\mathbb{C})$] and $\{S_t\}$ be a strongly continuous semigroup on X expressed as*

$$S_t f(x) = q(x,t)f(e^{\gamma t}x),$$

where $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C})$ with $\left\|\dfrac{q_t(x,t)}{q(x,t)}\right\|_\infty < \infty$ and $\gamma < 0$.

(1) *If we put $\eta(x) = \dfrac{1}{\left|q(1,\frac{\log x}{\gamma})\right|}$ for $x \in (0,1]$, then η is a continuous admissible weight function on $(0,1]$.*

(2) *Let $\widetilde{X}$ be the space $C_{0,\eta}((0,1],\mathbb{C})$ [resp. $L^p_\eta([0,1],\mathbb{C})$] and $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ be a strongly continuous semigroup on $\widetilde{X}$ defined by $S_t f(x) = f(e^{\gamma t}x)$. Then*

(i) $\{S_t\}_{t\geq 0}$ *is hypercyclic on* X *iff* $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ *is hypercyclic on* $\widetilde{X}$.

(ii) $\{S_t\}_{t\geq 0}$ *is chaotic on* X *iff* $\left\{\widetilde{S}_t\right\}_{t\geq 0}$ *is chaotic on* $\widetilde{X}$.

Proof. (1) By using the equations (3.2) and (3.10), $\eta(x) = \frac{1}{\left|q(1,\frac{\log x}{\gamma})\right|}$ implies that $\rho(\psi^{-1}(x)) = \frac{1}{|q(\psi^{-1}(1),\psi^{-1}(x))|}$, that is, $\rho(s) = \frac{1}{|q(0,s)|}$. So by Proposition 2.2 (1), η is a continuous admissible weight function on $(0,1]$.
(2) It is obtained by the same way as Proposition 2.2 (2). □

By Theorem 2.3, we have

Theorem 3.4. *Let* $\{S_t\}$ *be a strongly continuous semigroup on* $C_0([0,1],\mathbb{C})$ *expressed as*

$$S_t f(x) = q(x,t)f(e^{\gamma t}x),$$

where $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C})$ *with* $\left\|\dfrac{q_t(x,t)}{q(x,t)}\right\|_\infty < \infty$ *and* $\gamma < 0$. *Then the following are equivalent* :

(1) $\{S_t\}_{t\geq 0}$ *is hypercyclic if and only if* $\limsup\limits_{\tau\to\infty} |q(1,\tau)| = \infty$;

(2) $\{S_t\}_{t\geq 0}$ *is chaotic if and only if* $\lim\limits_{\tau\to\infty} |q(1,\tau)| = \infty$.

In case of $C_0([0,1],\mathbb{C})$, the property of η plays an essential role in proving that $\{S_t\}$ is chaotic or hypercyclic. However, in case of $L^p([0,1],\mathbb{C})$, we have not obtained any property of η for a strongly continuous semigroup to be chaotic. So we use the the following

Theorem D ([2]). *Let* X *be a separable Banach space and let* A *be the infinitesimal generator of a strongly continuous semigroup* $\{S_t\}_{t\geq 0}$ *on* X. *Let* U *be an open subset of the point spectrum of* A, *which intersects the imaginary axis, and for each* $\lambda \in U$ *let* x_λ *be a nonzero eigenvector, i.e.* $Ax_\lambda = \lambda x_\lambda$. *For each* $\phi \in X^*$ *we define a function* $F_\phi\colon\ U \to \mathbb{C}$ *by* $F_\phi(\lambda) = \langle \phi, x_\lambda\rangle$. *Assume that for each* $\phi \in X^*$ *the function* F_ϕ *is analytic and that* F_ϕ *does not vanish identically on* U *unless* $\phi = 0$. *Then* $\{S_t\}_{t\geq 0}$ *is chaotic.*

Theorem 3.5. *Let* $\{S_t\}$ *be a strongly continuous semigroup on* $L^p([0,1],\mathbb{C})$ *expressed as*

$$S_t f(x) = q(x,t)f(e^{\gamma t}x),$$

where $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C})$ *with* $\left\|\dfrac{q_t(x,t)}{q(x,t)}\right\|_\infty < \infty$ *and* $\gamma < 0$.

