SUT Journal of Mathematics
Vol. 40, No. 1 (2004), 75–90

Nonlinear nonlocal Schrödinger type equations on a segment

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(Received June 15, 2004)

Abstract. We study the global existence and large time asymptotic behavior of solutions to the initial-boundary value problem for the nonlinear nonlocal Schrödinger equation on a segment $(0,a)$

$$\begin{cases} u_t + i|u|^2 u + \mathbb{K}u = 0, \ t>0, x\in(0,a) \\ u(x,0) = u_0(x), \ x\in(0,a), \end{cases} \tag{0.1}$$

where the pseudodifferential operator $\mathbb{K}$ has the dissipation propery and the symbol of order $\alpha\in(0,1)$. We prove that if the initial data $u_0\in\mathbf{L}^\infty$ are small, then there exists a unique solution $u\in\mathbf{C}([0,\infty);\mathbf{L}^\infty)$ of the initial-boundary value problem (0.1) Moreover there exists a function $A\in\mathbf{L}^\infty$ such that the solution has the following large time asymptotics

$$u(x,t) = A(x)\, t^{-\frac{1}{\alpha}} \Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + O\left(t^{-\frac{1+\delta}{\alpha}}\right),$$

where $\Lambda(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^\alpha + zx} dz$.

AMS 2000 *Mathematics Subject Classification.* 35Q55; 35B40.

Key words and phrases. Nonlinear Schroedinger equation, initial-boundary value problem, large time asymptotics.

§1. Introduction

Our aim in the present paper is to study the global existence and large time asymptotic behavior of solutions to the initial-boundary value problem for the nonlinear Shrödinger equation on a segment $[0,a]$

$$\begin{cases} u_t + i\,|u|^2\, u + \mathbb{K}u = 0, \ t>0, x\in(0,a), \\ u(x,0) = u_0(x), \ x\in(0,a), \end{cases} \tag{1.1}$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is given by

$$\mathbb{K}u = (1-\theta(x-a))\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p)\widehat{u}(p,t)dp, \tag{1.2}$$

where $K(p) = p^{\alpha}, \alpha \in (0,1)$.

The nonlinear nonlocal Schrödinger equation (1.1) is a simple model appearing as the first approximation in the description of the dispersive dissipative nonlinear waves. As far as we know the global existence and large time asymptotic behavior for solutions of the initial-boundary value problem for the nonlinear nonlocal Schrödinger equation (1.1) on a segment was not studied previously. In the case of the Cauchy problem global existence of solutions was proved in papers [7], [2] and the large time asymptotics of solutions was obtained in [10], [9], [4]. In the case of the boundary value problem on a half-line the large time asymptotics of solutions were studied in papers [1], [3], [6], [8].

Let us start with the following general linear initial-boundary value problem on a segment

$$\begin{cases} u_t + \mathbb{K}u = f(x,t),\ t>0,\ x\in(0,a)\,, \\ u(x,0) = u_0(x),\ x\in(0,a)\,, \\ \partial_x^j u(0,t) = h_{0j}(t),\ j=1,...,m, \\ \partial_x^l u(a,t) = h_{al}(t),\ l=1,...,n, \end{cases} \tag{1.3}$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is defined by the inverse Laplace transformation as follows

$$\begin{aligned} \mathbb{K}u &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p) \\ &\times \left(\widehat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1}u(0,t) - e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}\right) dp \\ &-\theta(x-a)\frac{1}{2\pi i}\int_{\Gamma_1} e^{px}K(p) \\ &\times \left(\widehat{u}(p,t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1}u(0,t) - e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}\right) dp, \end{aligned} \tag{1.4}$$

where the contour Γ_1 goes along the boundary of the domain of analyticity of the symbol $K(p)$, we assume that $K(p)$ is always analytic in the domain $\operatorname{Re} p > 0$. Note that in the case of holomorphic symbol $K(p)$ (for example, a polynomial) the last integral in the definition (1.4) is equal to zero, hence we get a usual differential operator. Also we can rewrite the definition (1.4) in the form

$$\begin{aligned}\mathbb{K}u &= (1-\theta(x-a))\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}K(p) \\ &\quad\times\left(\widehat{u}(p,t)-\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)-e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}\right)dp, \end{aligned}\tag{1.5}$$

if we take $K(p)=C_\alpha p^\alpha$, $\alpha>0$ for simplicity. We make a cut along the negative part of the real axis, that is we choose $\arg z\in[-\pi,\pi)$ for any complex $z\in\mathbb{C}$. Here $[\alpha]$ is the integer part of the number α, C_α will be chosen by the dissipation condition $\operatorname{Re}K(p)>0$ for all $\operatorname{Re}p=0$. Note that the inverse Laplace transform gives us a function, which is equal to 0 for all $x<0$, so that multiplication by the factor $(1-\theta(x-a))$ yields that the operator $\mathbb{K}u$ vanish outside of the interval $(0,a)$. Thus the solution $u(x,t)$ is considered for all $x\in\mathbf{R}$ prolonged by zero outside of the segment $[0,a]$. We expect that by analogy with the case of a half-line the integers n and m are defined by the number of regions, where $\operatorname{Re}K(p)<0$.

