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On the Well-posedness of the 1-D Quadratic Semilinear Schrödinger Equations

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Abstract. This paper deals with a slight improvement on the results of the 1-D semilinear Schrödinger equations with quadratic nonlinearities. We study the local well-posedness of the initial value problem in particular function spaces containing the Sobolev spaces H^s with $s > -1/4$ for the nonlinearity $u\bar{u}$, and with $s > -3/4$ for u^2 or $\bar{u}^2$, in which the local well-posedness was proved by Kenig, Ponce and Vega. Our improvement lies in the estimate of the Fourier restriction norm with a homogeneous weight $|\xi|^s$. It makes the behavior of the initial data at $\xi = 0$ in the phase space less restrictive.

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§1. Introduction

This paper is devoted to the well-posedness of the initial value problem (IVP) for the 1-D quadratic semilinear Schrödinger equations.

Well-posedness here means that the existence, the uniqueness, the persistence property of the solution and the continuous dependence of the solution on the initial data.

$$\begin{cases} \partial_t u = i\partial_x^2 u + N(u,\bar{u}), \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

1.1. The former results due to Kenig, Ponce and Vega

In [15], C.E. Kenig, G. Ponce and L. Vega verified the local well-posedness of the IVP (1.1) in the Sobolev space $H^s(\mathbb{R})$ with $s > -3/4$ for $N(u,\bar{u}) = cu^2$,

$c\bar{u}^2$, and with $s > -1/4$ for $N(u, \bar{u}) = cu\bar{u}$.

They showed these local well-posedness by combining the following nonlinear estimates. These estimates evaluate directly the quadratic nonlinearities of the equations:

Theorem 1.1 (Kenig, Ponce and Vega [15])
(i) *Let* $s \in (-3/4, 0]$. *Then there exist* $b \in (1/2, 1)$ *and* $C > 0$ *such that*

$$\|FG\|_{X_{s,b-1}} \leq C\|F\|_{X_{s,b}}\|G\|_{X_{s,b}}, \tag{1.2}$$

$$\|\overline{FG}\|_{X_{s,b-1}} \leq C\|F\|_{X_{s,b}}\|G\|_{X_{s,b}}. \tag{1.3}$$

for any $F, G \in X_{s,b}$.
(ii) *Let* $s \in (-1/4, 0]$. *Then there exist* $b \in (1/2, 1)$ *and* $C > 0$ *such that*

$$\|F\overline{G}\|_{X_{s,b-1}} \leq C\|F\|_{X_{s,b}}\|G\|_{X_{s,b}} \tag{1.4}$$

for any $F, G \in X_{s,b}$.

Here $X_{s,b}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|F\|_{X_{s,b}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \tau - \xi^2\rangle^{2b}\langle\xi\rangle^{2s}|\widetilde{F}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}, \tag{1.5}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. Here $\widetilde{f}$ and $\widehat{f}$ (or $\mathcal{F}_x(f)$) denote the Fourier transform of f with respect to the space-time and the space variables respectively.

On the other hand, it is known that the above nonlinear estimates do not generally hold for the other cases $s \leq -3/4$ and $s \leq -1/4$.

Theorem 1.2 (Kenig, Ponce and Vega [15], Nakanishi, Takaoka and Tsutsumi [19])
(i) *For any* $s \leq -3/4$ *and any* $b \in \mathbb{R}$, *the estimates* (1.2) *and* (1.3) *fail.*
(ii) *For any* $s < -1/4$ *and any* $b \in \mathbb{R}$, *the estimate* (1.4) *fails.*
(ii') *For* $s = -1/4$ *and any* b *with* $b \geq 1/2$, *the estimate* (1.4) *fails.*

The failure of the estimates of the critical exponents (i) $s = -3/4$ and (ii') $s = -1/4$ are proved in [19].

Following the argument by Bourgain [4, 5], Kenig, Ponce and Vega used the above Fourier restriction norm. However, the norm in the argument is not entirely new in the theory of partial differential equations. Actually in [1], M. Beals dealt with the propagation of the singularity in the analogous way.

The arguments with the Fourier restriction norm enable us to solve the initial value problem in weaker function spaces, and are presently applied to several equations. See e.g. [2], [8], [14], [16], [20], [23] and [24]. Also see [7].

The IVP (1.1) in the 2-dimensional setting are studied in [22] and [6].

Before stating our results, we shall present a few examples which is not contained as the initial data of the IVP (1.1) in the framework of the argument by Kenig, Ponce and Vega.

When we consider the IVP (1.1) with $N(u,\bar{u}) = cu^2$ (or $c\bar{u}^2$), the function $u_0(x) = |x|^{-k}$ with $3/4 < k < 1$ is an example as the initial data which is not in $H^s(\mathbb{R})$ with $s > -3/4$, indeed $\mathcal{F}_x(|x|^{-k}) = c_k|\xi|^{k-1}$, where $0 < 1-k < 1/4$. Hence it is clear that $\langle\xi\rangle^{-3/4+\epsilon}\mathcal{F}_x(|x|^{-k}) \notin L^2(\mathbb{R})$, which implies that $|x|^{-k} \notin H^s(\mathbb{R})$ with $s > -3/4$.

In the case of the IVP (1.1) with $N(u,\bar{u}) = cu\bar{u}$, we present an concrete example;

$$u_0(x) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\cos(|t|\xi)}{|x-t|^{3/4+\epsilon}\langle\xi\rangle^{2\epsilon}} d\xi dt. \tag{1.6}$$

Indeed, putting

$$f(x) = \frac{1}{|x|^{3/4+\epsilon}}, \quad g(x) = c\int_{\mathbb{R}} \langle\xi\rangle^{-2\epsilon} e^{i\xi|x|} d\xi, \tag{1.7}$$

we write $u_0(x) = (f * g)(x)$. Hence

$$\widehat{u}_0(\xi) = c\widehat{f}(\xi)\widehat{g}(\xi) = c\frac{\langle\xi\rangle^{-2\epsilon}}{|\xi|^{1/4-\epsilon}}, \tag{1.8}$$

which is not in $H^s(\mathbb{R})$ with $s > -1/4$.

However, in our framework stating below, we can treat these examples as the initial data of the IVP (1.1). We shall state our results including these topics in the next section.

1.2. Statement of our results

We shall improve the results of Kenig, Ponce and Vega by using the Fourier restriction norm with a homogeneous weight $|\xi|^s$. Our proofs are, however, analogous as that of Kenig, Ponce and Vega [15].

We set the function space below;

Definition 1.3 For $s, s', b \in \mathbb{R}$, $X^b_{s,s'}$ denotes the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|F\|_{X^b_{s,s'}} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\tau-\xi^2\rangle^{2b}\langle\xi\rangle^{2s}|\xi|^{2s'}|\widetilde{F}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}. \tag{1.9}$$

Also we shall use the following function space;

Definition 1.4 $H^{s,s'}(\mathbb{R})$ is defined by

$$\{f \in \mathcal{S}'(\mathbb{R})\colon \langle\xi\rangle^s |\xi|^{s'} \widehat{f}(\xi) \in L^2(\mathbb{R})\} \tag{1.10}$$

with the norm

$$\|f\|_{H^{s,s'}} = \|\langle\xi\rangle^s |\xi|^{s'} \widehat{f}(\xi)\|_{L^2}.$$

Remark 1.5 According to [17], we can not solve the initial value problem for the Benjamin-Ono equation in the Sobolev space $H^s(\mathbb{R})$ via iterative methods. Recently, it was announced by K. Kato [10] that it is possible to show the existence of the solution to the BO equation in $H^{1+\epsilon,-1/2}(\mathbb{R})$ via iterative methods.

The function spaces defined in Definitions 1.3 and 1.4 are also used in [3] in a different context.

We first state our results in the case of $N(u,\bar{u}) = cu\bar{u}$.

Theorem 1.6 *Let $s \in (-1/4, 0)$ and $s' \in (0, 1/4)$. Then there exists $b \in (1/2, 1)$ such that for any $u_0 \in H^{s-s',s'}(\mathbb{R})$ there exist $T = T(\|u_0\|_{H^{s-s',s'}}) > 0$ and a unique solution $u(t)$ of the IVP (1.1) with $N(u,\bar{u}) = cu\bar{u}$ satisfying*

$$u \in C([-T,T]; H^{s-s',s'}(\mathbb{R})), \tag{1.11}$$

$$u \in X^b_{s-s',s'}, \tag{1.12}$$

$$u\bar{u} \in X^{b-1}_{s-s',s'}, \quad \partial_t u,\ \partial_x^2 u \in X^{b-1}_{s-s'-2,s'} \cap X^{b-1}_{s-s',s'-2}. \tag{1.13}$$

Moreover, for a given $T' \in (0,T)$ there exists $R = R(T') > 0$ such that the map $v_0 \mapsto v(t)$ is Lipschitz, where the map is from $\{v_0 \in H^{s-s',s'}(\mathbb{R}) : \|v_0 - u_0\|_{H^{s-s',s'}} < R\}$ into the class $C([-T,T]; H^{s-s',s'}(\mathbb{R})) \cap X^b_{s-s',s'}$.

This theorem will be assured by the following nonlinear estimate.

Proposition 1.7 *Let $s = -\rho \in (-1/4, 0)$ and $s' \in (0, 1/4)$. Then there exist $b \in (1/2, 1-2\rho)$ and $b' \in (1/2, b]$ such that*

$$\|F\overline{G}\|_{X^{b-1}_{s-s',s'}} \le c\|F\|_{X^{b'}_{s-s',s'}} \|G\|_{X^{b'}_{s-s',s'}} \tag{1.14}$$

for any $F, G \in X^{b'}_{s-s',s'}$.

The results in the cases of u^2 and $\bar{u}^2$ are followings:

In the case of $N(u,\bar{u}) = cu^2$.

Theorem 1.8 *For the IVP (1.1) with the nonlinear term $N(u,\bar{u}) = cu^2$ the results in Theorem 1.6 (with u^2 in (1.13) instead of $u\bar{u}$) hold for $s \in (-3/4, -1/2)$ and $s' \in (0, 1/4)$.*

Proposition 1.9 *Let $s = -\rho \in (-3/4, -1/2)$ and $s' \in (0, 1/4)$. Then there exist $b \in (1/2, 5/4 - \rho)$ and $b' \in (1/2, b]$ such that*

$$\|FG\|_{X^{b-1}_{s-s',s'}} \leq c\|F\|_{X^{b'}_{s-s',s'}} \|G\|_{X^{b'}_{s-s',s'}} \tag{1.15}$$

for any $F, G \in X^{b'}_{s-s',s'}$.

In the case of $N(u, \bar{u}) = c\bar{u}^2$.

Theorem 1.10 *For the IVP* (1.1) *with the nonlinear term $N(u, \bar{u}) = c\bar{u}^2$ the results in Theorem* 1.6 *(with $\bar{u}^2$ in* (1.13) *instead of $u\bar{u}$) hold for $s \in (-3/4, -1/2)$ and $s' \in (0, 1/4)$.*

Proposition 1.11 *Let $s = -\rho \in (-3/4, -1/2)$ and $s' \in (0, 1/4)$. Then there exist $b \in (1/2, 5/4 - \rho)$ and $b' \in (1/2, b]$ such that*

$$\|\overline{FG}\|_{X^{b-1}_{s-s',s'}} \leq c\|F\|_{X^{b'}_{s-s',s'}} \|G\|_{X^{b'}_{s-s',s'}} \tag{1.16}$$

for any $F, G \in X^{b'}_{s-s',s'}$.

Remark 1.12 We note that

$$H^{s+s'}(\mathbb{R}) \subset H^{s,s'}(\mathbb{R}), \tag{1.17}$$

provided $s' > 0$. In this sense, we improve the results of Kenig, Ponce and Vega.

Furthermore, we find that the counterexamples considered in the previous section are indeed adopted as the initial data of the IVP (1.1) in our framework.

In the case of the IVP (1.1) with $N(u, \bar{u}) = cu^2$ (or $c\bar{u}^2$), the function $u_0(x) = |x|^{-k}$ with $3/4 < k < 1$ is not in $H^s(\mathbb{R})$, $s > -3/4$ as is seen in the previous section. Noting that $\mathcal{F}_x(|x|^{-k}) = c_k|\xi|^{k-1}$, we find that $\langle\xi\rangle^{-3/4+\epsilon}|\xi|^{1-k}\mathcal{F}_x(|\xi|^{-k}) = c_k\langle\xi\rangle^{-3/4+\epsilon} \in L^2(\mathbb{R})$, where $0 < 1 - k < 1/4$. This implies $|x|^{-k} \in H^{s-s',s'}(\mathbb{R})$ with $s > -3/4$ and $s' \in (0, 1/4)$.