Then if there is $\varepsilon > 0$ *satisfying* $|q(1,\tau)| > e^{(\frac{\gamma}{p}+\varepsilon)\tau}$ *for any* $\tau \in [0,\infty)$, *then* $\{S_t\}$ *is chaotic.*

Proof. Let A be the infinitesimal generator of a strongly continuous semigroup $\{S_t\}$. Then

$$AS_tf(x) = q_t(x,t)f(e^{\gamma t}x) + \gamma e^{\gamma t}xq(x,t)f'(e^{\gamma t}x).$$

In order to use Theorem D, we shall prove that the existence of an open set U of the point spectrum of the infinitesimal generator A which intersects the imaginary axis.

If $f_\lambda(x) = \dfrac{x^\lambda \gamma}{q(1, \frac{\log x}{\gamma})}$ belongs to $L^p([0,1],\mathbb{C})$, then $Af_\lambda = \lambda f_\lambda$ holds. Put

$$U = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < \varepsilon\}.$$

For $\lambda \in U$, by using the condition $|g(1,\tau)| > e^{(\frac{\gamma}{p}+\varepsilon)\tau}$, we have

$$\begin{aligned}\int_0^1 |f_\lambda(x)|^p\,dx &= \int_0^1 \left|\frac{x^\lambda\gamma}{q(1,\frac{\log x}{\gamma})}\right|^p dx \le \int_0^1 \left|\frac{x^\lambda\gamma}{e^{(\frac{\gamma}{p}+\varepsilon)\frac{\log x}{\gamma}}}\right|^p dx \\ &= \int_0^1 x^{\frac{p(\Re(\lambda)-\varepsilon)}{\gamma}-1}\,dx < \infty,\end{aligned}$$

since $\dfrac{p(\Re(\lambda)-\varepsilon)}{\gamma} > 0$. So f_λ belongs to $L^p([0,1],\mathbb{C})$ for $\lambda \in U$ and U is an open subset of the point spectrum of A, which intersects the imaginary axis. So we can prove in a similar way to the proof of [7, Theorem 2]. □

As for the relation among the strongly continuous semigroups $\{S_t\}$ mentioned above, we have

Proposition 3.6. *Consider the following strongly continuous semigroups* (1)–(4) :

(1) *The strongly continuous semigroup* $\{S_t\}$ *on* $C_0([0,1],\mathbb{C})$ *or* $L^p([0,1],\mathbb{C})$ *expressed as* $S_tf(x) = q(x,t)f(e^{\gamma t}x)$ *where* $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{C})$ *satisfies* $\left\|\dfrac{q_t(x,t)}{q(x,t)}\right\|_\infty < \infty$;

(2) *The strongly continuous semigroup* $\{S_t\}$ *on* $C_0([0,1],\mathbb{C})$ *or* $L^p([0,1],\mathbb{C})$ *expressed as* $S_tf(x) = \dfrac{\kappa(x)}{\kappa(e^{\gamma t}x)}f(e^{\gamma t}x)$, *where* $\eta(x) = |\kappa(x)|$ *is an admissible weight function on* $(0,1]$;

(2') *The strongly continuous semigroup* $\{S_t\}$ *on* $C_0([0,1],\mathbb{C})$ *or* $L^p([0,1],\mathbb{C})$ *expressed as* $S_tf(x) = \dfrac{\eta(x)}{\eta(e^{\gamma t}x)}f(e^{\gamma t}x)$, *where* $\eta(x)$ *is an admissible weight function on* $(0,1]$;

(3) *The solution semigroup* $\{S_t\}$ *to the following initial value problem* :

$$\begin{cases} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + k(x)v & (x \in [0,1],\ t > 0) \\ v(x,0) = f(x) & (x \in [0,1]), \end{cases}$$

where $\gamma < 0$, $k \in C([0,1],\mathbb{C})$ *and* $f \in C_0([0,1],\mathbb{C})$ *or* $L^p([0,1],\mathbb{C})$;

(4) *The strongly continuous semigroup* $\{\widetilde{S}_t\}$ *expressed as* $\widetilde{S}_t f(x) = f(e^{\gamma t}x)$ *on* $C_{0,\eta}((0,1],\mathbb{C})$ *or* $L^p_\rho([0,1],\mathbb{C})$ *with an admissible weight function* η.