Taking the Laplace transform of the operator $\mathbb{K}u$ we get

$$\begin{aligned}\int_0^a e^{-px}\mathbb{K}u dx &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-p)a}-1}{q-p}K(q)\widehat{\widehat{u}}(q,t)dq \\ &= \frac{e^{-pa}}{2\pi i}\int_\Gamma\frac{e^{qa}}{q-p}K(q)\widehat{\widehat{u}}(q,t)dq+K(p)\widehat{\widehat{u}}(p,t),\end{aligned}\tag{1.6}$$

where we denote the contour

$$\Gamma=\left\{q\in\mathbb{C};\ q\in(\infty e^{-i\pi},-i0)\cup\left(+i0,e^{i\pi}\infty\right)\right\}\tag{1.7}$$

and

$$\widehat{\widehat{u}}(p,t)=\widehat{u}(p,t)-\sum_{j=1}^{[\alpha]}\frac{\partial_x^{j-1}u(0,t)-e^{-pa}\partial_x^{j-1}u(a,t)}{p^j}.$$

Applying the Laplace transformation with respect to x to problem (1.3) we get

$$\begin{cases}\widehat{u}_t+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-p)a}-1}{q-p}K(q)\widehat{\widehat{u}}(q,t)dq=\widehat{f}(p,t), \quad t>0,\\ \widehat{u}(p,0)=\widehat{u}_0(p),\\ \partial_x^j u(0,t)=h_{0j}(t), j=1,...,n,\\ \partial_x^l u(a,t)=h_{al}(t), l=1,...,m.\end{cases}$$

Integrating with respect to time t in view of (1.6) we obtain for the Laplace transform $\widehat{u}(p,t)$

$$\widehat{u}(p,t) = e^{-K(p)t}\hat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)}\widehat{f}_1(p,\tau)d\tau, \tag{1.8}$$

where

$$\begin{aligned}\widehat{f}_1(p,t) &= \widehat{f}(p,t) + K(p)\sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1}u(0,t) - e^{-pa}\partial_x^{j-1}u(a,t)}{p^j} \\ &\quad - \frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}K(q)\widehat{\widehat{u}}(q,\tau)dq.\end{aligned}$$

In order to get the integral formula for solutions of (1.3), we need to know the boundary values $\partial_x^{j-1}u\,(0,t)$, $\partial_x^{j-1}u(a,t)$. Some of the boundary values we put in the problem as given boundary data and the rest we will find from the equation using the growth condition

$$|\hat{u}(p,t)| \le M(1+|p|)^{\beta}\left(1+\left|e^{-pa}\right|\right) \text{ for all } |p| \ge 1, \tag{1.9}$$

with some M, $\beta > 0$, which guarantee us that the inverse Laplace transform $u(x,t)$ vanish for all $x < 0$ and $x > a$. It is easy to prove that condition (1.9) is fulfilled in domains $\operatorname{Re} K(p) > 0$. In domains, where $\operatorname{Re} K(p) < 0$, we rewrite formula (1.8) as

$$\begin{aligned}\widehat{u}(p,t) &= e^{-K(p)t}\left(\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau}f_1(p,\tau)d\tau\right) \\ &\quad - \int_t^{+\infty} e^{-K(p)(t-\tau)}f_1(p,\tau)d\tau.\end{aligned}$$

Clearly the last integral

$$\int_t^{+\infty} e^{-K(p)(t-\tau)}f_1(p,\tau)d\tau$$

satisfies condition (1.9) for all $|p| \ge 1$, such that Re $K(p) < 0$. However the first summand with exponentially growing factor $e^{-K(p)t}$ does not satisfy condition (1.9), therefore we have to put the following conditions

$$\hat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau}f_1(p,\tau)d\tau = 0 \tag{1.10}$$

for all $|p| > 1$ in the domains, where Re $K(p) < 0$. We use equations (1.10) to find some of the boundary values $\partial_x^j u(0,t)$, $\partial_x^j u(a,t)$ involved in formula

(1.8). Making a change of the independent variable $K(p) = -\xi$ we transform the domains $\operatorname{Re} K(p) < 0$ to the half-complex plane $\operatorname{Re}\, \xi > 0$ by $[\alpha]$ different roots $\phi_1(\xi), \phi_2(\xi), \ldots, \phi_{[\alpha]}(\xi)$, which are analytic functions for all $\operatorname{Re}\xi > 0$ and transform the half-complex plane $\operatorname{Re}\, \xi > 0$ to domains, where $\operatorname{Re}\, K(p) < 0$. Then condition (1.10) can be written as a system of $[\alpha]$ equations in the half-complex plane $\operatorname{Re}\, \xi > 0$

$$
\begin{aligned}
&\widehat{u}_0(\phi_l) + \widehat{\widehat{f}}(\phi_l, \xi) \\
&-\xi \int_0^{+\infty} e^{-\xi\tau} \left(\sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t) - e^{-\phi_l a} \partial_x^{j-1} u(a,t)}{\phi_l^j} \right) d\tau \\
= \ & -\frac{1}{2\pi i} \int_\Gamma \frac{e^{(q-\phi_l(\xi))a}}{q - \phi_l(\xi)} K(q) \int_0^{+\infty} e^{-\xi\tau} \left(\widehat{u}(q,\tau) \right. \\
&\left. - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1} u(0,t) - e^{-qa} \partial_x^{j-1} u(a,t)}{q^j} \right) d\tau dq,
\end{aligned}
\tag{1.11}
$$

for $l = 1, 2, ..., [\alpha]$, where $\widehat{u}(p,t)$ is the solution of problem (1.8) and

$$
\widehat{u}_0(\phi_l) = \int_0^a e^{-\phi_l y} u_0(y) dy,\ \widehat{\widehat{f}}(\phi_l, \xi) = \int_0^{+\infty} \int_0^a e^{-(\phi_l y + \xi t)} f(y,t) dy dt.
$$