In the case of the IVP (1.1) with $N(u, \bar{u}) = cu\bar{u}$, the function

$$u_0(x) = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\cos(|t|\xi)}{|x-t|^{3/4+\epsilon}\langle\xi\rangle^{2\epsilon}} d\xi dt \tag{1.18}$$

is not in $H^s(\mathbb{R})$ but in $H^{s-s',s'}(\mathbb{R})$ with $s > -1/4$ and $s' \in (0, 1/4)$. Indeed, as is seen in the previous section

$$\widehat{u}_0(\xi) = c\frac{\langle\xi\rangle^{-2\epsilon}}{|\xi|^{1/4-\epsilon}}, \tag{1.19}$$

from which $u_0(x)$ is in $H^{s-s',s'}(\mathbb{R})$ with $s > -1/4$ and $s' \in (0, 1/4)$.

1.3. The related known results

Extensions of the results of Kenig, Ponce and Vega

Recently, T. Muramatu and S. Taoka proved in [18] the existence of the unique solution in Besov-type spaces, which are extensions of the results of Kenig, Ponce and Vega.

Let $B^{s,\#}_{2,q}(\mathbb{R})$ denote the completion of $\mathcal{S}(\mathbb{R})$ with the norm

$$\|u\|_{B^{s,\#}_{2,q}} = \|u_0^{\#}\|_{L^2} + \left\|2^{sj}\|u_j\|_{L^2}\right\|_{l^q(\mathbb{N})},$$

where $\widehat{u}_0^{\#}(\xi) = \varphi_0(|\xi|)(1+|\log|\xi||)^{-2}\widehat{u}(\xi), \widehat{u}_j(\xi) = \varphi_j(|\xi|)\widehat{u}(\xi)$, and $\varphi_j(z) \in C^\infty(\mathbb{R})$ $(j = 0, 1, \dots)$ have the following properties;

$$\begin{aligned}
&\varphi_j(z) = \varphi_j(-z) \geq 0 \\
&\varphi_j(z) = \varphi_1(2^{-j+1}z) \quad \textit{for } j \geq 1 \\
&\operatorname{supp} \varphi_0 \subset \{z \colon |z| < 2\}, \quad \operatorname{supp} \varphi_1 \subset \{z \colon 1 < |z| < 4\}, \\
&\sum_{j=0}^{\infty} \varphi_j(z) = 1.
\end{aligned}$$

Theorem 1.13 (Muramatu and Taoka [18], cf. [25])
(i) *For any* $u_0 \in B^{-3/4}_{2,1}(\mathbb{R})$, *there exist* $T = T(\|u_0\|_{B^{-3/4}_{2,1}}) > 0$ *and a unique solution* $u(x,t)$ *of the IVP* (1.1) *with* $N(u,\bar{u}) = cu^2$ *or* $c\bar{u}^2$.
(ii) *For any* $u_0 \in B^{-1/4,\#}_{2,1}(\mathbb{R})$, *there exist* $T = T(\|u_0\|_{B^{-1/4,\#}_{2,1}}) > 0$ *and a unique solution* $u(x,t)$ *of the IVP* (1.1) *with* $N(u,\bar{u}) = cu\bar{u}$.

Note that $B^{-1/4,\#}_{2,1}(\mathbb{R}) \supset B^{-1/4}_{2,1}(\mathbb{R}) \supset H^s(\mathbb{R})$ with $s > -1/4$.

Remark 1.14 The space $B^{-1/4,\#}_{2,1}(\mathbb{R})$ does not properly include our space $H^{s-s',s'}(\mathbb{R})$ with $s \in (-1/4, 0)$ and $s' \in (0, 1/4)$ (cf. Theorem 1.6). Indeed letting $u \in \mathcal{S}'(\mathbb{R})$ such that $\widehat{u} \in L^2(\mathbb{R})$ and putting $\widehat{f}(\xi) \equiv (1+|\xi|)^{-s+s'}|\xi|^{-s'}\widehat{u}(\xi)$, we clearly find that $f \in H^{s-s',s'}(\mathbb{R})$. On the contrary, for $|\xi| < 2$

$$|\widehat{f}^{\#}(\xi)| = \left|\varphi(|\xi|)\frac{(1+|\xi|)^{-s+s'}\widehat{u}(\xi)}{(1+|\log|\xi||)^2|\xi|^{s'}}\right| < \frac{|\widehat{u}(\xi)|}{|\xi|^{s'}},$$

which means the first term of right-hand side of the norm $\|\cdot\|_{B^{-1/4,\#}_{2,1}}$ is not finite. Hence $f \notin B^{-1/4,\#}_{2,1}(\mathbb{R})$.

Thus we find that the improvement by Muramatu and Taoka is different from ours.

Nonlinear terms and the asymptotic behavior as $t \to \infty$

As Kenig, Ponce and Vega already raised in [13], the Sobolev exponent s is affected by the kinds of the nonlinear terms in the studies of the well-posedness of the nonlinear Schrödinger equations in H^s (gauge-invariant or not, e.g. $|u|u$ or u^2, $\bar{u}^2$). Also, it is gradually made clear that the asymptotic behavior of the solutions as $t \to \infty$ vary with nonlinearities, which are accompanied by the well-posedness results.

Recently several studies show the relations between the nonlinearities and the asymptotic behavior. Hayashi, Naumkin, Shimomura and Tonegawa [9], Shimomura and Tonegawa [21], and Kawahara [11] deal with these problems in the scattering theory for the nonlinear Schrödinger equations with non-gauge-invariant nonlinearities.

This paper is organized as follows: Sections 2 and 3 treat the proofs of Proposition 1.7 and Theorem 1.6 respectively. Sections 4 and 5 contain the proofs of the nonlinear estimates of the nonlinearities $N(u, \bar{u}) = cu^2$ and $c\bar{u}^2$ respectively. Since the proofs of Theorems 1.8 and 1.10 are almost the same as that of Theorem 1.6, they are omitted.

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§2. Proof of Proposition 1.7

Let $s = -\rho \in (-1/4, 0)$ and $s' \in (0, 1/4)$. Putting

$$f(\xi,\tau) = \langle \tau - \xi^2\rangle^b \langle \xi\rangle^s |\xi|^{s'} \widetilde{F}(\xi,\tau)$$

$$\text{and} \quad g(\xi,\tau) = \langle \tau + \xi^2\rangle^b \langle \xi\rangle^s |\xi|^{s'} \widetilde{\overline{G}}(-\xi,-\tau)$$

for $F, G \in X^b_{s,s'}$, we have $\|f\|_{L^2_\xi L^2_\tau} = \|F\|_{X^b_{s,s'}}$ and $\|g\|_{L^2_\xi L^2_\tau} = \|\overline{G}\|_{X^b_{s,s'}}$. We note that $\widetilde{\overline{G}}(\xi,\tau) = \overline{\widetilde{G}}(-\xi,-\tau)$. Thus we write for $F, G \in X^{b'}_{s-s',s'}$

$$\begin{aligned}
\|F\overline{G}\|_{X^{b-1}_{s-s',s'}} &= \left\| \langle \tau - \xi^2\rangle^{b-1} \langle \xi\rangle^{s-s'} |\xi|^{s'} \widetilde{F\overline{G}}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} \\
&= c \left\| \frac{|\xi|^{s'}}{\langle \tau - \xi^2\rangle^{1-b} \langle \xi\rangle^{\rho+s'}} \right. \\
&\quad \left. \times \iint \frac{f(\xi-\xi_1, \tau-\tau_1)\langle \xi-\xi_1\rangle^{\rho+s'}}{\langle \tau-\tau_1-(\xi-\xi_1)^2\rangle^{b'} |\xi-\xi_1|^{s'}} \frac{\overline{g}(\xi_1,\tau_1)\langle \xi_1\rangle^{\rho+s'}}{\langle \tau_1+\xi_1^2\rangle^{b'} |\xi_1|^{s'}} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

Next we note the following algebraic relation;

$$(\tau_1 + \xi_1^2) + (\tau - \tau_1 - (\xi-\xi_1)^2) - (\tau - \xi^2) = 2\xi\xi_1,$$

and consequently

$$\max\{|\tau-\xi^2|,\ |\tau_1+\xi_1^2|,\ |\tau-\tau_1-(\xi-\xi_1)^2|\} \geq 2|\xi\xi_1|/3. \tag{2.1}$$

Lemma 2.1 ([14], [15]) *If* $1/2 < b < 1$, $b' > 1/2$, *there exists* $c > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-\alpha\rangle^{2b}\langle x-\beta\rangle^{2b}} \leq \frac{c}{\langle \alpha-\beta\rangle^{2b}}, \tag{2.2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x\rangle^{2b}|\sqrt{\alpha-x}|} \leq \frac{c}{\langle \alpha\rangle^{1/2}}, \tag{2.3}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-\alpha\rangle^{2(1-b)}\langle x-\beta\rangle^{2b}} \leq \frac{c}{\langle \alpha-\beta\rangle^{2(1-b)}}, \tag{2.4}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-\alpha\rangle^{2(1-b)}\langle x-\beta\rangle^{2b'}} \leq \frac{c}{\langle \alpha-\beta\rangle^{2(1-b)}} \tag{2.5}$$

and

$$\int_{|x|\leq\beta} \frac{dx}{\langle x\rangle^{2(1-b)}|\sqrt{\alpha-x}|} \leq c\frac{\langle \beta\rangle^{2(b-1/2)+\epsilon}}{\langle \alpha\rangle^{1/2}}. \tag{2.6}$$

Remark 2.2 The hypothesis in Lemma 2.1 that $b < 1$ is not necessary for (2.2) and (2.3).

Remark 2.3 If there are some positive constants $c > 0$ such that $A \leq cB$ and $B \leq cA$, we shall often write $A \sim B$ to denote these relations.

Remark 2.4 To establish Proposition 1.7, we shall use Lemmas 2.5-2.8 in the region $|\xi_1| \geq 1$ and $|\xi-\xi_1| \geq 1$. In this region, it follows that $\langle\xi_1\rangle\langle\xi-\xi_1\rangle \leq 4|\xi_1(\xi-\xi_1)|$. In particular,

$$\frac{\langle\xi_1\rangle^{2(\rho+s')}\langle\xi-\xi_1\rangle^{2(\rho+s')}}{|\xi_1|^{2s'}|\xi-\xi_1|^{2s'}} \leq c|\xi_1(\xi-\xi_1)|^{2\rho}.$$

Lemma 2.5 *If* $\rho = -s \in (0, 1/4)$ *and* $s' > 0$, *then there exist* $b > 1/2$ *and* $b' > 1/2$ *such that*

$$\sup_{|\xi|\geq 1}\sup_{\tau} \frac{1}{\langle\tau-\xi^2\rangle^{1-b}}\frac{|\xi|^{s'}}{\langle\xi\rangle^{\rho+s'}} \times \left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle\tau_1+\xi_1^2\rangle^{2b'}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}}d\xi_1 d\tau_1\right)^{1/2} < \infty, \tag{2.7}$$

where

$$A = \left\{\begin{array}{ll}(\xi_1,\tau_1)\in\mathbb{R}^2: & |\tau-\tau_1-(\xi-\xi_1)^2| \leq |\tau-\xi^2| \\ & |\tau_1+\xi_1^2| \leq |\tau-\xi^2| \\ & |\xi_1|\geq 1, \quad |\xi-\xi_1|\geq 1\end{array}\right\}.$$

Proof. From the definition of A,

$$|\tau - \xi^2 + 2\xi\xi_1| = |(\tau_1 + \xi_1^2) + (\tau - \tau_1 - (\xi - \xi_1)^2\,)| \leq 2|\tau - \xi^2| \tag{2.8}$$

holds. By (2.1)

$$|\xi_1(\xi - \xi_1)| < 2|\xi\xi_1|^2 < 9|\tau - \xi^2|^2/2 \tag{2.9}$$

holds in A with $|\xi| \geq 1$. If $4\rho < 1$, then there exists $b > 1/2$ such that

$$|\xi_1(\xi - \xi_1)|^{2\rho} \leq c|\tau - \xi^2|^{4\rho} \leq c|\tau - \xi^2|^{2(1-b)}. \tag{2.10}$$

It follows from (2.10) that

$$\begin{aligned} &\frac{1}{\langle \tau - \xi^2\rangle^{1-b}} \frac{\langle \xi\rangle^{s'}}{\langle \xi\rangle^{\rho+s'}} \left(\iint_A \frac{|\xi_1(\xi - \xi_1)|^{2\rho}}{\langle \tau_1 + \xi_1^2\rangle^{2b'} \langle \tau - \tau_1 - (\xi - \xi_1)^2\rangle^{2b'}} d\xi_1 d\tau_1 \right)^{1/2} \\ &\qquad \leq C \frac{\langle \tau - \xi^2\rangle^{1-b}}{\langle \tau - \xi^2\rangle^{1-b}} \left(\iint_A \frac{d\xi_1 d\tau_1}{\langle \tau_1 + \xi_1^2\rangle^{2b'} \langle \tau - \tau_1 - (\xi - \xi_1)^2\rangle^{2b'}} \right)^{1/2}. \end{aligned} \tag{2.11}$$