Then (1) $\Leftrightarrow$ (3) $\Rightarrow$ (2) *and* (2') $\Leftrightarrow$ (4) *holds, which means that there is a bijection between* (1) *and* (3)[*resp.* (2') *and* (4)] *and any* T_t *defined by* (1) *or* (3) *corresponds to some* T_t *defined by* (2).

If we replace (2) *by the following* ;

(2") *The strongly continuous semigroup* $\{S_t\}$ *on* $C_0([0,1],\mathbb{C})$ *or* $L^p([0,1],\mathbb{C})$ *expressed as* $S_t f(x) = \dfrac{\kappa(x)}{\kappa(e^{\gamma t}x)} f(e^{\gamma t}x)$, *where* $\eta(x) = |\kappa(x)|$ *is a differentiable admissible weight function on* $(0,1]$ *satisfying* $\left\| \dfrac{\kappa'(t)}{\kappa(t)} \right\|_\infty < \infty$,

then there is a one-to-one onto correspondence among (1), (2") *and* (3).

Proof. In case of $C_0([0,1],\mathbb{C})$, we get the result by using the relation (3.6) and Proposition 2.4.

In case of $L^p([0,1],\mathbb{C})$,

(1) $\Rightarrow$ (3): For $q(x,t)$ defined in (1), put $k(x) = \frac{q_t(1,\frac{\log x}{\gamma})}{q(1,\frac{\log x}{\gamma})}$. Then the solution semigroup $\{S_t\}$ is obtained by $S_t f(x) = \exp\left\{ \displaystyle\int_0^t \frac{q_t(1,t-s+\frac{\log x}{\gamma})}{q(1,t-s+\frac{\log x}{\gamma})} ds \right\} f(e^{\gamma t}x)$ $= \frac{q(1,t+\frac{\log x}{\gamma})}{q(1,\frac{\log x}{\gamma})} f(e^{\gamma t}x) = q(x,t)f(e^{\gamma t}x)$ by Lemma 2.(4).

The other parts will be proved in a similar way to Proposition 2.4. $\square$

If we consider real-valued functions $q(x,t)$, $k(x)$ and we assume $\eta(x)$ is differentiable, then we have

Corollary. There is a one-to-one onto correspondence among the following strongly continuous semigroups (1) – (4) :

(1) The strongly continuous semigroup $\{S_t\}$ on $C_0([0,1],\mathbb{C})$ or $L^p([0,1],\mathbb{C})$ expressed as $S_t f(x) = q(x,t)f(e^{\gamma t}x)$ where $q(x,t) \in C^1([0,1]\times[0,\infty),\mathbb{R})$ with $\left\| \dfrac{q_t(x,t)}{q(x,t)} \right\|_\infty < \infty$;

(2) The strongly continuous semigroup $\{S_t\}$ on $C_0([0,1],\mathbb{C})$ or $L^p([0,1],\mathbb{C})$ expressed as $S_tf(x) = \dfrac{\eta(x)}{\eta(e^{\gamma t}x)} f(e^{\gamma t}x)$, where $\eta(x)$ is a differentiable admissible weight function on $(0,1]$ satisfying $\left\|\dfrac{\eta'(t)}{\eta(t)}\right\|_\infty < \infty$;

(3) The solution semigroup $\{S_t\}$ to the following initial value problem:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + k(x)v & (x \in [0,1]),\ t > 0 \\ v(x,0) = f(x) & (x \in [0,1]), \end{cases}$$

where $\gamma < 0$, $k \in C([0,1],\mathbb{C})$ and $f \in C_0([0,1],\mathbb{C})$ or $L^p([0,1],\mathbb{C})$;

(4) The strongly continuous semigroup $\{\widetilde{S}_t\}$ expressed as $\widetilde{S}_tf(x) = f(e^{\gamma t}x)$ on $C_{0,\eta}((0,1],\mathbb{C})$ or $L^p_\rho([0,1],\mathbb{C})$ with a differentiable admissible weight function η satisfying $\left\|\dfrac{\eta'(t)}{\eta(t)}\right\|_\infty < \infty$.

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