We have $[\alpha]$ equations with $2\,[\alpha]$ unknowns $u_x^{(j-1)}(0,t)$, $u_x^{(j-1)}(a,t)$ so we need to put $[\alpha]$ boundary data in the problem (1.3) and the rest $[\alpha]$ boundary values can be found from system (1.11).

In the case $\alpha \in (0,1)$, which is under the consideration in the present paper, we do not need to solve system (1.11), because condition (1.9) is fulfilled automatically for any complex p, due to the estimate $\left| e^{-K(p)t} \right| \le C\,(1 + |e^{-pa}|)$.

In the present paper we consider problem (1.1) in the case of the initial data belonging to space $\mathbf{L}^\infty$. For obtaining $\mathbf{L}^p$ -estimates of the Green function we use the method of our previous papers [3] and [6].

Let us denote the space $\mathbf{L}^\infty(0,a) = \{\varphi \in \mathbf{L}^\infty\,(0,a)\,; \|\varphi\|_{\mathbf{L}^\infty} < +\infty\}$. Let $\|\phi\|_{\mathbf{L}^p(\mathbf{R}^+)} = \|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^p(0,a)} = \|\phi\|_p$, $1 \le p \le \infty$.

We state the main result of this paper.

Theorem 1. *Let the initial data $u_0 \in \mathbf{L}^\infty(0,a)$ and the norm $\|u_0\|_\infty < \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Then there exists a unique solution $u \in \mathbf{C}\,([0,\infty)\,; \mathbf{L}^\infty(0,a))$ of problem (1.1). Moreover there exists a function $A \in \mathbf{L}^\infty(0,a)$ such that the solution has the following asymptotics*

$$
u(x,t) = (1 - \theta(x-a)) A(x) t^{-\frac{1}{\alpha}} \Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}} \right) + O\left(t^{-\frac{1}{\alpha} - \omega} \right)
$$

for $t \to +\infty$ uniformly with respect to $x \in (0,a)$, where

$$\Lambda(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-z^{\alpha}+z\xi} dz$$

and

$$A(x) = \int_0^x u_0(y)dy + \int_0^{+\infty} d\tau \int_0^x |u(y,\tau)|^2 u(y,\tau)dy < +\infty,$$

here $\xi \in \mathbf{R}^+$, $\delta \in (0, 1-\alpha)$, $\omega = \frac{\delta}{\alpha}$ if $\delta \leq \min(\alpha, 1)$, and $\omega = 1$ if $\alpha \leq \delta \leq 1$.

Remark 1. Note that the symbols $K(p)$ under consideration are not analytic in the left half-complex plane (see definition (1.5)), so the contour of integration in the inverse Laplace transform could not be shifted in order to obtain some more rapid time decay (see formula (2.4) below). As a consequence, the solutions of nonlocal equation (1.3) have a potential decay rate such as $t^{-\frac{1}{\alpha}}$, in comparison with the case of purely differential operator $\mathbb{K}$. For example, it is well-known that solutions of the heat equation on a segment decay exponentially with respect to time.

Remark 2. By the method of this paper we also can consider more general nonlinearities of the form $|u|^{\rho} u$ with super critical power $\rho > \alpha$.

We organize our paper as follows. In Section 2 we solve the linear initial-boundary value problem corresponding to (1.1) and prove some preliminary estimates in Lemma 3. Section 3 is devoted to the proof of Theorem 1. Everywhere below by the same letter C we denote different positive constants.

§2. Linear problem

We consider the following linear initial-boundary value problem

$$\begin{cases} u_t + \mathbb{K}u = f(x,t), \ t > 0, x \in (0,a), \\ u(x,0) = u_0(x), \ x \in (0,a), \end{cases} \tag{2.1}$$

where the pseudodifferential operator $\mathbb{K}u$ on a segment $[0,a]$ is defined in (1.2).

We have for the Laplace transform of operator $\mathbb{K}u$, $p \notin (-\infty, 0)$

$$\begin{aligned} \int_0^a e^{-px} \mathbb{K}u dx &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a} - 1}{q-p} K(q)\widehat{u}(q,t)dq \\ = \quad & \frac{e^{-pa}}{2\pi i} \int_{\Gamma} \frac{e^{qa}}{q-p} K(q)\widehat{u}(q,t)dq + K(p)\widehat{u}(p,t), \end{aligned}$$

where the contour Γ was defined by the formula (1.7).