From (2.2) and (2.8), we get

$$\int_{\tau_1 \in A} \frac{d\tau_1}{\langle \tau_1 + \xi_1^2\rangle^{2b'} \langle \tau - \tau_1 - (\xi - \xi_1)^2\rangle^{2b'}} \leq c \frac{\psi\left((\tau - \xi^2 + 2\xi\xi_1)/2(\tau - \xi^2)\right)}{\langle \tau - \xi^2 + 2\xi\xi_1\rangle^{2b'}}, \tag{2.12}$$

where $\psi \in C_0^\infty(\mathbb{R})$ with supp $\psi \subset [-2, 2]$, $\psi \equiv 1$ on $[-1, 1]$. We shall often use this cut-off function ψ hereafter. Hence the right-hand side of (2.11) is dominated by

$$C \left(\int_{|\tau - \xi^2 + 2\xi\xi_1| \leq 2|\tau - \xi^2|} \frac{d\xi_1}{\langle \tau - \xi^2 + 2\xi\xi_1\rangle^{2b'}} \right)^{1/2}.$$

Changing variables $\eta_1 = \tau - \xi^2 + 2\xi\xi_1$ and $d\eta_1 = 2\xi d\xi_1$, we obtain

$$\begin{aligned} &C \left(\int_{|\tau - \xi^2 + 2\xi\xi_1| \leq 2|\tau - \xi^2|} \frac{d\xi_1}{\langle \tau - \xi^2 + 2\xi\xi_1\rangle^{2b'}} \right)^{1/2} \\ &= C \left(\frac{1}{|\xi|} \int_{|\eta_1| \leq 2|\tau - \xi^2|} \frac{d\eta_1}{\langle \eta_1\rangle^{2b'}} \right)^{1/2} \\ &\leq \frac{C}{|\xi|^{1/2}}. \end{aligned} \tag{2.13}$$

Noting that $|\xi| \geq 1$, we conclude Lemma 2.5. $\square$

Lemma 2.6 *If* $\rho = -s \in (0, 1/4)$ *and* $s' > 0$, *then there exist* $b \in (1/2, 1-2\rho)$ *and* $b' \in (1/2, b]$ *such that*

$$\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \times \left(\iint_B \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} < \infty, \tag{2.14}$$

where

$$B = \left\{ \begin{array}{c} (\xi, \tau) \in \mathbb{R}^2 : \quad |\tau - \tau_1 - (\xi - \xi_1)^2| \leq |\tau_1 + \xi_1^2| \\ |\tau - \xi^2| \leq |\tau_1 + \xi_1^2| \\ |\xi| \geq 1, \quad |\xi - \xi_1| \geq 1 \end{array} \right\}.$$

Proof. It follows from the definition of B that

$$|\tau_1 + \xi_1^2 - 2\xi\xi_1| = |(\tau - \xi^2) - (\tau - \tau_1 - (\xi - \xi_1)^2)| \leq 2|\tau_1 + \xi_1^2|. \tag{2.15}$$

By (2.1)

$$|\xi_1(\xi - \xi_1)| \leq 2|\xi\xi_1|^2 \leq 9|\tau_1 + \xi_1^2|^2/2 \tag{2.16}$$

holds in B with $|\xi_1| \geq 1$. From (2.16), we have

$$\begin{aligned} &\frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{\langle \xi \rangle^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'} \langle \tau - \xi^2 \rangle^{2(1-b)}} d\xi d\tau \right)^{1/2} \\ &\qquad \leq C \frac{\langle \tau_1 + \xi_1^2 \rangle^{2\rho}}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{d\xi d\tau}{\langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'} \langle \tau - \xi^2 \rangle^{2(1-b)}} \right)^{1/2}. \end{aligned} \tag{2.17}$$

From (2.5) and (2.15), it follows that

$$\int_{\tau \in B} \frac{d\tau}{\langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'} \langle \tau - \xi^2 \rangle^{2(1-b)}} \leq c \frac{\psi(\tau_1 + \xi_1^2 - 2\xi\xi_1/2(\tau_1 + \xi_1^2))}{\langle \tau_1 + \xi_1^2 - 2\xi\xi_1 \rangle^{2(1-b)}}. \tag{2.18}$$

Hence the right-hand side of (2.17) is bounded by

$$C \langle \tau_1 + \xi_1^2 \rangle^{2\rho - b'} \left(\int_{|\tau_1 + \xi_1^2 - 2\xi\xi_1| \leq 2|\tau_1 + \xi_1^2|} \frac{d\xi}{\langle \tau_1 + \xi_1^2 - 2\xi\xi_1 \rangle^{2(1-b)}} \right)^{1/2}.$$

Changing variables $\eta = \tau_1 + \xi_1^2 - 2\xi\xi_1$ and $d\eta = -2\xi_1 d\xi$, we obtain

$$\begin{aligned} &C\langle \tau_1 + \xi_1^2 \rangle^{2\rho - b'} \left(\int_{|\eta| \le 2|\tau_1 + \xi_1^2|} \frac{d\eta}{\langle \eta \rangle^{2(1-b)} |\xi_1|} \right)^{1/2} \\ &\le \frac{C}{|\xi_1|^{1/2}} \langle \tau_1 + \xi_1^2 \rangle^{2\rho - b' + b - 1/2 + \epsilon}. \end{aligned} \tag{2.19}$$

Since

$$\sup_{|\xi_1| \ge 1} \sup_{\tau_1} \frac{\langle \tau_1 + \xi_1^2 \rangle^{2\rho + b - b' - 1/2 + \epsilon}}{|\xi_1|^{1/2}} < \infty, \tag{2.20}$$

we establish our statement. □

Lemma 2.7 *If $\rho = -s \in (0, 1/2)$ and $s' > 0$, then there exist $b \in (1/2, 1-\rho)$ and $b' \in (1/2, b]$ such that*

$$\begin{aligned} &\sup_{|\xi_1| \ge 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \\ &\quad \times \left(\iint_D \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho + s')}} \frac{|\xi_1(\xi - \xi_1)|^{2\rho}}{\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b'} \langle \tau - \xi^2 \rangle^{2(1-b)}} d\xi d\tau \right)^{1/2} < \infty, \end{aligned} \tag{2.21}$$

where

$$D = \left\{ \begin{array}{ll} (\xi, \tau) \in \mathbb{R}^2 : & |\tau_1 + \xi_1^2| \le |\tau - \tau_1 - (\xi - \xi_1)^2| \\ & |\tau - \xi^2| \le |\tau - \tau_1 - (\xi - \xi_1)^2| \\ & |\xi| \ge 1, \quad |\xi - \xi_1| \ge 1 \end{array} \right\}.$$

Proof. We change variables $\tau_1' = \tau - \tau_1$ and $\xi_1' = \xi - \xi_1$. It suffices to show that

$$\begin{aligned} &\sup_{|\xi_1'| \ge 1} \sup_{\tau_1} \frac{1}{\langle \tau_1' - (\xi_1')^2 \rangle^{b'}} \\ &\quad \times \left(\iint_{D'} \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho + s')}} \frac{|\xi_1'(\xi - \xi_1')|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1' + (\xi - \xi_1')^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} < \infty, \end{aligned} \tag{2.22}$$

where

$$D' = \left\{ \begin{array}{ll} (\xi, \tau) \in \mathbb{R}^2 : & |\tau - \tau_1' + (\xi - \xi_1')^2| \le |\tau_1' - (\xi_1')^2| \\ & |\tau - \xi^2| \le |\tau_1' - (\xi_1')^2| \\ & |\xi| \ge 1, \quad |\xi - \xi_1| \ge 1 \end{array} \right\}.$$

For simplicity, we put $\tau_1 = \tau_1'$ and $\xi_1 = \xi_1'$.
It follows from the definition of D' that

$$|\tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1)| = |(\tau - \xi^2) - (\tau - \tau_1 + (\xi - \xi_1)^2)| \le 2|\tau_1 - \xi_1^2|. \quad (2.23)$$

And it follows from (2.1) that

$$|\xi(\xi - \xi_1)| \le c|\tau_1 - \xi_1^2|. \quad (2.24)$$

The left-hand side of (2.22) is bounded by

$$C \sup_{|\xi_1| \ge 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\iint_{D'} \frac{|\xi_1(\xi - \xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2}. \quad (2.25)$$

From (2.5) and (2.23), it follows that

$$\begin{aligned} &\int_{\tau \in D'} \frac{d\tau}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b'}} \\ &\qquad \le c \frac{\psi\left((\tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1)/2(\tau_1 - \xi_1^2) \right)}{\langle \tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1) \rangle^{2(1-b)}}. \end{aligned} \quad (2.26)$$

For a domain E, we define

$$I(E) = \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\int_E \frac{|\xi_1(\xi - \xi_1)|^{2\rho} d\xi}{\langle \tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1) \rangle^{2(1-b)}} \right)^{1/2}. \quad (2.27)$$

And we put

$$\not{D} = \left\{ \begin{matrix} \xi \in D' : & |\tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1)| \le 2|\tau_1 - \xi_1^2| \\ & |\xi| \ge 1, \quad |\xi - \xi_1| \ge 1 \end{matrix} \right\}$$

and divide $\not{D}$ into three regions,

$$\begin{aligned} \not{D}_1 &= \{\, \xi \in \not{D} \colon |\xi_1| \le 100|\xi| \,\}, \\ \not{D}_2 &= \{\, \xi \in \not{D} \colon |\xi_1| \ge 100|\xi|,\ |\xi_1| \le 500|\tau_1 - \xi_1^2| \,\}, \\ \not{D}_3 &= \{\, \xi \in \not{D} \colon |\xi_1| \ge 100|\xi|,\ 500|\tau_1 - \xi_1^2| \le |\xi_1| \,\}. \end{aligned}$$

Estimate in $\not{D}_1$.
From (2.24), $|\xi_1(\xi - \xi_1)| \le c|\xi(\xi - \xi_1)| \le c|\tau_1 - \xi_1^2|$ holds in $\not{D}_1$. We change variables

$$\eta = \tau_1 - \xi_1^2 - 2\xi(\xi - \xi_1), \quad d\eta = 2(\xi_1 - 2\xi)d\xi = 2\sqrt{2\tau_1 - 2\eta - \xi_1^2}\, d\xi. \quad (2.28)$$

With the aid of (2.6), we obtain

$$\begin{aligned}I(\not{D}_1) &\leq \langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \left(\int_{|\eta| \leq 2|\tau_1 - \xi_1^2|} \frac{d\eta}{\langle \eta \rangle^{2(1-b)} \sqrt{|2\tau_1 - 2\eta - \xi_1^2|}} \right)^{1/2} \\ &\leq C \langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \frac{\langle \tau_1 - \xi_1^2 \rangle^{b-1/2+\epsilon}}{\langle 2\tau_1 - \xi_1^2 \rangle^{1/4}} \\ &< \infty. \end{aligned} \tag{2.29}$$

Estimate in $\not{D}_2$.
With $|\xi| \geq 1$ and (2.24),

$$\begin{aligned}|\xi_1(\xi - \xi_1)|^{2\rho} &\leq |\xi_1|^{2\rho} |\xi(\xi - \xi_1)|^{2\rho} \\ &\leq c|\xi_1|^{2\rho} |\tau_1 - \xi_1|^{2\rho} \end{aligned} \tag{2.30}$$

holds in $\not{D}_2 \cup \not{D}_3$. In the region $\not{D}_2$, $d\eta = c\xi_1 d\xi$ holds since $|\xi_1 - 2\xi| \sim |\xi_1|$. Under the change of variable (2.28), we get

$$\begin{aligned}I(\not{D}_2) &\leq C \frac{|\xi_1|^{\rho} \langle \tau_1 - \xi_1^2 \rangle^{\rho}}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\int_{|\eta| \leq 2|\tau_1 - \xi_1^2|} \frac{d\eta}{|\xi_1|^{1/2} \langle \eta \rangle^{2(1-b)}} \right)^{1/2} \\ &\leq C |\xi_1|^{\rho - 1/2} \langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \langle \tau_1 - \xi_1^2 \rangle^{b - 1/2 + \epsilon}. \end{aligned} \tag{2.31}$$

With $\rho + b - b' - 1/2 + \epsilon < 0$, we obtain

$$I(\not{D}_2) \leq C |\xi_1|^{\rho - 1/2 + \rho + b - b' - 1/2 + \epsilon} < \infty. \tag{2.32}$$

Estimate in $\not{D}_3$.
We can also estimate $I(\not{D}_3)$ as in $\not{D}_2$:

$$\begin{aligned}I(\not{D}_3) &\leq C \frac{|\xi_1|^{\rho - 1/2} \langle \tau_1 - \xi_1^2 \rangle^{\rho}}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\int_{|\eta| \leq 2|\tau_1 - \xi_1^2|} \frac{d\eta}{\langle \eta \rangle^{2(1-b)}} \right)^{1/2} \\ &\leq C |\xi_1|^{\rho - 1/2} \langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \langle \tau_1 - \xi_1^2 \rangle^{b - 1/2 + \epsilon}. \end{aligned} \tag{2.33}$$

With $\rho - 1/2 < 0$, we obtain

$$I(\not{D}_3) \leq C \langle \tau_1 - \xi_1 \rangle^{\rho - 1/2 + \rho + b - b' - 1/2 + \epsilon} < \infty. \tag{2.34}$$

Therefore it suffices to collect (2.29), (2.32) and (2.34) to conclude Lemma 2.7. □

Lemma 2.8 *If $\rho = -s \in (0, 1/4)$ and $s' > 0$, then there exist $b \in (1/2, 1-2\rho)$ and $b' \in (1/2, b]$ such that*

$$\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \times \left(\iint_{|\xi|\leq 1} \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} < \infty. \tag{2.35}$$

Proof. In this case, from $|\xi| \leq 1$ and $|\xi_1| \geq 1$, we have

$$|\xi_1(\xi - \xi_1)| \leq |\xi_1| + |\xi_1|^2 \leq 2|\xi_1|^2.$$

The left-hand side of (2.35) is bounded by

$$\begin{aligned} &\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_{|\xi|\leq 1} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} \\ &\leq C \sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{|\xi_1|^{2\rho}}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_{|\xi|\leq 1} \frac{d\xi d\tau}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} \right)^{1/2}. \end{aligned} \tag{2.36}$$

Using (2.5), we find that

$$\iint_{|\xi|\leq 1} \frac{d\tau d\xi}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} \leq C \int_{|\xi|\leq 1} \frac{d\xi}{\langle \tau_1 + \xi_1^2 - 2\xi\xi_1 \rangle^{2(1-b)}}. \tag{2.37}$$

Changing variables $\eta = \tau_1 + \xi_1^2 - 2\xi\xi_1$ and $d\eta = -2\xi_1 d\xi$, we get

$$\begin{aligned} &\left(\int_{|\xi|\leq 1} \frac{d\xi}{\langle \tau_1 + \xi_1^2 - 2\xi\xi_1 \rangle^{2(1-b)}} \right)^{1/2} \\ &\leq C \frac{1}{|\xi_1|^{1/2}} \left(\int_{|\eta|\leq |\tau_1+\xi_1^2|+2|\xi_1|} \frac{d\eta}{\langle \eta \rangle^{2(1-b)}} \right)^{1/2} \\ &\leq C|\xi_1|^{-1/2} [\langle \tau_1 + \xi_1^2 \rangle^{b-1/2+\epsilon} + |\xi_1|^{b-1/2+\epsilon}]. \end{aligned} \tag{2.38}$$

Hence the left-hand side of in (2.36) is dominated by

$$\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{|\xi_1|^{2\rho-1/2}}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} [\langle \tau_1 + \xi_1^2 \rangle^{b-1/2+\epsilon} + |\xi_1|^{b-1/2+\epsilon}], \tag{2.39}$$

which yields the result. □

Lemma 2.9 *Let $\rho = -s > 0$ and $0 < s' < 1/4$. Then there exist $b > 1/2$ and $b' > 1/2$ such that*

$$\sup_{\xi} \sup_{\tau} \frac{|\xi|^{s'}}{\langle \tau - \xi^2 \rangle^{1-b} \langle \xi \rangle^{\rho+s'}} \times \left(\iint_E \frac{\langle \xi_1 \rangle^{2(\rho+s')}}{\langle \tau_1 + \xi_1^2 \rangle^{2b'} |\xi_1|^{2s'}} \frac{\langle \xi - \xi_1 \rangle^{2(\rho+s')}}{\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b'} |\xi - \xi_1|^{2s'}} d\xi_1 d\tau_1 \right)^{1/2} < \infty, \tag{2.40}$$

where $E = \{(\xi_1, \tau_1) \in \mathbb{R}^2 \colon |\xi_1| < 1$ or $|\xi - \xi_1| < 1\}$.

Proof. If $|\xi_1| < 1$ or $|\xi - \xi_1| < 1$, then we find that $\langle \xi_1 \rangle \langle \xi - \xi_1 \rangle < 4 \langle \xi \rangle$. Using this inequality, we obtain

$$\begin{aligned} \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{\langle \xi_1 \rangle^{2(\rho+s')}}{|\xi_1|^{2s'}} \frac{\langle \xi - \xi_1 \rangle^{2(\rho+s')}}{|\xi - \xi_1|^{2s'}} &< \frac{1}{\langle \xi \rangle^{2\rho}} \frac{\langle \xi_1 \rangle^{2(\rho+s')}}{|\xi_1|^{2s'}} \frac{\langle \xi - \xi_1 \rangle^{2(\rho+s')}}{|\xi - \xi_1|^{2s'}} \\ &< C \left(1 + \frac{1}{|\xi_1|} \right)^{2s'} \left(1 + \frac{1}{|\xi - \xi_1|} \right)^{2s'}. \end{aligned} \tag{2.41}$$

The left-hand side of (2.40) is bounded by

$$\begin{aligned} &\sup_{\xi,\tau} \frac{C}{\langle \tau - \xi^2 \rangle^{1-b}} \left(\iint_E \left(1 + \frac{1}{|\xi_1|} \right)^{2s'} \left(1 + \frac{1}{|\xi - \xi_1|} \right)^{2s'} \right. \\ &\qquad \left. \times \frac{d\xi_1 d\tau_1}{\langle \tau_1 + \xi_1^2 \rangle^{2b'} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b'}} \right)^{1/2} \\ &< \sup_{\xi,\tau} \frac{C}{\langle \tau - \xi^2 \rangle^{1-b}} \left(\int_{\{|\xi_1|<1 \text{ or } |\xi - \xi_1| < 1\}} \left(1 + \frac{1}{|\xi_1|} \right)^{2s'} \left(1 + \frac{1}{|\xi - \xi_1|} \right)^{2s'} \right. \\ &\qquad \left. \times \frac{d\xi_1}{\langle \tau - \xi^2 + 2\xi\xi_1 \rangle^{2b'}} \right)^{1/2} < \infty, \end{aligned} \tag{2.42}$$

from which the conclusion follows when $0 < s' < 1/4$, $b, b' > 1/2$. □

Proof of Proposition 1.7. By duality argument, it suffices to show that for any $v \in X^{1-b}_{s'-s,s'}$ with $\|v\|_{X^{1-b}_{s'-s,s'}} \le 1$,

$$|I| = |\langle F\overline{G}, v \rangle| \le C \|F\|_{X^{b'}_{s-s',s'}} \|G\|_{X^{b'}_{s-s',s'}} \|v\|_{X^{1-b}_{s'-s,s'}}. \tag{2.43}$$

Putting

$$\begin{aligned} f(\xi,\tau) &= \langle \tau-\xi^2\rangle^{b'}\langle\xi\rangle^{s-s'}|\xi|^{s'}\widetilde{F}(\xi,\tau),\\ g(\xi,\tau) &= \langle \tau+\xi^2\rangle^{b'}\langle\xi\rangle^{s-s'}|\xi|^{s'}\overline{\widetilde{G}(-\xi,-\tau)}\\ \text{and}\quad h(\xi,\tau) &= \langle \tau-\xi^2\rangle^{1-b}\langle\xi\rangle^{s'-s}|\xi|^{-s'}\widetilde{v}(\xi,\tau), \end{aligned}$$

we find that our aim is to show

$$|I| \le C\|f\|_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau}\|h\|_{L^2_\xi L^2_\tau}. \tag{2.44}$$

And we can rewrite I as follows:

$$\begin{aligned} I &= \int_{\mathbb{R}^2} \widetilde{F\overline{G}}(\xi,\tau)\overline{\widetilde{v}(\xi,\tau)}d\xi d\tau\\ &= \int_{\mathbb{R}^4} \overline{\widetilde{v}(\xi,\tau)}\,\widetilde{\overline{G}}(\xi_1,\tau_1)\widetilde{F}(\xi-\xi_1,\tau-\tau_1)d\xi d\tau d\xi_1 d\tau_1\\ &= \int_{\mathbb{R}^4} \overline{\widetilde{v}(\xi,\tau)}\,\overline{\widetilde{G}(-\xi_1,-\tau_1)}\widetilde{F}(\xi-\xi_1,\tau-\tau_1)d\xi d\tau d\xi_1 d\tau_1\\ &= \int_{\mathbb{R}^4} \frac{\overline{h(\xi,\tau)}|\xi|^{s'}}{\langle\tau-\xi^2\rangle^{1-b}\langle\xi\rangle^{\rho+s'}}\frac{g(\xi_1,\tau_1)\langle\xi_1\rangle^{\rho+s'}}{\langle\tau_1+\xi_1^2\rangle^{b'}|\xi_1|^{s'}}\\ &\qquad\quad \times \frac{f(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{\rho+s'}}{\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{b'}|\xi-\xi_1|^{s'}}d\xi d\tau d\xi_1 d\tau_1. \end{aligned} \tag{2.45}$$

We divide $\mathbb{R}^4$ into the following two integral regions:

$$\begin{aligned} \widetilde{D}_1 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4\colon\ |\xi_1|\ge 1 \text{ and } |\xi-\xi_1|\ge 1\ \},\\ \widetilde{D}_2 &= \{(\xi,\tau,\xi_1,\tau_1)\in\mathbb{R}^4\colon\ |\xi_1|< 1 \text{ or } |\xi-\xi_1|< 1\ \}. \end{aligned}$$

Moreover we split $\widetilde{D}_1$ into four regions:

$$\begin{aligned} \widetilde{D}_{11} &= \{(\xi,\tau,\xi_1,\tau_1)\in\widetilde{D}_1\colon\ |\xi|>1,\ |\sigma|=\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\ \},\\ \widetilde{D}_{12} &= \{(\xi,\tau,\xi_1,\tau_1)\in\widetilde{D}_1\colon\ |\xi|>1,\ |\sigma_1|=\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\ \},\\ \widetilde{D}_{13} &= \{(\xi,\tau,\xi_1,\tau_1)\in\widetilde{D}_1\colon\ |\xi|>1,\ |\sigma_2|=\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\ \},\\ \widetilde{D}_{14} &= \{(\xi,\tau,\xi_1,\tau_1)\in\widetilde{D}_1\colon\ |\xi|\le 1\ \}, \end{aligned}$$

where $\quad\sigma=\tau-\xi^2,\quad \sigma_1=\tau_1+\xi_1^2,\quad \sigma_2=\tau-\tau_1-(\xi-\xi_1)^2.$

For these integral regions, the integral I is divided into each part;

$$I = I_{\widetilde{D}_{11}} + I_{\widetilde{D}_{12}} + I_{\widetilde{D}_{13}} + I_{\widetilde{D}_{14}} + I_{\widetilde{D}_2},$$

where

$$I_{\widetilde{D}} = \int_{\widetilde{D}} \frac{\overline{h(\xi,\tau)}|\xi|^{s'}}{\langle \tau-\xi^2\rangle^{1-b}\langle\xi\rangle^{\rho+s'}} \frac{g(\xi_1,\tau_1)\langle\xi_1\rangle^{\rho+s'}}{\langle\tau_1+\xi_1^2\rangle^{b'}|\xi_1|^{s'}} \times \frac{f(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{\rho+s'}}{\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{b'}|\xi-\xi_1|^{s'}} d\xi d\tau d\xi_1 d\tau_1.$$

Each $I_{\widetilde{D}}$ is estimated according to the following two cases:

Case I

This case applies to the integral regions $\widetilde{D} = \widetilde{D}_{12} \cup \widetilde{D}_{13} \cup \widetilde{D}_{14}$. By using Schwarz inequality with respect to (ξ,τ), $|I_{\widetilde{D}}|$ is dominated by

$$\begin{aligned}&\int \frac{\langle\xi_1\rangle^{\rho+s'} g(\xi_1,\tau_1)}{\langle\tau_1+\xi_1^2\rangle^{b'}|\xi_1|^{s'}}\\ &\times\left(\int_{\widetilde{D}} \frac{|\xi|^{2s'}}{\langle\tau-\xi^2\rangle^{2(1-b)}\langle\xi\rangle^{2(\rho+s')}} \frac{\langle\xi-\xi_1\rangle^{2(\rho+s')}}{\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}|\xi-\xi_1|^{2s'}} d\xi d\tau\right)^{1/2}\\ &\times\left(\int_{\widetilde{D}} |h(\xi,\tau)|^2|f(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau\right)^{1/2} d\xi_1 d\tau_1. \qquad (2.46)\end{aligned}$$