To derive an integral representation for solutions of the problem (2.1) we suppose that there exists a solution $u(x,t)$ of problem (2.1), which we prolonged by zero outside the interval $(0,a)$, that is

$$u(x,t) = 0 \text{ for all } x \notin [0,a]\,. \tag{2.2}$$

Applying the Laplace transformation with respect to x to the problem (2.1) we get

$$\begin{cases} \widehat{u}_t + \frac{e^{-pa}}{2\pi i}\int_\Gamma \frac{e^{qa}}{q-p} K(q)\widehat{u}(q,t)dq + K(p)\widehat{u}(p,t) = \widehat{f}(p,t), \quad t>0, \\ \widehat{u}(p,0) = \widehat{u}_0(p). \end{cases}$$

Integrating with respect to time t we obtain for the Laplace transform $\widehat{u}(p,t)$

$$\widehat{u}(p,t) = e^{-K(p)t}\hat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)}\widehat{f}_1(p,\tau)d\tau, \tag{2.3}$$

where

$$\widehat{f}_1(p,t) = \widehat{f}(p,t) + \frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}K(q)\widehat{u}(q,\tau)dq.$$

Note that by virtue of (2.2) the function $\widehat{u}(p,t)$ is analytic for all complex p and the condition $0<\alpha<1$ implies the condition (1.9) .

Taking the inverse Laplace transform of (2.3) with respect to space variable we get

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} e^{px-K(p)t}\widehat{u}_0(p)dp \\ &\quad + \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dpe^{px}\int_0^t e^{-K(p)(t-\tau)}\widehat{f}(p,\tau)d\tau \\ &\quad + \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dpe^{px}\int_0^t d\tau e^{-K(p)(t-\tau)} \\ &\quad \times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{(q-p)a}}{q-p}K(q)\widehat{u}(q,\tau)dq \\ &\equiv I_1 + I_2 + I_3, \end{aligned} \tag{2.4}$$

where $\varepsilon > 0$.

Now we prove that the last integral in (2.4) is equal to zero for all $x \in [0,a]$. Indeed, since Re $K(p) > 0$ for all Re $p > 0$ by the Cauchy Theorem we get for Re $q = 0, x \in [0,a]$, $\tau \in (0,t)$

$$\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dpe^{p(x-a)}e^{-K(p)(t-\tau)}\frac{1}{q-p}dp = 0.$$

Therefore changing the order of integration we obtain for $x \in [0,a]$ (we can change the order of integration since all integrals converge absolutely)

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi i}\int_0^t d\tau \int_{-i\infty}^{i\infty} e^{qa} K(q)\widehat{u}(q,\tau)dq \\
&\quad \times \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dp e^{p(x-a)} e^{-K(p)(t-\tau)} \frac{1}{q-p}dp \\
&= 0. \qquad (2.5)
\end{aligned}
$$

Since $u(x,t) = 0$ for all $x > a$ and for $x < 0$ substituting the Laplace transforms $\widehat{u}_0(p)$ and $\widehat{f}(p,\tau)$ into (2.4) and using (2.5), we obtain the following integral representation for solutions $u(x,t)$ of the problem (2.1)

$$
u(x,t) = \int_0^a u_0(y)G(x,y,t)dy + \int_0^t d\tau \int_0^a f(y,\tau)G(x,y,t-\tau)d\tau, \qquad (2.6)
$$

where Green function $G(x,y,t)$ is defined by

$$
G(x,y,t) = (1-\theta(x-a))\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K(p)t}dp.
$$

Thus in supposition that there exist solutions of problem (2.1) we get the integral representation (2.6) for these solutions.

Now we prove that the function $u(x,t)$ defined by formula (2.6) gives us a solution to problem (2.1). Indeed, takin the Laplace transformation of (2.6) we get for $\operatorname{Re} p = 0$

$$
\begin{aligned}
\widehat{u}(p,t) &= \int_0^\infty dx e^{-px} \int_0^a u_0(y)G(x,y,t)dy \qquad (2.7) \\
&\quad + \int_0^\infty dx e^{-px} \int_0^t d\tau \int_0^a f(y,\tau)G(x,y,t-\tau)dy \\
&= \int_0^a dx e^{-px} \left(\int_0^a u_0(y)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{q(x-y)-K(q)t}dqdy \right. \\
&\quad \left. + \int_0^t d\tau \int_0^a f(y,\tau)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{q(x-y)-K(q)(t-\tau)}dqd\tau \right).
\end{aligned}
$$

By analyticity of the symbol $K(p)$ in the complex half-plane $\operatorname{Re} p > 0$ and $\alpha < 1$ we have for all $\operatorname{Re} p = 0$ and $y \in [0,a)$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(q)t-qy}\frac{e^{(q-p)a}-1}{q-p}dq \\
&= \frac{1}{2\pi i}e^{-pa}\int_{-i\infty}^{i\infty} e^{-K(q)t}\frac{e^{q(a-y)}}{q-p}dq - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-K(q)t-qy}\frac{1}{q-p}dq \\
&= e^{-K(p)t-py} + \frac{1}{2\pi i}e^{-pa}\int_\Gamma e^{-K(q)t+q(a-y)}\frac{1}{q-p}dq.
\end{aligned}
$$