With the aid of Lemmas 2.6, 2.7, 2.8 and Schwarz inequality, (2.46) is dominated by

$$\begin{aligned}&C\int_{\mathbb{R}^2} g(\xi_1,\tau_1)\left(\int_{\mathbb{R}^2} |h(\xi,\tau)|^2|f(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau\right)^{1/2} d\xi_1 d\tau_1\\ &\le C\left(\int_{\mathbb{R}^2} |g(\xi_1,\tau_1)|^2 d\xi_1 d\tau_1\right)^{1/2}\\ &\qquad\times\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} |h(\xi,\tau)|^2|f(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau\right) d\xi_1 d\tau_1\right)^{1/2}\\ &= C\|f\|_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau}\|h\|_{L^2_\xi L^2_\tau}. \qquad (2.47)\end{aligned}$$

Case II

This case applies to the integral regions $\widetilde{D} = \widetilde{D}_{11} \cup \widetilde{D}_2$. By using Schwarz inequality with respect to (ξ_1,τ_1), $|I_{\widetilde{D}}|$ is dominated by

$$\begin{aligned}&\int \frac{|\xi|^{s'}\overline{h}(\xi,\tau)}{\langle\tau-\xi^2\rangle^{1-b}\langle\xi\rangle^{\rho+s'}}\\ &\times\left(\int_{\widetilde{D}} \frac{\langle\xi_1\rangle^{2(\rho+s')}}{\langle\tau_1+\xi_1^2\rangle^{2b'}|\xi_1|^{2s'}} \frac{\langle\xi-\xi_1\rangle^{2(\rho+s')}}{\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}|\xi-\xi_1|^{2s'}} d\xi_1 d\tau_1\right)^{1/2}\\ &\times\left(\int_{\widetilde{D}} |g(\xi_1,\tau_1)|^2|f(\xi-\xi_1,\tau-\tau_1)|^2 d\xi_1 d\tau_1\right)^{1/2} d\xi d\tau. \qquad (2.48)\end{aligned}$$

By virtue of Lemmas 2.5, 2.9 and Schwarz inequality, (2.48) is dominated by

$$C\int_{\mathbb{R}^2}\overline{h}(\xi,\tau)\left(\int_{\mathbb{R}^2}|g(\xi_1,\tau_1)|^2|f(\xi-\xi_1,\tau-\tau_1)|^2d\xi_1d\tau_1\right)^{1/2}d\xi d\tau$$
$$\le C\|f\|_{L^2_\xi L^2_\tau}\|g\|_{L^2_\xi L^2_\tau}\|h\|_{L^2_\xi L^2_\tau}. \tag{2.49}$$

Summing up, we establish our statement. □

§3. Proof of Theorem 1.6

Lemma 3.1 *If* $s, s' \in \mathbb{R}$ *and* $b \in (1/2, 1]$, *then for* $\delta \in (0,1]$

$$\left\|\psi(\delta^{-1}t)e^{it\partial_x^2}u_0\right\|_{X^b_{s,s'}} \le c\delta^{(1-2b)/2}\|u_0\|_{H^{s,s'}}, \tag{3.1}$$

$$\left\|\psi(\delta^{-1}t)F\right\|_{X^b_{s,s'}} \le c\delta^{(1-2b)/2}\|F\|_{X^b_{s,s'}}, \tag{3.2}$$

$$\left\|\psi(\delta^{-1}t)\int_0^t e^{i(t-t')\partial_x^2}F(t')dt'\right\|_{X^b_{s,s'}} \le c\delta^{(1-2b)/2}\|F\|_{X^{b-1}_{s,s'}} \tag{3.3}$$

and

$$\left\|\psi(\delta^{-1}t)\int_0^t e^{i(t-t')\partial_x^2}F(t')dt'\right\|_{H^{s,s'}} \le c\delta^{(1-2b)/2}\|F\|_{X^{b-1}_{s,s'}}. \tag{3.4}$$

Here $\psi \in C_0^\infty(\mathbb{R})$ *with* $\psi \equiv 1$ *on* $[-1,1]$ *and* $\operatorname{supp}\psi \subseteq (-2,2)$.

Proof. Replacing $\langle\xi\rangle^s$ and the unitary group $\{W(t)\}_{-\infty}^{\infty}$ associated to the linearized KdV equation by $\langle\xi\rangle^s|\xi|^{s'}$ and $\{e^{it\partial_x^2}\}_{-\infty}^{\infty}$ respectively, it suffices to follow the proofs of Lemmas 3.1-3.3 of [12]. Therefore the proofs are omitted. □

3.1. Existence

For $u_0 \in H^{s-s',s'}(\mathbb{R})$ with $s \in (-1/4, 0)$ and $s' \in (0, 1/4)$, and for $b \in (1/2, 1)$, we define

$$\mathcal{B}_M = \{u \in X^b_{s-s',s'} : \|u\|_{X^b_{s-s',s'}} \le M\}, \tag{3.5}$$

where $M = 2C_0\|u_0\|_{H^{s-s',s'}}$. For $\omega \in \mathcal{B}_M$, we define the map

$$T_{u_0}(\omega) = T(\omega) = \psi(t)e^{it\partial_x^2}u_0 + c\psi(t)\int_0^t e^{i(t-t')\partial_x^2}|\psi_\delta|^2\omega\overline{\omega}(t')dt', \tag{3.6}$$

where $\psi_\delta(t) = \psi(\delta^{-1}t)$. We shall show that T_{u_0} is a contraction map on $\mathcal{B}_M$.

Following the similar argument in [14], we get

$$\|\psi(\rho^{-1}\cdot)u\|_{X^{b'}_{s-s',s'}} \leq c\rho^{(b-b')/8}\|u\|_{X^{b}_{s-s',s'}} \tag{3.7}$$

for $1/2 < b' \leq b$.

With the aid of (3.1), (3.3) of Lemma 3.1 with $\delta = 1$, Proposition 1.7 and (3.7),

$$\begin{aligned}
\|T(\omega)\|_{X^{b}_{s-s',s'}} &\leq \left\|\psi(t)e^{it\partial_x^2}u_0\right\|_{X^{b}_{s-s',s'}} \\
&\quad + c\left\|\psi(t)\int_0^t e^{i(t-t')\partial_x^2}|\psi_\delta(t')|^2\omega\overline{\omega}(t')dt'\right\|_{X^{b}_{s-s',s'}} \\
&\leq C_0\|u_0\|_{H^{s-s',s'}} + c\|(\psi_\delta\omega)\overline{(\psi_\delta\omega)}\|_{X^{b-1}_{s-s',s'}} \\
&\leq C_0\|u_0\|_{H^{s-s',s'}} + c\|\psi_\delta\omega\|^2_{X^{b'}_{s-s',s'}} \\
&\leq C_0\|u_0\|_{H^{s-s',s'}} + C_1\delta^\mu\|\omega\|^2_{X^{b}_{s-s',s'}} \\
&\leq M/2 + C_1\delta^\mu M^2,
\end{aligned} \tag{3.8}$$

where $\mu = (b-b')/4$. Choosing $\delta^\mu \leq 1/4C_1M$, we obtain

$$\|T(\omega)\|_{X^{b}_{s-s',s'}} \leq 3M/4 < M, \tag{3.9}$$

which means $T(\omega) \in \mathcal{B}_M$.

Similar calculation yields

$$\begin{aligned}
&\|T(u_1) - T(u_2)\|_{X^{b}_{s-s',s'}} \\
&\quad\leq c\left\|\psi(t)\int_0^t e^{i(t-t')\partial_x^2}|\psi_\delta(t')|^2(u_1\overline{u_1}(t') - u_2\overline{u_2}(t'))dt'\right\|_{X^{b}_{s-s',s'}} \\
&\quad\leq c\||\psi_\delta|^2(u_1\overline{u_1} - u_2\overline{u_2})\|_{X^{b-1}_{s-s',s'}} \\
&\quad\leq c\left(\|\overline{\psi_\delta u_1}\psi_\delta(u_1-u_2)\|_{X^{b-1}_{s-s',s'}} + \|\psi_\delta u_1\overline{\psi_\delta(u_1-u_2)}\|_{X^{b-1}_{s-s',s'}}\right) \\
&\quad\leq c\left(\|\psi_\delta u_1\|_{X^{b'}_{s-s',s'}} + \|\psi_\delta u_2\|_{X^{b'}_{s-s',s'}}\right)\|\psi_\delta(u_1-u_2)\|_{X^{b'}_{s-s',s'}} \\
&\quad\leq C_1\delta^\mu\left(\|u_1\|_{X^{b}_{s-s',s'}} + \|u_2\|_{X^{b}_{s-s',s'}}\right)\|u_1-u_2\|_{X^{b}_{s-s',s'}} \\
&\quad\leq 2C_1\delta^\mu M\|u_1-u_2\|_{X^{b}_{s-s',s'}} \\
&\quad\leq \frac{1}{2}\|u_1-u_2\|_{X^{b}_{s-s',s'}}.
\end{aligned} \tag{3.10}$$

Hence T_{u_0} is a contraction, thus there exists a unique solution $u(t)$ in $\mathcal{B}_M$ for $T < \delta$ such that

$$u(t) = \psi(t)\left[e^{it\partial_x^2}u_0 + c\int_0^t e^{i(t-t')\partial_x^2}\psi_\delta(t')u\overline{u}(t')dt'\right]. \tag{3.11}$$

Therefore $u(t)$ solves the integral equation associated to the IVP (1.1) with $N(u,\bar{u}) = cu\bar{u}$ in the time interval $[-T,T]$.

3.2. Uniqueness

We define

$$\|u\|_{X_T} = \inf_w \{\, \|w\|_{X^b_{s-s',s'}} : w \in X^b_{s-s',s'} \quad \text{such that} \\ u(t) = w(t) \quad t \in [0,T] \quad \text{in } H^{s-s',s'}\}. \tag{3.12}$$

Let u_1 be the solution obtained above. And let u_2 be a solution to the integral equation with the same initial data u_0. We assume for some $M > 0$

$$\|u_1\|_{X^b_{s-s',s'}},\ \|\psi u_2\|_{X^b_{s-s',s'}} \leq M. \tag{3.13}$$

We may assume $M > 1$, $T < 1$. For some $T^* < T$, we get

$$\psi u_2(t) = \psi(t)e^{it\partial_x^2}u_0 + c\psi(t)\int_0^t e^{i(t-t')\partial_x^2}|\psi_{T^*}|^2|\psi|^2 u_2\overline{u_2}(t')dt' \tag{3.14}$$

for $t \in [0,T^*]$. From the definition of the norm, it follows that for any $\epsilon > 0$, there exists $w \in X^b_{s-s',s'}$ such that for $t \in [0,T^*]$,

$$w(t) = u_1(t) - \psi(t)u_2(t) \tag{3.15}$$

and

$$\|w\|_{X^b_{s-s',s'}} \leq \|u_1 - \psi u_2\|_{X_T} + \epsilon. \tag{3.16}$$

We define for $t \in \mathbb{R}$

$$\omega(t) = c\psi(t)\int_0^t e^{i(t-t')\partial_x^2}|\psi_{T^*}(t')|^2(u_1\overline{w}(t') + w\overline{\psi u_2}(t'))dt'. \tag{3.17}$$

For $t \in [0,T^*]$, we get

$$\omega(t) = w(t) = u_1(t) - \psi(t)u_2(t). \tag{3.18}$$

Hence

$$\|u_1 - \psi u_2\|_{X_{T^*}} \le \|\omega\|_{X^b_{s-s',s'}}. \tag{3.19}$$

From the similar calculation as in Section 3.1, it follows that

$$\begin{aligned}\|u_1 - \psi u_2\|_{X_{T^*}} &\le \|\omega\|_{X^b_{s-s',s'}} \\ &\le c\||\psi_{T^*}|^2(u_1\overline{w}(t') + w\overline{\psi u_2})\|_{X^{b-1}_{s-s',s'}} \\ &\le C_1(T^*)^\mu \left(\|u_1\|_{X^b_{s-s',s'}} + \|\psi u_2\|_{X^b_{s-s',s'}}\right)\|w\|_{X^b_{s-s',s'}} \\ &\le 2C_1(T^*)^\mu M\|w\|_{X^b_{s-s',s'}},\end{aligned} \tag{3.20}$$

where $\mu = (b-b')/4$. If $(T^*)^\mu \le 1/4C_1M$, for any $\epsilon > 0$ we have

$$\begin{aligned}\|u_1 - \psi u_2\|_{X_{T^*}} &\le \frac{1}{2}\|w\|_{X^b_{s-s',s'}} \\ &\le \frac{1}{2}\left(\|u_1 - \psi u_2\|_{X_{T^*}} + \epsilon\right).\end{aligned} \tag{3.21}$$

Therefore

$$\|u_1 - \psi u_2\|_{X_{T^*}} \le \epsilon, \tag{3.22}$$

which implies $u_1 = u_2$ on $[0, T^*]$. Repeating this procedure, we obtain the uniqueness result for any existence interval.

Remark 3.2 In [12], [14] and [15], Kenig, Ponce and Vega do not carry out the proof of the uniqueness result completely. We referred to the proof by Bekiranov, Ogawa and Ponce [2].

3.3. Other properties

We shall prove the persistence property;

$$u(t) \in C([-T,T]; H^{s-s',s'}(\mathbb{R})). \tag{3.23}$$

Using the integral equation (3.11), (3.4), Proposition 1.7 and (3.2), for $0 \le \tilde{t} < t \le 1$ and $t - \tilde{t} \le \Delta t$, we obtain

$$\begin{aligned}\|u(t) - u(\tilde{t})\|_{H^{s-s',s'}} &\le \|e^{i(t-\tilde{t})\partial_x^2}u(\tilde{t}) - u(\tilde{t})\|_{H^{s-s',s'}} \\ &\quad + c\left\|\int_{\tilde{t}}^t e^{i(t-t')\partial_x^2}\left|\psi\left(\frac{t'-\tilde{t}}{\Delta t}\right)\right|^2 u\bar{u}(t')dt'\right\|_{H^{s-s',s'}} \\ &\le \|e^{i(t-\tilde{t})\partial_x^2}u(\tilde{t}) - u(\tilde{t})\|_{H^{s-s',s'}} + c(\Delta t)^{(b-b')/4}\|u\|^2_{X^b_{s-s',s'}} \\ &= o(1)\end{aligned} \tag{3.24}$$

as $\Delta t \to 0$, which is the persistence property.

Next we find the continuous dependence on the initial data from the following (3.25) and (3.26):
Using the integral equation (3.11), (3.1), (3.3) and Proposition 1.7, we get

$$\begin{aligned}
\|u-v\|_{X^b_{s-s',s'}} &\le c\|u_0-v_0\|_{H^{s-s',s'}} \\
&\quad + c(\|u\|_{X^b_{s-s',s'}} + \|v\|_{X^b_{s-s',s'}})\|u-v\|_{X^b_{s-s',s'}} \\
&\le c\|u_0-v_0\|_{H^{s-s',s'}} + \frac{1}{2}\|u-v\|_{X^b_{s-s',s'}}, \\
\text{i.e.} \quad \|u-v\|_{X^b_{s-s',s'}} &\le 2c\|u_0-v_0\|_{H^{s-s',s'}} \qquad (3.25)
\end{aligned}$$

and similarly by (3.1), (3.4) and Proposition 1.7

$$\begin{aligned}
\|u(t)-v(t)\|_{H^{s-s',s'}} &\le \|\psi(t)e^{it\partial_x^2}(u_0-v_0)\|_{H^{s-s',s'}} \\
&\quad + \frac{1}{2}\|u-v\|_{X^b_{s-s',s'}} \\
&\le c\|u_0-v_0\|_{H^{s-s',s'}} + \frac{1}{2}\|u-v\|_{X^b_{s-s',s'}} \\
&\le c'\|u_0-v_0\|_{H^{s-s',s'}}, \\
\text{i.e.} \quad \sup_{t\in[-T,T]} \|u(t)-v(t)\|_{H^{s-s',s'}} &\le c'\|u_0-v_0\|_{H^{s-s',s'}}. \qquad (3.26)
\end{aligned}$$

Therefore the map $v_0 \mapsto v(t)$ is Lipschitz from $\{v_0 \in H^{s-s',s'}(\mathbb{R}) : \|v_0 - u_0\|_{H^{s-s',s'}} < R\}$ into $X^b_{s-s',s'} \cap C([-T,T]; H^{s-s',s'}(\mathbb{R}))$.

Thus the proof of Theorem 1.6 is completed. □

§4. Proof of Proposition 1.9 (the case of $N(u,\bar{u}) = cu^2$)

Let $s = -\rho \in (-3/4, -1/2)$ *and* $s' \in (0, 1/4)$. Putting

$$\begin{aligned}
f(\xi,\tau) &= \langle \tau - \xi^2\rangle^b \langle \xi\rangle^s |\xi|^{s'} \widetilde{F}(\xi,\tau) \\
\text{and} \quad g(\xi,\tau) &= \langle \tau - \xi^2\rangle^b \langle \xi\rangle^s |\xi|^{s'} \widetilde{G}(\xi,\tau)
\end{aligned}$$

for $F, G \in X^b_{s,s'}$, we have $\|f\|_{L^2_\xi L^2_\tau} = \|F\|_{X^b_{s,s'}}$ and $\|g\|_{L^2_\xi L^2_\tau} = \|G\|_{X^b_{s,s'}}$. Thus we write for $F, G \in X^{b'}_{s-s',s'}$

$$\begin{aligned}
&\|FG\|_{X^{b-1}_{s-s',s'}} \\
&= \left\| \langle \tau - \xi^2 \rangle^{b-1} \langle \xi \rangle^{s-s'} |\xi|^{s'} \widetilde{FG}(\xi,\tau) \right\|_{L^2_\xi L^2_\tau} \\
&= c \left\| \frac{|\xi|^{s'}}{\langle \tau - \xi^2 \rangle^{1-b} \langle \xi \rangle^{\rho+s'}} \right. \\
&\times \left. \iint \frac{f(\xi-\xi_1, \tau-\tau_1)\langle \xi-\xi_1\rangle^{\rho+s'}}{\langle \tau-\tau_1-(\xi-\xi_1)^2\rangle^{b'}|\xi-\xi_1|^{s'}} \frac{g(\xi_1,\tau_1)\langle \xi_1\rangle^{\rho+s'}}{\langle \tau_1-\xi_1^2\rangle^{b'}|\xi_1|^{s'}} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

Next we note the following algebraic relation;

$$(\tau_1 - \xi_1^2) + (\tau - \tau_1 - (\xi - \xi_1)^2) - (\tau - \xi^2) = 2\xi_1(\xi - \xi_1).$$

Consequently we have

$$\max\{|\tau - \xi^2|,\ |\tau_1 - \xi_1^2|,\ |\tau - \tau_1 - (\xi-\xi_1)^2|\} \geq \frac{2}{3}|\xi_1(\xi - \xi_1)|. \tag{4.1}$$

We may assume that $|\tau - \tau_1 - (\xi - \xi_1)^2| \leq |\tau_1 - \xi_1^2|$ without loss of generality.

Remark 4.1 To establish Proposition 1.9, we use Lemmas 4.2 and 4.3 in the region $|\xi_1| \geq 1$ and $|\xi - \xi_1| \geq 1$. In this region, it follows that $\langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \leq 4|\xi_1(\xi - \xi_1)|$. In particular,

$$\frac{\langle \xi_1 \rangle^{2(\rho+s')} \langle \xi - \xi_1 \rangle^{2(\rho+s')}}{|\xi_1|^{2s'}|\xi - \xi_1|^{2s'}} \leq c|\xi_1(\xi - \xi_1)|^{2\rho}.$$

Lemma 4.2 *If $\rho = -s \in (1/2, 3/4)$ and $s' > 0$, then there exist $b \in (1/2, 5/4-\rho)$ and $b' \in (1/2, b]$ such that*

$$\begin{aligned}
&\sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^2 \rangle^{1-b}} \frac{|\xi|^{s'}}{\langle \xi \rangle^{\rho+s'}} \\
&\quad \times \left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau_1 - \xi_1^2\rangle^{2b'} \langle \tau - \tau_1 - (\xi-\xi_1)^2\rangle^{2b'}} d\xi_1 d\tau_1 \right)^{1/2} < \infty,
\end{aligned} \tag{4.2}$$

where

$$A = \{(\xi_1, \tau_1) \in \mathbb{R}^2 \colon\ |\tau - \tau_1 - (\xi - \xi_1)^2| \leq |\tau_1 - \xi_1^2| \leq |\tau - \xi^2|\}.$$

Proof. Since $|\xi_1(\xi-\xi_1)| \le c|\tau-\xi^2|$ holds in this case from (4.1), it follows that

$$\begin{aligned}&\frac{|\xi|^{s'}}{\langle\tau-\xi^2\rangle^{1-b}\langle\xi\rangle^{\rho+s'}}\left(\iint_A \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle\tau_1-\xi_1^2\rangle^{2b'}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}}d\xi_1 d\tau_1\right)^{1/2}\\ &\le C\frac{\langle\tau-\xi^2\rangle^{\rho+b-1}}{\langle\xi\rangle^{\rho}}\left(\iint_A \frac{d\xi_1 d\tau_1}{\langle\tau_1-\xi_1^2\rangle^{2b'}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}}\right)^{1/2}. \qquad (4.3)\end{aligned}$$

From the definition of A,

$$|\tau-\xi^2+2\xi_1(\xi-\xi_1)| = |(\tau_1-\xi_1^2)+(\tau-\tau_1-(\xi-\xi_1)^2)| \le 2|\tau-\xi^2| \qquad (4.4)$$

holds. And we get from (2.2) and (4.4)

$$\int_{\tau_1\in A}\frac{d\tau_1}{\langle\tau_1-\xi_1^2\rangle^{2b'}\langle\tau-\tau_1-(\xi-\xi_1)^2\rangle^{2b'}} \le c\frac{\psi(\tau-\xi^2+2\xi_1(\xi-\xi_1))/2(\tau-\xi^2)}{\langle\tau-\xi^2+2\xi_1(\xi-\xi_1)\rangle^{2b'}}. \qquad (4.5)$$

Hence the right-hand side of (4.3) is dominated by

$$C\frac{\langle\tau-\xi^2\rangle^{\rho+b-1}}{\langle\xi\rangle^{\rho}}\left(\int_{|\tau-\xi^2+2\xi_1(\xi-\xi_1)|\le 2|\tau-\xi^2|}\frac{d\xi_1}{\langle\tau-\xi^2+2\xi_1(\xi-\xi_1)\rangle^{2b'}}\right)^{1/2}. \qquad (4.6)$$

We change variables

$$\eta = \tau-\xi^2+2\xi_1(\xi-\xi_1), \quad d\eta = 2(\xi-2\xi_1)d\xi_1.$$

Moreover from $\xi = \frac{1}{2}(\xi \pm \sqrt{2\tau-\xi^2-2\eta})$, or $|2\xi_1-\xi| = \sqrt{2\tau-\xi^2-2\eta}$, we get

$$d\eta = c\sqrt{2\tau-\xi^2-2\eta}\,d\xi_1. \qquad (4.7)$$

With the aid of (2.3), the left-hand side of (4.2) is dominated by

$$\begin{aligned}&C\sup_{\xi,\tau}\frac{\langle\tau-\xi^2\rangle^{\rho+b-1}}{\langle\xi\rangle^{\rho}}\left(\int_{|\eta|\le 2|\tau-\xi^2|}\frac{d\eta}{\langle\eta\rangle^{2b'}|\sqrt{2\tau-2\eta-\xi^2}|}\right)^{1/2}\\ &\le C\sup_{\xi,\tau}\frac{\langle\tau-\xi^2\rangle^{\rho+b-1}}{\langle\xi\rangle^{\rho}\langle\tau-\xi^2/2\rangle^{1/4}},\end{aligned}$$

which yields the result. □

Lemma 4.3 *If $\rho = -s \in (1/2, 3/4)$ and $s' > 0$, then there exist $b \in (1/2, 5/4-\rho)$ and $b' \in (1/2, b]$ such that*

$$\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \times \left(\iint_B \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} < \infty, \tag{4.8}$$

where

$$B = \left\{ \begin{array}{ll} (\xi,\tau) \in \mathbb{R}^2 : & |\tau - \tau_1 - (\xi-\xi_1)^2| \leq |\tau_1 - \xi_1^2| \\ & |\tau - \xi^2| \leq |\tau_1 - \xi_1^2| \end{array} \right\}.$$

Proof. It is clear that

$$\begin{aligned} &\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau_1 - \xi_1^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} \\ &\leq \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{1}{\langle \xi \rangle^{2\rho}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 - (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2}. \end{aligned} \tag{4.9}$$

It follows from the definition of B that

$$|\tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)| = |(\tau - \tau_1 - (\xi-\xi_1)^2) - (\tau - \xi^2)| \leq 2|\tau_1 - \xi_1^2|. \tag{4.10}$$

By virtue of (2.5), the left-hand side of (4.9) is bounded by

$$C \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\int_D \frac{1}{\langle \xi \rangle^{2\rho}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1-\xi) \rangle^{2(1-b)}} d\xi \right)^{1/2} \tag{4.11}$$

with $D = \{\xi \in \mathbb{R} : |\tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)| \leq 2|\tau_1 - \xi_1^2|\}$.