So changing the order of integration in formula (2.7) and calculating the integrals with respect to x we get

$$\begin{aligned}
\widehat{u}(p,t) &= \frac{1}{2\pi i}\int_0^a u_0(y)dy\int_{-i\infty}^{i\infty} e^{-K(q)t-qy}\frac{e^{(q-p)a}-1}{q-p}dq \\
&\quad +\frac{1}{2\pi i}\int_0^t d\tau\int_0^a f(y,\tau)dy\int_{-i\infty}^{i\infty} e^{-qy-K(q)(t-\tau)}\frac{e^{(q-p)a}-1}{q-p}dq \\
&= e^{-K(p)t}\left(\int_0^a e^{-py}u_0(y)dy+\int_0^t e^{K(p)\tau}d\tau\int_0^a e^{-py}f(y,\tau)dy\right) \\
&\quad +\frac{1}{2\pi i}e^{-pa}\int_0^a u_0(y)dy\int_\Gamma e^{-K(q)t+q(a-y)}\frac{1}{q-p}dq \\
&\quad +\frac{1}{2\pi i}e^{-pa}\int_0^t d\tau\int_0^a f(y,\tau)dy\int_\Gamma e^{-K(q)(t-\tau)+q(a-y)}\frac{1}{q-p}dq. \qquad (2.8)
\end{aligned}$$

Substituting (2.8) into the definition of the pseudodifferential operator $\mathbb{K}u$ (see formula (1.2)) we obtain for all $x\in(0,a)$

$$\begin{aligned}
\mathbb{K}u &= \int_0^a u_0(y)dy\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)}e^{-K(p)t}K(p)dp \\
&\quad +\int_0^t d\tau\int_0^a f(y,\tau)dy\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)}e^{-K(p)(t-\tau)}K(p)dp \\
&\quad +\int_0^a u_0(y)dy\frac{1}{2\pi i}\int_\Gamma e^{-K(q)t+q(a-y)}dq\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{p(x-a)}\frac{K(p)}{q-p}dp \\
&\quad +\int_0^t d\tau\int_0^a f(y,\tau)dy\frac{1}{2\pi i}\int_\Gamma e^{-K(q)(t-\tau)+q(a-y)}dq \\
&\quad \times\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{p(x-a)}\frac{K(p)}{q-p}dp,
\end{aligned}$$

whence using the fact that

$$\int_{-i\infty}^{+i\infty} e^{p(x-a)}\frac{K(p)}{q-p}dp=0$$

for all $x\in(0,a)$ and $q\in\Gamma$ we obtain via formula (2.6)

$$\begin{aligned}
\mathbb{K}u &= \Big(-\frac{\partial}{\partial t}\int_0^a u_0(y)dy\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)}e^{-K(p)t}dp \\
&\quad -\frac{\partial}{\partial t}\int_0^t d\tau\int_0^a f(y,\tau)dy\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)}e^{-K(p)(t-\tau)}dp \\
&\quad +\int_0^a f(y,\tau)dy\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)}dp\Big) = -u_t(x,t)+f(x,t).
\end{aligned}$$

So that the function $u(x,t)$ given by (2.6) satisfies equation $u_t(x,t)+\mathbb{K}u = f(x,t)$. Also it is easy to see that the initial condition is fulfilled

$$\begin{aligned} u(x,0) &= (1-\theta(x-a))\int_0^a u_0(y)G(x-y,0)dy \\ &= (1-\theta(x-a))\int_0^{+\infty} u_0(y)\delta(x-y)dy = u_0(x). \end{aligned}$$

Thus there exists a solution to the problem (2.1), which is given by formula (2.6). The uniqueness follows from the fact that all solutions have representation (2.6).

Note that by the Cauchy Theorem the Green function $G(x,y,t)=0$ for all $x<y$ and $t<0$, therefore formula (2.6) can be written as

$$u(x,t) = \int_0^x u_0(y)G(x,y,t)dy + \int_0^t d\tau \int_0^x f(y,\tau)G(x,y,t-\tau)d\tau, \tag{2.9}$$

where

$$G(x,y,t) = (1-\theta(x-a))\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K(p)t}dp. \tag{2.10}$$

Thus we have proved the following result.

Theorem 2. *Let the initial data* $u_0 \in \mathbf{L}^1(0,a)$ *and a source* $f(x,t) \in \mathbf{L}^1_{loc}\left(0,\infty;\mathbf{L}^1(0,a)\right)$. *Then there exists a unique solution* $u(x,t)$ *of the initial-boundary value problem* (2.1)*, which has representation* (2.9)*.*

Remark 3. By the representation (2.9) we see that $\lim_{x\to+0} u(x,t)=0$ for all $t>0$. We emphasize however that we do not need to put the boundary condition $u(0,t)=0$ into the problem (2.1) for its well-posedness, since this is an inherent property of solutions. For example if we put the boundary condition $u(0,t)=1$ into the problem (2.1), then there does not exist any solution.

Remark 4. Note that the Green function $G(x,y,t)$ is similar to that for the cases of a half-line and the full line. It can be obtained from the full line Green function via multiplication by the step function $(1-\theta(x-a))$.

In the next lemma we estimate the kernel $G(x,y,t)$. Denote $\Lambda(\xi) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^\alpha+z\xi}dz$.