Since $2|\xi_1(\xi_1 - \xi)| \leq |\tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)| + |\tau_1 - \xi_1^2| \leq 3|\tau_1 - \xi_1^2|$ in D, we can divide D into two domains D_1 and D_2:

$$\begin{aligned} D_1 &= \{\xi \in D \colon |\xi_1(\xi - \xi_1)| \leq |\tau_1 - \xi_1^2|/4\}, \\ D_2 &= \{\xi \in D \colon |\tau_1 - \xi_1^2|/4 \leq |\xi_1(\xi - \xi_1)| \leq 3|\tau_1 - \xi_1^2|/2\}. \end{aligned}$$

For a domain C, we put

$$I(C) = \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{b'}} \left(\int_C \frac{1}{\langle \xi \rangle^{2\rho}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1-\xi) \rangle^{2(1-b)}} d\xi \right)^{1/2}. \tag{4.12}$$

Estimate in D_1.
From $|\xi_1(\xi-\xi_1)| \le |\tau_1-\xi_1^2|/4$, we get

$$\begin{aligned}|\tau_1-\xi_1^2| &\le |\tau_1-\xi_1^2+2\xi_1(\xi-\xi_1)| + 2|\xi_1(\xi-\xi_1)| \\ &\le |\tau_1-\xi_1^2+2\xi_1(\xi-\xi_1)| + |\tau_1-\xi_1^2|/2.\end{aligned}$$

Hence

$$|\tau_1-\xi_1^2| \le 2|\tau_1-\xi_1^2+2\xi_1(\xi-\xi_1)|.$$

On the other hand,

$$3|\tau_1-\xi_1^2|/2 \ge |\tau_1-\xi_1^2+2\xi_1(\xi-\xi_1)|.$$

Therefore it follows in D_1 that

$$|\tau_1-\xi_1^2| \sim |\tau_1-\xi_1^2+2\xi_1(\xi_1-\xi)|. \tag{4.13}$$

With the aid of (4.13), we get

$$I(D_1) \le C\frac{\langle\tau_1-\xi_1^2\rangle^\rho}{\langle\tau_1-\xi_1^2\rangle^{b'-b+1}}\left(\int_{D_1}\frac{d\xi}{\langle\xi\rangle^{2\rho}}d\xi\right)^{1/2} < \infty. \tag{4.14}$$

Estimate in D_2.
In D_2, it follows that

$$|\xi_1(\xi-\xi_1)| \sim |\tau_1-\xi_1^2|. \tag{4.15}$$

We subdivide D_2 into three regions:

$$\begin{aligned}D_{2,1} &= \{\xi\in D_2\colon\ |\xi|/4 \le |\xi_1| \le 100|\xi|\}, \\ D_{2,2} &= \{\xi\in D_2\colon\ 1 \le |\xi_1| \le |\xi|/4\}, \\ D_{2,3} &= \{\xi\in D_2\colon\ 100|\xi| \le |\xi_1|\}.\end{aligned}$$

In $D_{2,1}$, it follows that $|\xi_1(\xi-\xi_1)| \le c|\xi_1|^2$ and $|\xi_1| \ge 1$. Therefore we obtain

$$|\tau_1-\xi_1^2| \sim |\xi_1(\xi-\xi_1)| \le c|\xi_1|^2$$

and moreover

$$\langle\tau_1-\xi_1^2\rangle \le c|\xi_1|^2. \tag{4.16}$$

Changing variables

$$\eta = \tau_1-\xi_1^2+2\xi_1(\xi_1-\xi), \quad d\eta = -2\xi_1 d\xi, \tag{4.17}$$

and noting (4.15) and (4.16), we obtain

$$
\begin{aligned}
I(D_{2,1}) &\le c\langle \tau_1 - \xi_1^2\rangle^{\rho-b'} \left(\int_{D_{2,1}} \frac{d\xi}{\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)\rangle^{2(1-b)}} \right)^{1/2} \\
&= c\langle \tau_1 - \xi_1^2\rangle^{\rho-b'} \left(\int_{|\eta|\le 2|\tau_1-\xi_1^2|} \frac{d\eta}{|\xi_1|\langle \eta\rangle^{2(1-b)}} \right)^{1/2} \\
&\le c\langle \tau_1 - \xi_1^2\rangle^{\rho-b'-1/4+b-1/2+\epsilon} < \infty. \qquad (4.18)
\end{aligned}
$$

We should estimate $I(D_{2,2})$ more carefully. It follows in $D_{2,2}$ that

$$|\xi - \xi_1| \le |\xi| + |\xi|/4 = 5|\xi|/4.$$

Also it follows that

$$
\begin{aligned}
|\xi| &\le |\xi - \xi_1| + |\xi_1| \\
&\le |\xi - \xi_1| + |\xi|/4 \\
\text{i.e.} \quad 3|\xi|/4 &\le |\xi - \xi_1|.
\end{aligned}
$$

Hence we get

$$|\xi| \sim |\xi - \xi_1|. \qquad (4.19)$$

In particular, from (4.19) and (4.15)

$$|\xi| \le |\xi_1\xi| \sim |\xi_1(\xi - \xi_1)| \sim |\tau_1 - \xi_1^2|. \qquad (4.20)$$

Therefore we obtain

$$|\xi_1| \le |\xi|/4 \le c|\tau_1 - \xi_1^2|. \qquad (4.21)$$

By virtue of (4.19), we get

$$
\begin{aligned}
I(D_{2,2}) &\le C\langle \tau_1 - \xi_1^2\rangle^{-b'} \left(\int_{D_{2,2}} \frac{|\xi_1\xi|^{2\rho}}{\langle \xi\rangle^{2\rho}\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)\rangle^{2(1-b)}} d\xi \right)^{1/2} \\
&\le C\langle \tau_1 - \xi_1^2\rangle^{-b'} \left(\int_{D_{2,2}} \frac{|\xi_1|^{2\rho}}{\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)\rangle^{2(1-b)}} d\xi \right)^{1/2} \qquad (4.22)
\end{aligned}
$$

Making the change of variables (4.17) and noting (4.21), we obtain

$$
\begin{aligned}
I(D_{2,2}) &\le C\langle \tau_1 - \xi_1^2\rangle^{-b'} \left(\int_{|\eta|\le 2|\tau_1-\xi_1^2|} \frac{|\xi_1|^{2\rho-1}}{\langle \eta\rangle^{2(1-b)}} d\eta \right)^{1/2} \\
&\le C\langle \tau_1 - \xi_1^2\rangle^{-b'+\rho-1/2} \left(\int_{|\eta|\le 2|\tau_1-\xi_1^2|} \frac{d\eta}{\langle \eta\rangle^{2(1-b)}} \right)^{1/2} \\
&\le C\langle \tau_1 - \xi_1^2\rangle^{-b'+\rho-1/2+b-1/2+\epsilon} < \infty. \qquad (4.23)
\end{aligned}
$$

Finally noting that $|\xi_1| \sim |\xi - \xi_1|$ holds in $D_{2,3}$, we get $|\xi_1|^2 \sim |\tau_1 - \xi_1^2|$ from (4.15). Therefore making the change of variables (4.17), we obtain

$$\begin{aligned} I(D_{2,3}) &\leq C\langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \left(\int_{D_{2,3}} \frac{d\xi}{\langle \tau_1 - \xi_1^2 + 2\xi_1(\xi_1 - \xi)\rangle^{2(1-b)}} d\xi \right)^{1/2} \\ &\leq C\langle \tau_1 - \xi_1^2 \rangle^{\rho - b'} \left(\int_{|\eta| \leq 2|\tau_1 - \xi_1^2|} \frac{d\eta}{|\xi_1| \langle \eta \rangle^{2(1-b)}} \right)^{1/2} \\ &\leq C\langle \tau_1 - \xi_1^2 \rangle^{\rho - b' - 1/4 + b - 1/2 + \epsilon} < \infty. \end{aligned} \tag{4.24}$$

Summing up, we complete the proof. □

Lemma 4.4 *Let $\rho = -s > 0$ and $0 < s' < 1/4$. Then there exist $b > 1/2$ and $b' > 1/2$ such that*

$$\begin{aligned} &\sup_{\xi} \sup_{\tau} \frac{|\xi|^{s'}}{\langle \tau - \xi^2 \rangle^{1-b} \langle \xi \rangle^{\rho + s'}} \\ &\times \left(\iint_E \frac{\langle \xi_1 \rangle^{2(\rho + s')}}{\langle \tau_1 - \xi_1^2 \rangle^{2b'} |\xi_1|^{2s'}} \frac{\langle \xi - \xi_1 \rangle^{2(\rho + s')}}{\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b'} |\xi - \xi_1|^{2s'}} d\xi_1 d\tau_1 \right)^{1/2} < \infty, \end{aligned} \tag{4.25}$$

where $E = \{(\xi_1, \tau_1) \in \mathbb{R}^2 \colon\ |\xi_1| < 1$ or $|\xi - \xi_1| < 1\}$.

Proof. The proof is same as that of Lemma 2.9. Therefore it is omitted. □

§5. Proof of Proposition 1.11 (the case of $N(u, \bar{u}) = c\bar{u}^2$)

Let $s = -\rho \in (-3/4, -1/2)$ *and* $s' \in (0, 1/4)$. Putting

$$f(\xi, \tau) = \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s |\xi|^{s'} \overline{\widetilde{F}}(-\xi, -\tau)$$
$$\text{and} \quad g(\xi, \tau) = \langle \tau + \xi^2 \rangle^b \langle \xi \rangle^s |\xi|^{s'} \overline{\widetilde{G}}(-\xi, -\tau)$$

for $F, G \in X^b_{s,s'}$, we have $\|f\|_{L^2_\xi L^2_\tau} = \|\overline{F}\|_{X^b_{s,s'}}$ and $\|g\|_{L^2_\xi L^2_\tau} = \|\overline{G}\|_{X^b_{s,s'}}$. Thus we write for $F, G \in X^{b'}_{s-s',s'}$

$$\begin{aligned} \|\overline{FG}\|_{X^{b-1}_{s-s',s'}} &= \left\| \langle \tau - \xi^2 \rangle^{b-1} \langle \xi \rangle^{s-s'} |\xi|^{s'} \widetilde{\overline{FG}}(\xi, \tau) \right\|_{L^2_\xi L^2_\tau} \\ &= c \left\| \frac{|\xi|^{s'}}{\langle \tau - \xi^2 \rangle^{1-b} \langle \xi \rangle^{\rho + s'}} \right. \\ &\quad \left. \times \iint \frac{\bar{f}(\xi - \xi_1, \tau - \tau_1) \langle \xi - \xi_1 \rangle^{\rho + s'}}{\langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{b'} |\xi - \xi_1|^{s'}} \frac{\bar{g}(\xi_1, \tau_1) \langle \xi_1 \rangle^{\rho + s'}}{\langle \tau_1 + \xi_1^2 \rangle^{b'} |\xi_1|^{s'}} d\xi_1 d\tau_1 \right\|_{L^2_\xi L^2_\tau}. \end{aligned}$$

Next we note the following algebraic relation;

$$(\tau_1 + \xi_1^2) + (\tau - \tau_1 + (\xi - \xi_1)^2) - (\tau - \xi^2) = \xi^2 + (\xi - \xi_1)^2 + \xi_1^2.$$

Consequently we have

$$\max\{|\tau - \xi^2|,\ |\tau_1 + \xi_1^2|,\ |\tau - \tau_1 + (\xi - \xi_1)^2|\} \geq \frac{1}{3}(\xi^2 + (\xi - \xi_1)^2 + \xi_1^2). \quad (5.1)$$

We may assume that $|\tau - \tau_1 + (\xi - \xi_1)^2| \leq |\tau_1 + \xi_1^2|$ without loss of generality.

Remark 5.1 We note that $|\xi_1(\xi - \xi_1)| \leq \xi_1^2 + (\xi - \xi_1)^2 + \xi^2$ holds.