Lemma 3. *We have the asymptotics for large time*

$$G(x,y,t) = (1-\theta(x-a))\, t^{-\frac{1}{\alpha}}\Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + y^\delta O\left(t^{-\frac{1+\delta}{\alpha}}\right), \tag{2.11}$$

for $y\in(0,x)$ *.*

Proof. Changing the variable of integration $p^\alpha t = q^\alpha$ we get

$$\begin{aligned} G(x,y,t) &= (1-\theta(x-a))\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-K(p)t}dp \\ &= t^{-\frac{1}{\alpha}}(1-\theta(x-a))\frac{1}{2\pi i}\left(\int_{-i\infty}^{i\infty} e^{-q^\alpha+q\widetilde{x}}dq + R(\widetilde{x},\widetilde{y})\right), \end{aligned}$$

where $\widetilde{x} = xt^{-\frac{1}{\alpha}}, \widetilde{y} = yt^{-\frac{1}{\alpha}}$, and

$$R(\widetilde{x},\widetilde{y}) = \int_{-i\infty}^{i\infty} e^{-q^\alpha+q\widetilde{x}}(e^{-q\widetilde{y}}-1)dq.$$

Using estimates $\left|e^{-q\widetilde{y}}-1\right| < C\left|q\widetilde{y}\right|^\delta$ and Re $q^\alpha > 0$ for Re $q = 0$ we easily get

$$t^{-\frac{1}{\alpha}}\left|R(\widetilde{x},\widetilde{y})\right| \le Ct^{-\frac{1}{\alpha}}\left|\int_{-i\infty}^{i\infty} e^{-\mathrm{Re}q^\alpha}\left|q\widetilde{y}\right|^\delta dq\right| = y^\delta O\left(t^{-\frac{1+\delta}{\alpha}}\right).$$

Lemma 3 is proved. □

Denote $\mathcal{G}(t)\phi = \int_0^x G(x,y,t)\phi(y)\,dy$, where $G(x,t)$ is defined in formula (2.10).

Lemma 4. *Suppose that the function $\phi \in \mathbf{L}^\infty(0,a)$. Then the estimate*

$$\left\|\mathcal{G}(t)\phi\right\|_\infty \le C(1+t)^{-\frac{1}{\alpha}}\left\|\phi\right\|_\infty$$

is valid for all $t > 0$.

Proof. Denote $\widetilde{G}(x) = \mathcal{L}^{-1}\left(e^{-p^\alpha}\right)$. Note that the function $\widetilde{G}(x)$ is a smooth function $\widetilde{G}(x) \in \mathbf{C}^\infty(\mathbf{R}^+)$ and decays at infinity so that

$$\sup_{x\in\mathbf{R}^+}\langle x\rangle^{1+\gamma}\left|\widetilde{G}(x)\right| \le C, \tag{2.12}$$

for all $0 < \gamma < 1$. Indeed, since $\mathrm{Re}\,p^\alpha > 0$ for $\mathrm{Re}\,p = 0$ we have

$$\left|\widetilde{G}(x)\right| = \left|\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-p^\alpha}dp\right| \le C\left\|e^{-p^\alpha}\right\|_{\mathbf{L}^1} \le C.$$

For all $x \ge 1$, integrating by parts and changing the contour of integration we get

$$\begin{aligned} \left|\widetilde{G}(x)\right| &= \left|\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px-p^\alpha}dp\right| = \left|\frac{\alpha}{2\pi i x}\int_{\infty e^{-\frac{i\pi}{2}+i\varepsilon}}^{\infty e^{\frac{i\pi}{2}-i\varepsilon}} e^{px-p^\alpha}p^{\alpha-1}dp\right| \\ &\le Cx^{-1-\gamma}\left|\int_{\infty e^{-\frac{i\pi}{2}+i\varepsilon}}^{\infty e^{\frac{i\pi}{2}-i\varepsilon}} e^{-p^\alpha}p^{-1+\alpha-\gamma}dp\right| \le Cx^{-1-\gamma}, \end{aligned}$$

where $\varepsilon > 0, 0 < \gamma < 1$. Therefore estimate (2.12) is true. By virtue of (2.12) we find

$$t^{-\frac{1}{\alpha}} \left\| \widetilde{G}\left(t^{-\frac{1}{\alpha}}(\cdot)\right) \right\|_{\mathbf{L}^1} = \left\| \widetilde{G}(\cdot) \right\|_{\mathbf{L}^1} \le C \left\| \langle x \rangle^{-1-\mu} \right\|_{\mathbf{L}^1_x} \le C,$$

hence by the Young inequality and using estimate $\|\phi\|_1 < C \|\phi\|_\infty$ we obtain

$$\|\mathcal{G}(t)\phi\|_\infty \le C \left\| t^{-\frac{1}{\alpha}} \widetilde{G}\left(t^{-\frac{1}{\alpha}}(\cdot)\right) \right\|_{\mathbf{L}^1} \|\phi\|_\infty \le C \|\phi\|_\infty$$

and

$$\begin{aligned} \|\mathcal{G}(t)\phi\|_\infty &\le C \left\| t^{-\frac{1}{\alpha}} \widetilde{G}\left(t^{-\frac{1}{\alpha}}(\cdot)\right) \right\|_{\mathbf{L}^\infty} \|\phi\|_1 \\ &\le C t^{-\frac{1}{\alpha}} \|\phi\|_1 \le C t^{-\frac{1}{\alpha}} \|\phi\|_\infty \end{aligned}$$

for all $t > 0$. Whence the estimate of the lemma follows. Lemma 4 is proved. □

§3. Global existence

We prove Theorem 1. We consider the linearized version of problem (1.1)

$$\left\{ \begin{array}{c} u_t + \mathbb{K}u = -i|v|^2 v, \quad t > 0, x \in (0,a), \\ u(x,0) = u_0(x), x \in (0,a). \end{array} \right. \tag{3.1}$$

We suppose that

$$\|u_0\|_\infty < \varepsilon_1$$

and $v \in X_\varepsilon$, where $\varepsilon_1 > 0$ is small enough, $\varepsilon = 100C\varepsilon_1$ with the constant C from (2.12) and

$$\mathbf{X}_\varepsilon = \{v \in X, \|v\|_{\mathbf{X}} < \varepsilon\},$$

$$\mathbf{X} = \left\{ v \in \mathbf{C}([0,+\infty); \mathbf{L}^\infty(0,a)), \|v\|_{\mathbf{X}} = \sup_{t>0} \langle t \rangle^{\frac{1}{\alpha}} \|v(t)\|_\infty < \varepsilon \right\}.$$

We have from (2.9)

$$u(x,t) = \mathcal{G}(t)u_0 - i\int_0^t \mathcal{G}(t-\tau)|v(\tau)|^2 v(\tau)d\tau, \tag{3.2}$$

where

$$\mathcal{G}(t)\phi(\tau) = \int_0^x G(x,y,t)\phi(y,\tau)dy$$

and

$$G(x,y,t) = (1-\theta\,(x-a))\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p(x-y)-p^{\alpha}t}dp.$$

Via estimate

$$\left\| |v|^2\, v(t) \right\|_1 \leq \|v(t)\|_\infty^3 \leq C\varepsilon^3(1+t)^{-\frac{3}{\alpha}}$$

applying $L^\infty(0,a)$ norm to formula (3.2) and using results of Lemma (4) we get

$$\begin{aligned}
& \|u(t)\|_\infty \leq C\,\|\mathcal{G}\,(t)\,u_0\|_\infty + C\int_0^t \left\|\mathcal{G}\,(t-\tau)\,|v(\tau)|^2\,v(\tau)\right\|_\infty d\tau \\
\leq\ & C\,(1+t)^{-\frac{1}{\alpha}}\,\|u_0\|_\infty \\
& +C\int_0^{\frac{t}{2}} d\tau \left\| |v|^2\,v(\tau)\right\|_1 (t-\tau)^{-\frac{1}{\alpha}}d\tau + C\int_{\frac{t}{2}}^t d\tau \left\| |v|^2\,v(\tau)\right\|_\infty d\tau \\
\leq\ & \varepsilon_1\,(1+t)^{-\frac{1}{\alpha}} + \varepsilon^3 C\left(\int_0^{\frac{t}{2}} \langle\tau\rangle^{-\frac{3}{\alpha}}(t-\tau)^{-\frac{1}{\alpha}}d\tau + \int_{\frac{t}{2}}^t \langle\tau\rangle^{-\frac{3}{\alpha}}d\tau\right) \\
\leq\ & \varepsilon\,(1+t)^{-\frac{1}{\alpha}}. \qquad (3.3)
\end{aligned}$$

We introduce the distance in $\mathbf{X}$

$$d(f,g) = \sup_{t>0}(1+t)^{\frac{1}{\alpha}}\,\|f\,(t) - g\,(t)\|_\infty\,.$$

Then in the same way as in the proof of (3.3) we have

$$d(u_1,u_2) = d(\mathbb{M}v_1,\mathbb{M}v_2) \leq \frac{1}{2}d(v_1,v_2), \qquad (3.4)$$

where u_1 and u_2 are solutions of the problems

$$\begin{cases} \partial_t u_j + \mathbb{K}u_j = -i\,|v_j|^2\,v_j,\ t>0, x\in(0,a)\,, \\ u_j(x,0) = u_0(x),\ x\in(0,a)\,. \end{cases}$$

Estimates (3.3) and (3.4) show that $\mathbb{M}$ is a contraction mapping from $\mathbf{X}$ into itself. Therefore there exist a unique solution $u(x,t)\in\mathbf{X}$ satisfying estimate $\|u\|_{\mathbf{X}} < \varepsilon$. This completes the proof of the first part of Theorem 1.

Now using estimate (3.3) we prove that the solution has the following asymptotics

$$u(x,t) = (1-\theta(x-a))A\,(x)\,t^{-\frac{1}{\alpha}}\Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + O\left(t^{-\frac{1+\delta}{\alpha}}\right)$$

for $t \longrightarrow +\infty$ uniformly with respect to x, where $\delta \in (0, 1-\alpha)$

$$\Lambda(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-z^{\alpha}+zx} dz$$

and

$$A(x) = \int_0^x u_0(y)dy - i\int_0^{+\infty} d\tau \int_0^x |u(y,\tau)|^2 u(y,\tau)dy$$

is a bounded function. Indeed, in view of asymptotics (2.11) of Lemma 3 we have

$$u(x,t) = (1-\theta(x-a))A(x)\, t^{-\frac{1}{\alpha}}\Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}}\right) + R(x,t), \tag{3.5}$$

where

$$\begin{aligned} |R(x,t)| &\le Ct^{-\frac{1+\delta}{\alpha}}\left\|(\cdot)^{\delta}u_0(\cdot)\right\|_1 + Ct^{-\frac{1+\delta}{\alpha}}\int_0^t d\tau\int_0^a y^{\delta}|u|^3 dy \\ &\quad + t^{-\frac{1}{\alpha}}\left|\Lambda\left(\frac{x}{t^{\frac{1}{\alpha}}}\right)\right|\int_t^{+\infty} d\tau\int_0^a |u|^3 dy \\ &\quad + \int_0^t d\tau \int_0^a |u(y,\tau)|^3 |G(x,y,t-\tau) - G(x,y,t)|\, dy. \end{aligned}$$