Remark 5.2 To establish Proposition 1.11, we use Lemmas 5.3 and 5.4 in the region $|\xi_1| \geq 1$ and $|\xi - \xi_1| \geq 1$. In this region, it follows that $\langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \leq 4|\xi_1(\xi - \xi_1)|$. In particular,

$$\frac{\langle \xi_1 \rangle^{2(\rho+s')} \langle \xi - \xi_1 \rangle^{2(\rho+s')}}{|\xi_1|^{2s'} |\xi - \xi_1|^{2s'}} \leq c|\xi_1(\xi - \xi_1)|^{2\rho}.$$

Lemma 5.3 *If $\rho = -s \in (1/2, 3/4)$ and $s' > 0$, then there exist $b \in (1/2, 5/4-\rho)$ and $b' \in (1/2, b]$ such that*

$$\sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^2 \rangle^{1-b}} \frac{|\xi|^{s'}}{\langle \xi \rangle^{\rho+s'}} \times \left(\iint_A \frac{|\xi_1(\xi - \xi_1)|^{2\rho}}{\langle \tau_1 + \xi_1^2 \rangle^{2b'} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b'}} d\xi_1 d\tau_1 \right)^{1/2} < \infty, \quad (5.2)$$

where

$$A = \{(\xi_1, \tau_1) \in \mathbb{R}^2 \colon |\tau - \tau_1 + (\xi - \xi_1)^2| \leq |\tau_1 + \xi_1^2| \leq |\tau - \xi^2|\}.$$

Proof. In this case, $|\xi_1(\xi - \xi_1)| \leq 3|\tau - \xi^2|$ holds. Hence the left-hand side of (5.2) is dominated by

$$\begin{aligned} &C \sup_{\xi,\tau} \frac{1}{\langle \tau - \xi^2 \rangle^{1-b}} \frac{1}{\langle \xi \rangle^{\rho}} \left(\iint_A \frac{|\xi_1(\xi - \xi_1)|^{2\rho}}{\langle \tau + \xi_1^2 \rangle^{2b'} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b'}} d\xi_1 d\tau_1 \right)^{1/2} \\ &\quad \leq C \sup_{\xi,\tau} \frac{\langle \tau - \xi^2 \rangle^{\rho+b-1}}{\langle \xi \rangle^{\rho}} \left(\iint_A \frac{d\xi_1 d\tau_1}{\langle \tau_1 + \xi_1^2 \rangle^{2b'} \langle \tau - \tau_1 + (\xi - \xi_1)^2 \rangle^{2b'}} \right)^{1/2}. \end{aligned} \quad (5.3)$$

Changing (τ, τ_1) by $(-\tau, -\tau_1)$ and following the argument in Lemma 4.2, we find the bound of (5.3)

$$C \sup_{\xi,\tau} \frac{1}{\langle \xi \rangle^{\rho}} \frac{\langle \tau + \xi^2 \rangle^{\rho+b-1}}{\langle \tau - \xi^2/2 \rangle^{1/4}}, \quad (5.4)$$

which yields the result. $\square$

Lemma 5.4 *If $\rho = -s \in (1/2, 3/4)$ and $s' > 0$, then there exist $b \in (1/2, 5/4-\rho)$ and $b' \in (1/2, b]$ such that*

$$\sup_{|\xi_1|\geq 1} \sup_{\tau_1} \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \times \left(\iint_B \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 + (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} < \infty, \tag{5.5}$$

where

$$B = \left\{ \begin{array}{ll} (\xi,\tau) \in \mathbb{R}^2: & |\tau - \tau_1 + (\xi-\xi_1)^2| \leq |\tau_1 + \xi_1^2| \\ & |\tau - \xi^2| \leq |\tau_1 + \xi_1^2| \end{array} \right\}.$$

Proof. In B, $\xi^2 + (\xi-\xi_1)^2 + \xi_1^2 \leq 3|\tau_1 + \xi_1^2|$ holds.

With the aid of (2.5),

$$\begin{aligned} & \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{|\xi|^{2s'}}{\langle \xi \rangle^{2(\rho+s')}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 + (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} \\ & \leq C \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\iint_B \frac{1}{\langle \xi \rangle^{2\rho}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau - \xi^2 \rangle^{2(1-b)} \langle \tau - \tau_1 + (\xi-\xi_1)^2 \rangle^{2b'}} d\xi d\tau \right)^{1/2} \\ & \leq C \frac{1}{\langle \tau_1 + \xi_1^2 \rangle^{b'}} \left(\int_D \frac{1}{\langle \xi \rangle^{2\rho}} \frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle \tau_1 - \xi^2 - (\xi-\xi_1)^2 \rangle^{2(1-b)}} d\xi \right)^{1/2} \end{aligned} \tag{5.6}$$

with $D = \{\xi \in \mathbb{R} : \xi^2 + (\xi-\xi_1)^2 + \xi_1^2 \leq 3|\tau_1 + \xi_1^2|\}$.

We divide D into D_1 and D_2:

$$\begin{aligned} D_1 &= \{\xi \in D : \xi^2 + (\xi-\xi_1)^2 + \xi_1^2 \leq |\tau_1 + \xi_1^2|/2\}, \\ D_2 &= \{\xi \in D : |\tau_1 + \xi_1^2|/2 \leq \xi^2 + (\xi-\xi_1)^2 + \xi_1^2 \leq 3|\tau_1 + \xi_1^2|\}. \end{aligned}$$

Estimate in D_1.
We observe that

$$|\tau_1 + \xi_1^2| \sim |\tau_1 - \xi^2 - (\xi-\xi_1)^2| \tag{5.7}$$

holds in D_1.

Indeed it is clear that $|\tau_1 - \xi^2 - (\xi-\xi_1)^2| \leq |(\tau - \xi^2) - (\tau - \tau_1 + (\xi-\xi_1)^2)| \leq 2|\tau_1 + \xi_1^2|$.

Conversely, in D_1, we get

$$\begin{aligned}|\tau_1+\xi_1^2| &\le \big|(\tau_1+\xi_1^2)+(\tau-\tau_1+(\xi-\xi_1)^2)-(\tau-\xi^2)\big| \\ &\quad+\big|(\tau-\xi^2)-(\tau-\tau_1+(\xi-\xi_1)^2)\big| \\ &= \xi^2+(\xi-\xi_1)^2+\xi_1^2+|\tau_1-\xi^2-(\xi-\xi_1)^2| \\ &\le |\tau_1+\xi_1^2|/2+|\tau_1-\xi^2-(\xi-\xi_1)^2|.\end{aligned}$$

Hence $|\tau_1+\xi_1^2|/2 \le |\tau_1-\xi^2-(\xi-\xi_1)^2|$ holds.

With (5.7) and Remark 5.1, we obtain the bound of (5.6);

$$\begin{aligned}&\frac{1}{\langle\tau_1+\xi_1^2\rangle^{b'}}\left(\int_{D_1}\frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle\xi\rangle^{2\rho}\langle\tau_1-\xi^2-(\xi-\xi_1)^2\rangle^{2(1-b)}}d\xi\right)^{1/2} \\ &\qquad\sim\frac{1}{\langle\tau_1+\xi_1^2\rangle^{1-b+b'}}\left(\int_{D_1}\frac{|\xi_1(\xi-\xi_1)|^{2\rho}}{\langle\xi\rangle^{2\rho}}d\xi\right)^{1/2} \\ &\qquad\le C\langle\tau_1+\xi_1^2\rangle^{\rho+b-b'-1}\left(\int_{D_1}\frac{d\xi}{\langle\xi\rangle^{2\rho}}\right)^{1/2} \\ &\qquad\le C\langle\tau_1+\xi_1^2\rangle^{\rho+b-b'-1}<\infty.\end{aligned} \tag{5.8}$$

Estimate in D_2.
We split D_2 into $D_{2,1}$, $D_{2,2}$ and $D_{2,3}$;

$$\begin{aligned}D_{2,1}&=\{\xi\in D_2\colon |\xi|/4\le|\xi_1|\le 100|\xi|\},\\ D_{2,2}&=\{\xi\in D_2\colon 1\le|\xi_1|\le|\xi|/4\},\\ D_{2,3}&=\{\xi\in D_2\colon 100|\xi|\le|\xi_1|\}.\end{aligned}$$

In the region $D_{2,1}$, we make the change of variables

$$\eta=\tau_1-\xi^2-(\xi-\xi_1)^2,\quad d\eta=2(\xi_1-2\xi)d\xi. \tag{5.9}$$

From $\xi=\frac{1}{2}(\xi_1\pm\sqrt{2\tau_1-\xi_1^2-2\eta})$, we note that

$$|\xi_1-2\xi|=\sqrt{2\tau_1-\xi_1^2-2\eta},\quad d\eta=c\sqrt{2\tau_1-\xi_1^2-2\eta}d\xi. \tag{5.10}$$

In this domain, it follows that

$$|\xi|\sim|\xi_1|. \tag{5.11}$$

By virtue of (5.1), (5.11), (5.10) and (2.6), the left-hand side of (5.6) is bounded by

$$\begin{aligned} &\frac{1}{\langle \tau_1+\xi_1^2\rangle^{b'}}\frac{\langle \tau_1+\xi_1^2\rangle^{\rho}}{\langle \xi_1\rangle^{\rho}}\left(\int_{D_{2,1}}\frac{d\xi}{\langle \tau_1-\xi^2-(\xi-\xi_1)^2\rangle^{2(1-b)}}\right)^{1/2} \\ &= C\frac{\langle \tau_1+\xi_1^2\rangle^{\rho-b'}}{\langle \xi_1\rangle^{\rho}}\left(\int_{|\eta|\le 3|\tau_1+\xi_1^2|}\frac{d\eta}{\langle \eta\rangle^{2(1-b)}\sqrt{2\tau_1-\xi_1^2-2\eta}}\right)^{1/2} \\ &\le C\frac{\langle \tau_1+\xi_1^2\rangle^{\rho-b'}}{\langle \xi_1\rangle^{\rho}}\frac{\langle \tau_1+\xi_1^2\rangle^{b-1/2+\epsilon}}{\langle 2\tau_1-\xi_1^2\rangle^{1/4}}<\infty. \end{aligned} \tag{5.12}$$

In $D_{2,2}$, the proof is similar to that in $D_{2,1}$.

Finally we consider $D_{2,3}$. In this region $|\xi_1(\xi-\xi_1)|\le c|\xi_1|^2$ holds. Hence (5.6) is bounded by

$$C\frac{|\xi_1|^{2\rho}}{\langle \tau_1+\xi_1^2\rangle^{b'}}\left(\int_{D_{2,3}}\frac{d\xi}{\langle \tau_1-\xi^2-(\xi-\xi_1)^2\rangle^{2(1-b)}}\right)^{1/2}. \tag{5.13}$$

Since $|\xi_1|\sim|2\xi-\xi_1|$ holds in $D_{2,3}$, we change variables (5.9) and $d\eta=c|\xi_1|d\xi$. Hence (5.13) is dominated by

$$\begin{aligned} &C\frac{|\xi_1|^{2\rho}}{\langle \tau_1+\xi_1^2\rangle^{b'}}\left(\int_{|\eta|\le 3|\tau_1+\xi_1^2|}\frac{d\eta}{\langle \eta\rangle^{2(1-b)}|\xi_1|}\right)^{1/2} \\ &\le C\frac{|\xi_1|^{2\rho-1/2}}{\langle \tau_1+\xi_1^2\rangle^{b'}}\langle \tau_1+\xi_1^2\rangle^{b-1/2+\epsilon} \\ &\le C\langle \tau_1+\xi_1^2\rangle^{\rho-1/4-b'+b-1/2+\epsilon}<\infty, \end{aligned} \tag{5.14}$$

where we use, in the second term, $|\xi_1|^2\le 3|\tau_1-\xi_1^2|$ derived from (5.1). Summing up, we conclude this lemma. □

Lemma 5.5 *Let $\rho=-s>0$ and $0<s'<1/4$. Then there exist $b>1/2$ and $b'>1/2$ such that*

$$\begin{aligned} &\sup_{\xi}\sup_{\tau}\frac{|\xi|^{s'}}{\langle \tau-\xi^2\rangle^{1-b}\langle \xi\rangle^{\rho+s'}} \\ &\times\left(\iint_E\frac{\langle \xi_1\rangle^{2(\rho+s')}}{\langle \tau_1+\xi_1^2\rangle^{2b'}|\xi_1|^{2s'}}\frac{\langle \xi-\xi_1\rangle^{2(\rho+s')}}{\langle \tau-\tau_1+(\xi-\xi_1)^2\rangle^{2b'}|\xi-\xi_1|^{2s'}}d\xi_1 d\tau_1\right)^{1/2}<\infty, \end{aligned} \tag{5.15}$$

where $E=\{(\xi_1,\tau_1)\in\mathbb{R}^2\colon\ |\xi_1|<1$ or $|\xi-\xi_1|<1\}$.

Proof. The proof is the same as that of Lemma 2.9. Therefore it is omitted. □

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