We have

$$|G_t(x,y,t)| \le C\left|\int_{-i\infty}^{i\infty} e^{-\mathrm{Re}\ |p|^{\alpha}t}|p|^{\alpha} dp\right| \le Ct^{-1-\frac{1}{\alpha}}.$$

Therefore we obtain

$$|G(x,y,t-\tau) - G(x,y,t)| \le Ct^{-1-\frac{1}{\alpha}}\tau$$

and

$$\begin{aligned} &\int_0^t d\tau\int_0^a |u(y,\tau)|^3 |G(x,y,t-\tau) - G(x,y,t)|\, dy \\ \le\ & Ct^{-1-\frac{1}{\alpha}}\int_0^t \tau(1+\tau)^{-\frac{3}{\alpha}} d\tau \le Ct^{-1-\frac{1}{\alpha}} \end{aligned}$$

for all $t \ge 1$. Hence by virtue of (3.3) we have

$$\begin{aligned} |R(x,t)| &\le Ct^{-\frac{1+\delta}{\alpha}}\|u_0\|_{\infty} + Ct^{-\frac{1+\delta}{\alpha}}\int_0^t (1+\tau)^{-\frac{3}{\alpha}} d\tau \\ &\quad + t^{-\frac{1}{\alpha}}\left|\Lambda\left(xt^{-\frac{1}{\alpha}}\right)\right|\int_t^{+\infty}(1+\tau)^{-\frac{3}{\alpha}}d\tau + Ct^{-1-\frac{1}{\alpha}} \\ &\le Ct^{-\frac{1+\delta}{\alpha}} + Ct^{-\frac{1}{\alpha}+1-\frac{3}{\alpha}} + Ct^{-1-\frac{1}{\alpha}} \\ &\le \begin{cases} Ct^{-\frac{1+\delta}{\alpha}} \text{ if } \delta \le \min(\alpha, 1), \\ Ct^{-1-\frac{1}{\alpha}} \text{ if } \alpha \le \delta \le 1. \end{cases} \end{aligned}$$

Theorem 1 is proved.

Acknowledgement

We are grateful to an unknown referee for many useful suggestions and comments. This work is partially supported by CONACYT and COSNET.

References

[1] H. Aikawa and N. Hayashi, *Holomorphic solutions of semilinear heat equations,* Complex Variables, **16** (1991), pp. 115 - 125.

[2] N. Hayashi and E.I. Kaikina, *Local existence of solutions to the Cauchy problem for nonlinear Schrödinger equations,* SUT Journal of Mathematics, **34**, No. 2 (1998), pp. 111-137.

[3] N. Hayashi, E.I. Kaikina and R. Manzo, *Local and global existence of solutions to the nonlocal Whitham equation on half-line,* Nonlinear Analisys, **48** (2002), pp. 53-75.

[4] N. Hayashi, E.I. Kaikina and P.I. Naumkin, *Large time behavior of solutions to the dissipative nonlinear Schrödinger equation,* Proceedings of the Royal Soc. Edingburgh, **130**-A (2000), pp. 1029-1043.

[5] N. Hayashi, E.I. Kaikina and F.R. Paredez, *Boundary-value problem for the Korteweg-de Vries-Burgers type equation,* Nonlinear Differential Equations and Applications, **8** (2002), pp. 439-463.

[6] N. Hayashi, E.I. Kaikina and I.A. Shishmarev, *Asymptotics of Solutions to the Boundary-Value Problem for the Korteweg-de Vries-Burgers equation on a Half-Line,* Journal of Mathematical Analysis and Applications, **265** (2002), No. 2, pp. 343-370.

[7] E.I. Kaikina, *The Cauchy problem for the nonlocal Schrodinger equation. I, II* Matemat. Modelirovanie, 3 (11) (1991), pp. 83-95, 96-108.

[8] E.I. Kaikina, P.I. Naumkin and I.A. Shishmarev, *Asymptotic behavior for large time of solutions to the nonlinear nonlocal Schrödinger equation on half-line,* SUT Journal of Mathematics, **35** (1) (1999), pp. 37-79.

[9] P.I. Naumkin and I.A. Shishmarev, *Nonlinear Nonlocal Equations in the Theory of Waves,* Transl. of Math. Monographs, AMS, Providence, R.I., **133**, 1994.

[10] I. A. Shishmarev, M. Tsutsumi and E. I. Kaikina, *Asymptotics in time for the nonlinear nonlocal Schrödinger equations with a source*, J. Math. Soc. Japan, **51** (1999), pp. 463-484.

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